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ALGEBRA MADE EASY

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[Vol. I]

(MATRICULATION ALGEBRA)

(CONTAINING THE SYLLABUS PRESCRIBED FOR THE
MATRICULATION EXAMINATION OF ALL
THE INDIAN UNIVERSITIES)

BY

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PREFACE

THE present work is intended as a text-book in Algebra for all classes of students in our schools. It differs, however, in several respects from the existing text-books on the subject at present in use

Algebra like every other branch of Mathematics should be studied more as a subject for mental discipline than for anything else. An intelligent grasp of principles, therefore, is to be chiefly aimed at and not the mere learning by rote of a certain number of rules with some readiness in their application. This is the ideal I have ever kept in view in the preparation of this work.

The elementary principles of the subject have been dwelt upon at considerable length in the earlier chapters of the book. The full import of negative quantities has been explained, it is believed, with some degree of clearness, almost at the very outset, and rules for their addition and subtraction have subsequently been deduced therefrom by a very simple mode of reasoning.

The proposition of each article after being clearly demonstrated has been copiously illustrated by a number of select examples, a much larger number of other examples, arranged progressively, has then been added as an exercise for the student. The last article of each chapter consists of a number of miscellaneous examples fully worked out as interesting illustrations of special artifices; these again are followed by similar others for exercise.

The chapters on Formulæ and Factors will, it is hoped, be particularly acceptable to the young learner. The subject of factorisation has been treated exhaustively as far as the limits of this work would allow. The last chapter, on Elimination and Miscellaneous Artifices will, I hope, be of considerable use to the more advanced student.

Entrance Examination Papers of the Calcutta University from 1858 to 1890 will be found at the very end. The more important and difficult problems from these papers are fully worked out in the body of the work in illustration of the principles upon which their solutions depend, whilst others, comparatively simpler, have been suitably introduced among the exercises, just to give the student an opportunity of reassuring himself, when successful in working them out with unaided exertion, that his knowledge has, to some extent at least, come up to the University standard. With the examination papers are also given references to the pages where these problems are to be found in the body of the work.

Instead of ending the book with a collection of miscellaneous examples promiscuously arranged, I have added a number of miscellaneous examples in the form of separate examination papers, any one of which may be regarded as a good exercise for the student at a sitting of about two hours and a half.

The entire book contains nearly 3000 examples in all, of which over 400 are fully worked out. Many of these examples have been specially devised for this work whilst for the rest I am indebted to several of the standard works of English Universities.

I have attempted to make the work useful to the school student as a means of acquiring algebraical skill along with a sound knowledge of principles, and I have spared no pains for it. It is now for all experienced teachers of mathematics to judge as to how far I have been successful in my endeavour. To gentlemen interested in the cause of education I shall be much obliged if they will kindly communicate to me any corrections or suggestions that they may consider necessary for the improvement of the work.

DACCA : *March*, *1890* K. P. BASU

PREFACE TO THE SECOND EDITION

A FEW words of explanation seem to be necessary in connection with the publication of this edition. The first edition having been published rather unseasonably last year, I did not at all anticipate that a second edition would be in demand so soon. Accordingly the work of re-publication was not taken at hand earlier than January last. But the book beginning to be received with increased favour in different educational circles with the commencement of the new academic session, the first edition, consisting of 2250 copies, was found to be exhausted before the end of the last month. Hence, in the interests of the students of all those schools in which the book has been adopted as a text-book, my publisher had no other alternative than to hasten the work by all possible means. In consequence of this, I am sorry, I have not been able to give the book as thorough a revision as I intended, nor to effect such improvements as have been kindly suggested by some friends

DACCA : *March*, *1891* K. P. BASU

PREFACE TO THE FIFTH EDITION

IN this edition the bulk of the work has increased by about 60 pages. The additions that have been made are as follows : (1) an increase in the number of examples of exercises in the earlier chapters of the book ; (2) the insertion of examples with *Fractional Indices* in the chapters on *Multiplication* and *Division* ; (3) the introduction of three sets of Miscellaneous Exercises in suitable places in the body of the work ; (4) an article on the Method of finding the *Cube Root* of a Compound Algebraical Expression ; and (5) a chapter on

Quadratic Equations. For several of these improvements I am indebted to the kind and repeated suggestions of friends who are practical workers in the field of education. It is, therefore, hoped that the present edition will be found considerably more useful than its predecessors.

DACCA : *January*, *1894* K. P. BASU

PREFACE TO THE SIXTH EDITION

IN this edition the book has been thoroughly revised and answers to the examples in all the exercises have been carefully verified. Some additions and alterations have been occasionally made, but they do not deserve any special mention. I am indebted to several friends for their kindness in pointing out errors and misprints. My special thanks are due to Babu Bepinbihari Ganguly, B.A., Teacher, Jubilee School, Dacca, and to Moulvie Abdullah Khan, Teacher, D. B. School, Dipalpur (Montgomery).

DACCA : *April*, *1895* K. P. BASU

ALGEBRA MADE EASY

INTRODUCTION

1. How things are measured and represented by number. This will be best explained by taking up some particular instances familiar to the student.

(i) If we want to know the length of a piece of cloth, we are satisfied when we find how often this length contains a smaller length called a *metre* ($\frac{1}{10000000}$th of the distance between a pole and the equator of the earth—39·37011... inches).

(ii) If we want to know the distance between Dacca and Calcutta, we are satisfied when we are told how often this distance contains a smaller distance called a *kilometre*.

(iii) If we want to know the value of a sum of money, we are satisfied when we are told how often this sum contains a smaller sum called a *rupee*.

(iv) If we want to know the weight of a quantity of rice, we are satisfied when we find how often this weight contains a smaller weight called a *kilogram*.

From the above instances it is clear that whenever we have to measure a thing, we do so by finding how often it contains a smaller thing of the same kind. The 'smaller thing' chosen for this purpose is called the **unit** and the *number* which shows how often this unit is contained in the thing measured is called the *numerical measure* (or simply, the **measure**) of the latter ; thus, in the first instance, the *unit of length* is a *metre* ; in the second, the *unit of distance* is a *kilometre* ; in the third, the *unit of money* is a *rupee* ; and in the fourth instance, the *unit of weight* is a *kilogram*. Again, if we know that the piece of cloth is 10 metres long, that the distance between Dacca and Calcutta is 416 kilometres, that the sum of money is 500 rupees, and that the weight of the rice is 25 kilograms, then, 10 is the *measure* of the length of the cloth, 416 is the *measure* of the distance between Dacca and Calcutta, 500 is the *measure* of the sum of money, and 25 is the *measure* of the weight of the rice.

A thing is said to be *represented* by the number which shows how often that thing contains the unit of its kind ; thus, in the above instances, the length of the piece of cloth is represented by 10, the distance between the two places is represented by 416 ; and so on.

Note 1. *Such expressions as 'a sum of money estimated in pounds = 30', 'a distance estimated in kilometres = 25', and the like, respectively mean 'the numerical measure of a sum of money when a £ is the unit, is 30', 'the numerical measure of a distance when the unit is a kilometre, is 25', &c.*

Note 2. *It must be clearly understood that one and the same thing will be represented by different numbers when the units are different ; thus, taking a metre as the unit, a length of 10 metres is represented by 10, but if the unit be 2 metres, the same length is represented by 5.*

Example 1. If the *unit of length* be a metre, what will be the measure of 5 decametres and 2 metres ?

5 decametres and 2 metres, being equivalent to 52 metres, evidently contains the unit of length (*i.e.*, a metre) 52 times.

Hence, the required *measure* is 52.

Example 2. If a minute and a half be represented by 30, what is the *unit of time* ?

A minute and a half is equivalent to 90 seconds.

Now, since 30 is the *measure* of 90 seconds, it is clear that the *unit of time* is contained 30 times in 90 seconds.

Hence, the *unit of time* is $\frac{1}{30}$th part of 90 seconds, and is, therefore, equal to 3 seconds.

EXERCISE 1

1. What will be the *measure* of 2 quintals and 20 kilograms, when a kilogram is the *unit of weight* ?

2. What will be the measure of the same weight, when 10 kilograms is the *unit* ?

3. If a distance of 360 kilometres be represented by 30, what is the *unit of distance* ?

4. If the same distance be represented by 45, what is the *unit* ?

5. If a sum of 400 rupees be represented by 16, what will be the *measure* of Rs. 225 ?

6. If a length of 8 metres and 8 decimetres be represented by 22, what will be the *measure* of 4 metres and 8 decimetres ?

7. What must be the *unit of time* in order that 3 hours and 45 minutes may be represented by 5 ?

8. If the *unit of time* be 15 seconds, what time will be represented by 60 ?

9. If the *unit of weight* be $7\frac{1}{2}$ kilograms, what number will represent $1\frac{1}{2}$ quintals ?

10. If 8 square metres be the *unit of area*, what number will represent an area of 1250 square centimetres and what will represent 162 sq. metres ?

11. If an area of 125 sq. metres be represented by $8\frac{1}{3}$, how many square metres are there in 3 times the *unit of area* ?

12. What is the *unit of money* if a sum of £10. 2*s*. 6*d*. be represented by 27 ?

13. If 7*s*. 8*d*. be the *unit of money*, what will be the *measure* of £7. 13*s*. 4*d*. ?

14. If Rs. 5. 71 P. be the *unit of money*, what will be the *measure* of Rs. 51. 39 P. ?

15. If 23 kilograms 55 grams be the *unit of weight*, what will be the *measure* of 6 quintals 45 kilograms 540 grams ?

16. If Rs. 20. 2 P. be represented by $5\frac{1}{2}$, what will be the *measure* of Rs. 43. 68 P., supposing the new unit to be 3 times the former ?

17. If 273 be the *measure* of 9 cwt. 3 qrs., what number will represent one ton, supposing the new unit to be one-eighth of the former ?

18. If 84 be the *measure* of 11 metres 9 decimetres, what number will represent 22 metres 5 decimetres supposing the new unit to be three-seventeenths of the former ?

19. If 26 days 10 hours and 26 minutes be represented by 120, what number will represent a leap-year, supposing the new unit to be 47 minutes 13 seconds less than the former ?

20. In the preceding example what would be the answer if the latter unit exceeded the former by 6 hours 54 minutes 47 seconds ?

2. Different uses of the word Quantity.

(i) Anything that can be represented by *number* is called a *quantity*. Thus, time, weight, money, distance, &c., which all admit of numerical representation, as shown in the preceding article, are quantities.

(ii) *Quantity* is also often used in the sense of *number*, integral or fractional.

(iii) An algebraical *expression* also is sometimes called a *quantity*. [We shall refer to this again in its proper place.]

N. B. Quantities like weight, money, distance, area, &c., are often spoken of as ***concrete quantities*** *as distinguished from* ***numerical quantities*** *which mean only Arithmetical numbers, integral or fractional.*

[**Note.** *Any whole number is called an integer or an integral number.*]

3. What is Algebra? Algebra, like Arithmetic, is a science of *numbers* with this distinction that the numbers in Algebra are *generally* denoted by *letters* instead of by *figures.*

Hence, whenever concrete quantities come under the domain of Algebra, it is *only* their numerical *measures* (*i.e.*, the abstract numbers which represent them) with which we must concern ourselves.

Note. *The name 'Algebra' is derived from the title of a certain Arabian treatise* ***'Al-jebrw'al Muqabalah'****. This book was translated by early European scholars who first learnt of Algebra from the Arabs. But as in Arithmetic, so in Algebra, the Arabs got their first lessons from the ancient Hindus whose contributions to this science are of a fundamental character. Even some of the technical terms which are commonly used in modern Algebra are of Hindu origin.*

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CHAPTER I

SYMBOLS : SIGNS : SUBSTITUTIONS

4. Symbols. The *letters* of the alphabet a, b, c,………… &c. are used to denote numbers and the signs $+$, $-$, $\times$, $\div$, $=$,……… &c. are used either to denote operations to be performed upon the number to which they are attached or as abbreviations. These *letters* and *signs* are called *symbols*.

The letters as distinguished from the signs are called *symbols of quantity*.

5. The Plus Sign. The sign $+$ is read *plus* and when placed before a number indicates that the number is to be *added* to what precedes it. Thus, $a+b$ (which is read *a plus b*) means that the number denoted by b is to be *added* to that denoted by a ; hence, if a denote 5 and b denote 3, $a+b$ denotes 8. Again, $a+b+c$ means that the number denoted by b is to be added to that denoted by a, and to the result thus obtained, is to be added the number denoted by c ; hence, if a, b, c denote 5, 3, 2 respectively, $a+b+c$ denotes 10.

6. The Minus Sign. The sign $-$ is read *minus* and when placed before a number indicates that the number is to be *subtracted* from what precedes it. Thus, $a-b$ (which is read *a minus b*) means that the number denoted by b is to be *subtracted* from that denoted by a ; hence, if a denote 8 and b denote 3, $a-b$ denotes 5. Again, $a-b-c$ means that the number denoted by b is to be subtracted from that denoted by a, and from the result thus obtained, the number denoted by c is to be subtracted ; hence, if a, b, c denote 8, 3, 1 respectively, $a-b-c$ denotes 4.

N. B. When any number of quantities are connected with one another by the signs ***plus*** *and* ***minus****, the order of the operations is from* ***left to right****.* ***Thus****, $a-b+c$ means that the number denoted by b is to be subtracted from that denoted by a and to the result thus obtained, is to be added the number denoted by c.*

7. The Sign Plus or Minus. The sign $\pm$ is read *plus or minus* and when placed before a number indicates that the number is to be either *added to* or *subtracted from* what precedes it. Thus, if a denote 7 and b denote 2, $a \pm b$ (which is read *a plus or minus b*) denotes *either* 9 *or* 5.

8. The Sign of Difference. The sign $\sim$ when placed between two numbers indicates that the less of the two is to be subtracted from the greater. Thus, if a denote 5 and b denote 8, $a \sim b$ denotes 3.

9. The Sign of Multiplication. The sign × is read *into* and when placed between two numbers indicates that the number on the right of it is to be *multiplied* by that on the left.

Thus, $a \times b$ (which is read *a into b*) means that the number denoted by b is to be multiplied by that denoted by a; hence, if a denote 5 and b denote 3, $a \times b$ denotes 5 *times* 3, or 15.

The sign of multiplication is generally omitted when its position is between two numbers either (1) *both* of which are denoted by letters, or (2) the *first* of which is denoted by a *figure* and the second by a letter. Thus, ab is used for $a \times b$, and $4a$ for $4 \times a$.

Note. *The reason why* 83 *cannot be used for* 8×3 *is clear, because in Arithmetic* 83 *has already been understood to mean* $80+3$.

Sometimes the sign × is replaced by a dot, thus, $a.b$ and 5.4 respectively mean the same as $a \times b$ and 5×4. The dot so used is always placed as shown in the above instances in order to distinguish it from the decimal point which is put a little higher up; thus, 5.4 is read *five into four* whereas $5{\cdot}4$ is read *five decimal four.*

10. The Sign of Division. The sign ÷ is read *by* and when placed between two numbers indicates that the number on the left of it is to be *divided* by that on the right. Thus, $a \div b$ (which is read *a by b*) means that the number denoted by a is to be divided by that denoted by b; hence, if a denote 6 and b denote 3, $a \div b$ denotes 2. Similarly, $a \div b \div c$ means that the number denoted by a is to be divided by that denoted by b; and the result, thus obtained, is to be divided by that number denoted by c.

N. B. When any number of quantities are connected together by the signs of multiplication and division, the order of the operations is always from ***left to right.*** *Thus,* $a \times b \div c$ *means that the number denoted by* b *is to be multiplied by that denoted by* a*, and the result, thus obtained, is to be divided by the number denoted by* c*. Similarly,* $a \div b \times c$ *means that the number denoted by* a *is to be divided by that denoted by* b *and the result, thus obtained, is to be multiplied by the number denoted by* c*.*

Note. a *divided by* b *is also often expressed as* $\frac{a}{b}$; *thus,* $\frac{a}{b}$ *means the same as* $a \div b$.

11. Expression; Term. Any intelligible collection of letters, figures and *signs of operation* is called an *Algebraical Expression.* Such a collection is also sometimes called an *Algebraical Quantity*, or briefly, a *Quantity.* [See Art. 2]

Note. *Signs like* +, −, ×, ÷, *which indicate the operations to be performed upon the numbers to which they are attached, are called* ***signs of operation.***

The parts of an Algebraical Expression that are connected by the sign + or − are called its *terms.*

Thus, $5a+ab \div c \times d-8c \times f \div g$ is an algebraical expression of which the terms are $5a$, $ab \div c \times d$, $8c \times f \div g$.

Expressions are either simple or compound. A *simple* expression is one which has no parts connected by the sign + or −, *i.e.*, which consists of only one term, as $3ab$, $\frac{5x}{6y}$, $\frac{4x \times 3y}{2z}$, and is also called a *Monomial.* A *compound* expression consists of two or more terms ; if it consist of two terms, as $2a+5bcd$, $5x+\frac{6y}{7z}$, it is called a *Binomial* ; if of three terms, as $a+bc+8efg$, $x \times y+z+a \times c-\frac{8b}{9e}$, a *Trinomial* ; and if of more than three terms, a *Multinomial* or a *Polynomial.*

12. Functions ; Variables. Any expression involving a letter is called a *function* of that letter. Thus, x^3+5x+8 is a function of x ; a^2+ab+b^2 is a function of a and b ; $a^3+b^3+c^3+2abc$ is a function of a, b and c ; and so on.

The letters of which a function consists are called its *variables.* Thus, $x^2+5xy+y^2$ is a function of which the variables are x and y.

13. Sign of Equality. The sign = is read 'equals' or 'is equal to' and when placed between two *expressions* indicates that they are equal to one another. Thus, $b+c=a$ (which is read *b plus c equals a*) means that the number denoted by $b+c$ is equal to that denoted by a.

EXAMPLES

N. B. (1) A distinction must be observed between $a \div b \times c$ and $a \div bc$. The latter means that the number denoted by a is to be divided by that denoted by bc, whereas the former means that the number denoted by a is to be divided by that denoted by b, and the result, thus obtained, is to be multiplied by the number denoted by c. That is to say, when the sign of multiplication is omitted between any number of quantities, the result obtained by multiplying them together is to be regarded as a ***single quantity.***

N. B. (2) In finding the value of any ***expression*** *the values of the several* ***terms*** *which it contains must be first determined by the process mentioned in the Note of Art. 10 and* ***afterwards*** *the value of the whole expression is to be found by the process mentioned in the Note of Art. 6. Thus, in finding the value of the expression $a \times b-c \div d \times e+f \times g$ we must first of all find the values of the three terms, namely, $a \times b$, $c \div d \times e$ and $f \times g$; then subtract the value of the second term from that of the first, and to the result, thus obtained, add the value of the third.*

The above principles will be sufficiently illustrated by the following examples :

Example 1. If $a=2$, $b=3$, $c=5$, find the value of $5a+8b+7c$

$$5a=5\times a=5\times 2=10\,;$$
$$8b=8\times b=8\times 3=24\,;$$
$$7c=7\times c=7\times 5=35.$$

Therefore, $5a+8b+7c=10+24+35=34+35=69.$

Example 2. If $a=8$, $b=5$, $c=2$, find the value of $6a-5b+4c$.

$$6a=6\times a=6\times 8=48\,;$$
$$5b=5\times b=5\times 5=25\,;$$
$$4c=4\times c=4\times 2=8.$$

Therefore, $6a-5b+4c=48-25+8$
$=23+8=31.$

Example 3. If $m=3$, $n=7$, $t=9$, $v=4$, find the value of

$$7m\div 2n\times 8t\div 3v.$$

As the order of the operations is from *left to right*, we must proceed as follows: Divide $7m$ by $2n$; multiply $8t$ by the result; and then divide the product thus obtained by $3v$.

Now, (1) $7m\div 2n=\dfrac{7m}{2n}=\dfrac{7\times 3}{2\times 7}=\dfrac{3}{2}\,;$

(2) $\frac{3}{2}\times 8t=\frac{3}{2}\times 8\times 9=3\times 4\times 9\,;$

(3) $3\times 4\times 9\div 3v=\dfrac{3\times 4\times 9}{3\times 4}=9.$

Hence, the required value $=9$.

Example 4. If $a=1$, $b=2$, $c=3$, $d=6$, $e=5$, $f=0$, find the value of $abc-d\div b\times a+def+b\div a\times c-d\div bc$.

The given expression consists of 5 terms, namely, abc, $d\div b\times a$, def, $b\div a\times c$ and $d\div bc$.

Now, (1) $abc=a\times b\times c=1\times 2\times 3=6\,;$

(2) $d\div b\times a=6\div 2\times 1=3\times 1=3\,;$

(3) $def=d\times e\times f=6\times 5\times 0=0\,;$

(4) $b\div a\times c=2\div 1\times 3=2\times 3=6\,;$

(5) $d\div bc=\dfrac{d}{bc}=\dfrac{6}{2\times 3}=1.$

Hence, the required value $=6-3+0+6-1=3+6-1=8.$

EXERCISE 2

If $a=8$, $b=2$, $c=4$, find the numerical values of the following expressions:

1. $b+c\times a$. 2. $a-b\times c$. 3. $a\div c\times b$.

4. $a\div cb$. 5. $a\div 3\times b$. 6. $a\div 3b$.

7. $a-c\div b$. 8. $b+a\div c$. 9. $3a-4c+2b$

10. $a-c\div b+a\div c$. 11. $a\div c\div 2\times b$. 12. $a\div c\div 2b$.

13. $5a\div 2c$. 14. $5a\div 2\times c$.

15. $4bc-a\div 4\times b+c\div 2b$.

16. $80\div c\times ab+80\div ca\times b$.

17. $3ca\div 16b+5a\div 16\times b-a\div 2c\times c\div b\times 4$.

18. $48a\div c\div b\times 6\div 4c-3a\div 2c\div 4\times 3+b\times 8+6b\div a+2\times c\div 3\times 5$

If $m=2$, $n=3$, $p=4$, $q=0$, $r=7$, $s=10$, find the numerical values of the following expressions:

19. $8m-3p\div mn+q\times 3r+5s\div 2\times p$.

20. $s\times 6\div 5m\times 8p+16n$.

21. $mnr+5qs-3s\div m\div 5n+4r\div 3p\times 6m$.

22. $3\times r\div 5\times s\div 7\times p-8rs\div m\div 3\times n\div 7p+5m\div 2r\times 7$.

23. $4\times\dfrac{n-m}{p}-3\times\dfrac{p-m}{n}+2\times\dfrac{p-n}{m}$

24. $\dfrac{11r+n}{p+q}\times 2pm+\dfrac{14s+4n+2p}{sn+2}$.

25. $\dfrac{3m+2n}{q+p}-\dfrac{4p-3n}{q+r}+\dfrac{2p+3m}{q+m}$.

14. **Factor.** If any number be equal to the product of two or more numbers, each of the latter is called a *factor* of the former

[**Note.** *The* **product** *of two or more numbers is the result obtained by multiplying them together.*]

Thus, 3, 5 and 7 are the factors of 105, $\because 105=3\times 5\times 7$.

Similarly, 3, a, b and x are the factors of $3abx$, because

$$3abx=3\times a\times b\times x.$$

15. **Coefficient.** The number expressed in figures or symbols, which stands before an algebraical quantity as a multiplier, is called its *coefficient*. Thus, in $5abc$, 5 is the coefficient of abc, $5a$ is the coefficient of bc and $5ab$ is the coefficient of c.

A coefficient which is purely a numerical quantity is called a **numerical coefficient**; thus, in $5abc$, the coefficient of abc is numerical.

A coefficient which is not wholly numerical is called a **literal coefficient**, thus, in $5abc$, coefficients of bc and c are *literal.*

[Note. *When no arithmetical number stands before a quantity, the number* 1 *is understood ; thus,* a *is understood to mean* $1a$.]

16. Power ; Index ; Exponent. If a quantity be multiplied by itself any number of times, the product is called a *power* of that quantity. Thus, $a \times a$, $a \times a \times a$, $a \times a \times a \times a$, &c., are powers of a.

$a \times a$ is called the *second power* or *square* of a and is written a^2 ;

$a \times a \times a$ is called the *third power* or *cube* of a and is written a^3 ;

$a \times a \times a \times a \times a \times$ &c. to n factors is called the nth *power* of a and is written a^n.

The small figure or letter placed above a quantity and to the right of it to express its power is called the **Index** or **Exponent** of that power. Thus, 2, 3, 5, m are respectively the *indices* or *exponents* of a^2, a^3, a^5, a^m.

[Note. a^2 *is usually read* **'a squared'**, a^3 *is read* **'a cubed'**, a^4 *is read* **'a to the fourth'**, *or simply,* **'a fourth'** ; *and so on. Thus,* a^n *is read* **'a to the nth'** *or* **'a nth'**.

The quantity a *itself is called the* ***first*** *power of* a *and thus* a *is understood to mean* a^1.]

17. Dimensions and Degree of a Product. Each of the letters which occur as factors of an algebraical product is called a *dimension* of the product, and the number of the letters is called the *degree* of the product. Thus, a^2x^5y which is equivalent to $a \times a \times x \times x \times x \times x \times x \times y$, is said to be of *eight dimensions*, or of the *eighth degree* ; similarly, $ab^2c^4d^5$ is said to be of *twelve dimensions* or of the *twelfth degree.*

A numerical coefficient is not counted. Thus, $5ab^2c^3$ and ab^2c^3 are both said to be of *six dimensions* or of the *sixth degree.*

When an algebraical expression contains terms of different dimensions, the degree of the term which is of the highest dimensions is also called the *degree of the expression.*

18. Homogeneous Expression. An algebraical expression is said to be *homogeneous* when all its terms are of the same dimensions. Thus, the expression $5a^3b - 7a^2bc + 8b^2c^2$ is homogeneous, for each of its terms is of four dimensions.

EXAMPLES

Example 1. If $a=3$, find the numerical value of a^5-5a.

We have $a^5=a\times a\times a\times a\times a$

$=3\times 3\times 3\times 3\times 3=243$;

and $5a=5\times a$

$=5\times 3=15$.

Hence, the given expression $=243-15=228$.

Example 2. If $a=4$, find the numerical value of $2a^5-5a^2$.

We have $2a^5=2\times a\times a\times a\times a\times a$

$=2\times 4\times 4\times 4\times 4\times 4$

$=2048$;

and $5a^2=5\times a\times a$

$=5\times 4\times 4=80$.

Hence, the given expression $=2048-80=1968$.

Example 3. If $a=2$, $b=3$, $c=4$, $d=5$, find the numerical value of $\frac{a^5b^3d}{c^2}$.

$$\text{The given expression}=\frac{a\times a\times a\times a\times a\times b\times b\times b\times d}{c\times c}$$

$$=\frac{2\times 2\times 2\times 2\times 2\times 3\times 3\times 3\times 5}{4\times 4}$$

$$=2\times 3\times 3\times 3\times 5=270.$$

EXERCISE 3

If $a=8$, $b=12$, $c=4$, $m=7$, $n=6$, $x=2$, $y=3$, find the values of :

1. $3x^3$.
2. $7a^2+b$.
3. $2x^7-7n^2$.
4. $8cy^3-axy^2$.
5. $5c^5+3a^3$.
6. $7bx^3y^2+mn^4$.
7. a^x+c^y.
8. $9a^3b^2c^4+8n^2x^6y^2-b^2y+x$.
9. $2x^ab+a^2b^x$.
10. $3c^nx^b+a^my^3$.
11. Find the value of $y^6-65y^4+66y^2-21y+40$, when $y=8$.
12. Find the value of $8x^4+6x^3+11x^2+13x+29$, when $x=\cdot 75$.
13. Find the value of $15a^3-34a^4+7a-4a^2+35a^5-3$, when $a=\frac{1}{4}$.
14. Find the value of $23+20m+78m^5-199m^6+25m^8$, when $m=2\cdot 6$.
15. Find the value of $50y^7-51y^4+35y-563y^5-19$. when $y=3\cdot 4$.

16. Find the value of $64n^{10}-55n^4+32n^6-121n^8+64n^2-4n^4+79$, when $n=1{\cdot}375$.

Find the values of $a^3+b^3+c^3-3abc$:

17. When $a=29$, $b=24$, $c=27$.

18. When $a=5{\cdot}625$, $b=3{\cdot}625$, $c=4{\cdot}625$.

19. When $a=44\frac{3}{7}$, $b=51\frac{3}{7}$, $c=58\frac{3}{7}$.

20. When $a=1667$, $b=1674$, $c=1659$.

19. **Roots.** That quantity whose square (or second power) is equal to any given quantity a, is called the *square root* of a, and is denoted by the symbol $\sqrt[2]{a}$, or more simply, by $\sqrt{a}$. Thus, $3=\sqrt{9}$, because $3^2=9$.

That quantity whose cube (or third power) is equal to any given quantity a, is called the *cube root* of a, and is denoted by the symbol $\sqrt[3]{a}$. Thus, $2=\sqrt[3]{8}$, because $2^3=8$.

Generally, that quantity, whose nth power, where n is any whole number, is equal to any given quantity a, is called the nth *root* of a, and is denoted by the symbol $\sqrt[n]{a}$ Thus, $2=\sqrt[5]{32}$, because $2^5=32$; $3=\sqrt[4]{81}$, because $3^4=81$; and so on.

The sign $\sqrt{\ }$ is often called the *Radical sign*. It is said to be a corruption of the letter r, the first letter of the word *radix*.

Note. *$\sqrt{a}$, which means the square root of a, is often read simply as 'root a'.*

20. **Brackets.** Each of the symbols ( ), { }, and [ ] is called a *pair of brackets*. When an algebraical expression is enclosed within brackets, it is to be regarded as a *single* quantity by itself. Thus, $(a+b)x$ means that the number denoted by x is to be multiplied by that denoted by $a+b$, whereas $a+bx$ means that x is to be multiplied by b and the product added to a.

Hence, the expression $d+(a+b)x$ must be regarded as a *binomial*, the two terms being d and $(a+b)x$. Similarly, $c-\{d+(a+b)x\}$ also must be regarded as a *binomial*, the terms being c and $\{d+(a+b)x\}$, whereas, if the brackets be taken off, $c-d+a+bx$ is a *multinomial* consisting of four terms, namely, c, d, a and bx.

Sometimes instead of enclosing an expression within a pair of brackets a line called a **vinculum** is drawn over it.

Thus, $a-\overline{b-c}$ and $a-(b-c)$ have the same meaning.

N. B. *From the above it is easy to understand the distinction between $\sqrt{a+b}$ or $\sqrt{(a+b)}$ and $\sqrt{a}+b$; either of the first two expressions means the square root of the number denoted by $a+b$, whereas the last means that b is to be added to the square root of a. Similarly, $\sqrt{ab}$ or $\sqrt{(ab)}$ means the square root of the number denoted by ab, whereas $\sqrt{a}b$ means the product of b and the square root of a.*

Note. *The three different kinds of brackets ( ), { }, [ ] are often called respectively* ***parentheses, braces*** *and* ***crotchets.***

EXAMPLES

Example 1. If $a=2$, $b=4$, $c=9$, find the values of :

(i) $\surd cb+\surd b+5$, (ii) $\sqrt{cb}+\surd(b+5)$ and (iii) $\sqrt[3]{2b}+\surd 4a$.

(i) $\surd cb+\surd b+5=\surd 9\times 4+\surd 4+5$

$=3\times 4+2+5$

$=12+2+5=19.$

(ii) $\sqrt{cb}+\surd(b+5)=\surd 9\times 4+\surd(4+5)$

$=\sqrt{36}+\surd 9$

$=6+3=9.$

(iii) $\sqrt[3]{2b}+\surd 4a$ $=\sqrt[3]{2\times 4}+\surd 4\times 2$

$=\sqrt[3]{8}+2\times 2$

$=2+4=6.$

Example 2. If $a=3$, $b=5$, $c=8$, $d=12$, $e=20$, find the difference between the numerical values of :

$$a\{c+b^2-a(e-d)\} \text{ and } a\{c+\overline{b^2-a}(e-d)\}.$$

The 1st expression $=3\times\{8+5^2-3\times(20-12)\}$

$=3\times\{8+25-3\times 8\}$

$=3\times\{8+25-24\}$

$=3\times 9=27$;

and the 2nd expression $=3\times\{8+(5^2-3)\times(20-12)\}$

$=3\times\{8+22\times 8\}$

$=3\times\{8+176\}$

$=3\times 184=552.$

Thus, the reqd. diff. $=552-27=525.$

Example 3. If $m=10$, $n=8$, $p=2$, $q=12$, $r=15$, find the difference between the numerical values of the expressions

$$[\{rm-2q-n(pq-m)\}+p]\times(r-m-p)$$

$$\text{and}\quad [\{rm-2.\overline{q-n}(pq-m)\}+p]\times r-\overline{m-p}.$$

The first expression

$=[\{15\times 10-2\times 12-8\times(2\times 12-10)\}+2]\times(15-10-2)$

$=[\{150-24-8\times 14\}+2]\times 3$

$=[\{126-112\}+2]\times 3$

$=[14+2]\times 3=7\times 3=21$;

and the second expression

$$=[\{15\times10-2\times(12-8)(2\times12-10)\}\div2]\times15-(10-2)$$
$$=[\{150-2\times4\times14\}\div2]\times15-8$$
$$=[\{150-112\}\div2]\times15-8$$
$$=[38\div2]\times15-8$$
$$=19\times15-8=285-8=277.$$

Thus, the reqd. difference $=277-21=256$.

EXERCISE 4

If $a=7$, $b=3$, $c=8$, $d=9$, $e=4$, $f=0$, $m=5$, $n=2$, $p=1$, find the values of :

1. $\sqrt[3]{cen}$. 2. $\sqrt[5]{ce}$. 3. $\sqrt[3]{cb}$. 4. $6\sqrt[6]{b^4d}$.
5. $4\sqrt[4]{4e}$. 6. $4\sqrt[4]{e^4}$. 7. $2\sqrt{4c^2}$. 8. $2\sqrt{4}c^2$.
9. $m+n\sqrt{d}$. 10. $\overline{m+n}\sqrt{d}$. 11. $3\sqrt{p}+c$. 12. $3\sqrt{c+p}$.
13. $\sqrt[3]{3(c+p)}$. 14. $3\sqrt[3]{8}(b+3c)$. 15. $3\sqrt[3]{8(b+3c)}$
16. $f\sqrt{m}+d$. 17. $f\sqrt{a+d}$. 18. $3d-(2e-n)$.
19. $3d-2(e-n)$. 20. $3(d-2e)-n$. 21. $(3d-2)e-n$.
22. $(3d-2)\overline{e-n}$. 23. $3\{d-(2e-n)\}$. 24. $3(d-2)(e-n)$.
25. $7c-(b^2-n^2)$. 26. $(7c-b)^2-n^2$. 27. $7c-(b^2-n)^2$.
28. $7(c-b)^2-n^2$. 29. $\{7c-(b^2-n)\}^2$.
30. $\sqrt[3]{c+3p+4e}(p+b)^3$. 31. $\sqrt[3]{c}+3p+4e(p+b)^3$.
32. $\sqrt[3]{c}+\overline{3p+4e}(p+b)^3$. 33. $\sqrt[3]{c}+(3p+4e)p+b^3$.
34 $\sqrt[3]{c}+3\{(p+4)ep+b^3\}$.

If $x=2$, $y=3$, $z=4$, $a=6$, $d=8$, $c=5$, $n=9$, $p=1$, find the values of :

35. $a(x+y)^2(a-\overline{c-z})^3$. 36. $4\{n-a(d-\overline{a+p})\}\sim4\{\overline{n-a}(d-a)+p\}$.
37. $5\{c+\overline{x^2+y}(n-\overline{d-z})\}\sim5\{(c+x^2+y)\overline{n-d}-z\}$.
38. $[x+y^2\{ap-z(c-\overline{a-x})\}]\sim[\overline{x+y}^2\{(ap-z)c-a\}-x]$.
39. $\dfrac{x^x+y^y+z^z-7p^p}{3(x^2+p^2)+y^2+z^2}$. 40. $\sqrt{\left\{\dfrac{c^2+z^2+x^2}{c^2-(z^2+x^2)}-\dfrac{n(n^2-z^2+c^2)}{n^2+z^2-c^2}\right\}}$.

21. Like and Unlike Terms. Terms or simple expressions are said to be *like* when they do not differ at all or differ only in their numerical coefficients ; otherwise they are called *unlike*. Thus, $3ax^2y^5$ and $5ax^2y^5$ are *like* terms, whereas $3ax^2y^5$ and $5ax^2y^4$ are *unlike* ; similarly, abc, $5axbd$, $7a^2b^3$ and c^2d^3x are all *unlike*.

22. Special meaning of the word Sign : Like and Unlike Signs. The word *sign* is often used to denote exclusively the signs + and −. Thus, when we speak of the *sign* of a term we mean the *plus* or *minus* sign which stands before it.

Two signs are called *like* when they are *both* + or *both* −, otherwise they are called *unlike*. Thus, in the expression $ax^2+bx-cy+d^2-f$, the signs of the 3rd and 5th terms are *like* as also those of the 1st, 2nd and 4th, whereas the signs of the 2nd and 3rd terms as well as those of the 4th and 5th are *unlike*.

23. The Signs $>$, $<$, ∵ and ∴. The sign $>$ when placed between two quantities indicates that the quantity on the left of it is *greater than* that on the right. Thus, $a+b > c+d$ means that $a+b$ is greater than $c+d$.

The sign $<$ when placed between two quantities indicates that the quantity on the left of it is *less than* that on the right. Thus, $a+x < b+y$ means that $a+x$ is less than $b+y$.

The sign ∵ is used as an abbreviation for the word *because* or *since*.

The sign ∴ is used as an abbreviation for the word *therefore* or *hence*.

CHAPTER II

POSITIVE AND NEGATIVE QUANTITIES

24. Quantities of the same class, but of opposite character. When we speak of a quantity of money it may be either a *gain* or a *loss*, a receipt or a payment. Now, it is quite clear that whilst a gain adds to our stock, a loss lessens it ; moreover, gain and loss are so related that if we gain as much as we lose, the effect on our stock is nothing. Hence, a quantity of money which forms a *gain* is said to be *opposite in character* to a quantity which forms a *loss*.

When we speak of a distance measured from a point, it may be in either of two opposite directions, either towards the north or towards the south of the point, either towards the east or towards the west of the point, either towards the north-east or towards the south-west of the point ; and so on. It is also clear that distances measured towards the east are so related to those measured towards the west that if we first walk any distance towards the east and then walk an equal distance towards the west there will be no change in our position with respect to the starting point. Hence, a distance measured in any direction is said to be *opposite in character* to that measured in the opposite direction.

Thus, in the first illustration, in so far as a gain and a loss are both looked upon as portions of money, they are said to be quantities of the same class, but as they affect our stock in directly opposite ways (a gain increasing and a loss diminishing it), they are said to be of *opposite character*. In the second illustration, a distance measured towards the south of the point as well as one measured towards the north may both be styled *distance* and thus far they are said to be quantities *of the same class*; but when we consider the directions in which they are measured, they must be regarded as *opposite in character*.

25. The Signs Plus and Minus under a new aspect. It has been shown in the introduction how concrete quantities are represented by numbers. It now remains to be seen how quantities of the same class but of *opposite* character are distinguished in their numerical representation.

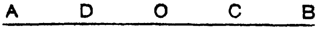
When we consider any pair of such quantities, we prefix the sign + before the numerical measures of one, and the sign − before those of the other. It is quite immaterial which of the two quantities, we select for representation by numbers preceded by the sign +, but when we have once made our choice, we must stick to it throughout any connected series of operations. The following example will illustrate the principle :

Income and *debt* are evidently quantities of opposite character. If then we choose to represent incomes by numbers preceded by the sign +, we must represent debts by numbers preceded by the sign −, and ***vice versa.***

Hence, if in any problem we choose the sign + for incomes and the sign − for debts, +30, +45, +90 will respectively represent incomes of £30, £45 and £90 whereas −30, −45, −90 will represent debts of £30, £45 and £90 respectively, a £ being the unit. But if the contrary choice be made +10, +25, +36 will respectively represent debts of £10, £25 and £36 and −10, −25, −36 will represent incomes of £10, £25 and £36 respectively.

Hence, generally, if a represent a portion of any quantity, $-a$ will represent an equal portion of the quantity opposite in character to it.

Graphical Illustration :



Suppose, AB is a road. If a person starting from any point O on it *travels towards* B to any point C and then *travels back* to O, it is evident that his position on the road is just the same at the end of his journey as at the commencement. Thus, it is clear that distances measured along the road from *left to right* are opposite in character to those measured from *right to left*. Accordingly, if distances measured from left to right be represented by numbers preceded by the sign +, those measured from right to left must be represented by numbers preceded by the sign −, and ***vice versa.***

On the other hand, if we choose the sign + for distances measured from *right to left*, distance of -3 kilometres from any point O will mean a distance of 3 kilometres measured from O *towards the right*; again, if a kilometre be the unit of distance, and if C and D be two points on opposite sides of O at distances of 5 kilometres and 4 kilometres respectively then the distances OD, OC, CD and DC will be respectively represented by $+4$, -5, $+9$ and -9.

From the above instances it is quite clear that the signs + and −, besides being used as signs of the operations of addition and subtraction, are also used as *signs of distinction* between quantities *of opposite character*. The signs when used in this sense are often called ***signs of affection.***

N. B. When no sign is prefixed to a number, the sign + is understood; thus, a and + a have the same meaning.

26. Positive and Negative Quantities. Numbers or symbols preceded by the sign + or no sign are called *positive quantities*. Whilst those preceded by the sign − are called *negative quantities*. Thus, each of the expressions 4, $+6$, a, $+b$, $+c$ is a *positive quantity*, whilst each of -4, -6, $-a$, $-b$, $-c$ is a *negative quantity.*

Hence, the signs + and − are often respectively called *the **positive** and negative signs.*

Note 1. *In 'positive and negative quantities' the word quantity is used in the sense of number. There is no difficulty however in understanding a **negative number**, when the explanation given in Art. 25 is remembered.*

Note 2. *The **absolute value** of a positive or a negative quantity is its value considered apart from its sign. Thus, if a stands for 5 and b for 3,* $+(ab)$ *and* $-(ab)$ *have the **same** absolute value, namely,* 15.

*N. B. It is important to bear in mind the meanings of such expressions as 'a **gain** of −£20', 'a **rise** of −8 centimetres', 'a distance of −5 kilometres to the **north**', &c. The expressions respectively mean 'a **loss** of £20', 'a **fall** of 8 centimetres', 'a distance of 5 kilometres to the **south**', &c.*

EXERCISE 5

1. If £4 be the unit, what is meant by "A's gain = −25"?

2. If a trader's loss of Rs. 30 be represented by 30, what will represent a gain of Rs. 70?

3. If an income of Rs. 60 be represented by 15, what will represent a debt of Rs. 100?

4. If a debt of £100 be represented by 25, what will represent an income of £400?

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5. If a distance of 75 kilometres to the north of a point be represented by 15, what will represent a distance of 150 kilometres to the south of it ?

6. If a river level rises 12 centimetres on any day, falls 9 centimetres the next day, and again rises 15 centimetres on the third, how would you represent the *rises* on successive days, taking 3 centimetres as the unit of length ?

7. A man gains Rs. 30 in one year, loses Rs. 20 in the second year, loses Rs. 40 in the third year, and gains Rs. 60 in the fourth year; how would you represent his *gains* in successive years, taking Rs. 2 as the unit ?

8. In the preceding question, how would the man's losses be represented ?

CHAPTER III

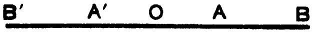
FOUR SIMPLE RULES

I. Addition

27. Definition. When two or more quantities are united together, the result is called their *sum* and the process of finding the result is called *addition*.

Note. *As negative numbers are not recognised in Arithmetic, there is clearly a difference between the Arithmetical and the Algebraical significance of the word addition. Hence, when we speak of an Algebraic sum, we mean that quantities added together are not necessarily all positive.*

28. The result when one positive quantity is added to another. Suppose $B'B$ is a road and that distances measured from left to right are reckoned positive whilst those measured in the opposite direction, negative.



Suppose, O, A and B are three points on the road such that OA is 2 kilometres and AB is 3 kilometres; then, if a kilometre be the unit of distance and if A and B be situated as shown in the figure, OA and AB will be respectively represented by $+2$ and $+3$.

If then a man starting from O travels to A in the first hour and from A to B in the second hour, his distance from O at the end of two hours is evidently OB and will, therefore, be represented by $+5$.

Hence, since (the distance travelled in the 1st hour) + (the distance travelled in the 2nd hour) = (the distance travelled *in two hours*), we have $(+2)+(+3)=5$.

Hence, generally speaking, $(+a)+(+b)=+(a+b)$, or more simply, $(a)+(b)=(a+b)$.

Thus, *when two positive quantities are added together, the sum is a positive quantity whose absolute value is equal to the arithmetical sum of the absolute values of those quantities.*

29. **The result when one negative quantity is added to another.** Suppose, in the above figure $OA'=2$ kilometres and $A'B'=3$ kilometres, and that A' is on the left of O and B' on the left of A' as shown in the figure. Then, the distances OA' and $A'B'$ are respectively represented by -2 and -3.

If a man starting from O travels to A' in the first hour and from A' to B' in the second hour, his distance from O at the end of the second hour, will evidently be OB' and will, therefore, be represented by -5.

Hence, since (the distance travelled in the 1st hour) + (the distance travelled in the 2nd hour) = (the distance travelled *in two hours*), we have $(-2)+(-3)=-5$.

Hence, generally speaking, $(-a)+(-b)=-(a+b)$.

Thus, *when two negative quantities are added together, the sum is a negative quantity whose absolute value is equal to the arithmetical sum of the absolute values of those quantities.*

Example 1. Find the sum of $-a$, $-bc$, $-a^2b$, when $a=2$, $b=3$, $c=5$.

We have $a=2$, $bc=3\times5=15$, $a^2b=2^2\times3=12$.

Hence, $(-a)+(-bc)+(-a^2b)=(-2)+(-15)+(-12)$

$=-(2+15+12)=-29.$

Example 2. Find the value of $(-3c)+(-a^3d)+(b+f+g)$, when $a=3$, $b=-2$, $c=4$, $d=5$, $f=-6$, $g=-8$.

We have $b+f+g=(-2)+(-6)+(-8)$

$=-(2+6+8)=-16$;

also, $3c=12$,

and $a^3d=3^3\times5=27\times5=135$.

Hence, the given expression $=(-12)+(-135)+(-16)$

$=-(12+135+16)=-163.$

EXERCISE 6

1. Find the sum of -2, -9 and -11.

2. Find the sum of $-5x$, $-y$ and $-z$, when $x=2$, $y=3$, $z=5$.

3. Find the sum of -7, x and y, and find the result of adding it to -10, when $x=-5$ and $y=-19$.

4. Find the value of $2a-3(b+c)$, when $a=-5$, $b=2$, $c=1$.

5. Find the value of $(-a^2c^4)+(-a^4b^2)+\{-(c^2-a^2)\}$, when $a=2$, $b=3$, $c=4$.

6. Find the sum of $-3a^3b^3$, d, e, $-20c^2$ and $(d+e)$, when $a=1$, $b=2$, $c=3$, $d=-4$, $e=-5$.

7. Find the sum of $-a^4(b-c)$, $-b^4(c-a)$ and $-c^4(b-a)$, when $a=2$, $b=5$, $c=4$.

8. Find the value of $\{-(a^2-b^2)\}+\{-(a^3-b^3)\}+\{-(a^4-b^4)\}$, when $a=3$, $b=5$.

9. Find the sum of $-x^3(y^2-z^2)$, $-y^3(z^2-x^2)$ and $-z^3(y^2-x^2)$, when $x=3$, $y=6$, $z=5$.

10. Find the sum of $-\{a^4+b^4-c^4\}$, $-\{a^4+(b^4-c^4)\}$, $-\{a^4-b^4\times c^4\}$ and $-\{(a^4-b^4)\times c^4\}$, when $a=60$, $b=4$, $c=2$.

30. The result when a negative quantity is added to a positive quantity. In the figure of Art. 28 suppose a man starting from O travels to B in the first hour and from B to A in the second hour; then, the distances travelled in the first and second hours will be respectively represented by $+5$ and -3, and therefore, the distance from O at the end of the second hour will be represented by $(+5)+(-3)$. But the distance of the man from O at the end of the second hour (*i.e.*, OA) is also evidently represented by $+2$. Hence, we have $(+5)+(-3)=+2$, that is, $=+(5-3)$.

Again, if the man starting from O travels to B in the 1st hour and from B to A' in the second hour, then, the distances travelled by him in the 1st and 2nd hours will be respectively represented by $+5$ and -7, and therefore, his distance from O at the end of the second hour will be represented by $(+5)+(-7)$. But his distance from O at the end of the second hour (*i.e.*, OA') is also represented by -2. Hence, we have $(+5)+(-7)=-2$, that is, $=-(7-5)$.

Thus, generally speaking, we have $(+a)+(-b)=+(a-b)$ or, $-(b-a)$ according as b is less or greater than a. In other words, *if a positive and a negative quantity be added together, the sign of the result is positive or negative according as the absolute value of the negative* *quantity is less or greater than that of the positive quantity and the absolute value of the result is always equal to the difference between the absolute values of the quantities.*

Cor. 1. Since, $a+(-b)=-(b-a)$ when b is greater than a, putting $a=0$, we have $+(-b)=-b$; that is, *to add a negative quantity is the same as to subtract its absolute value, and conversely, to subtract a positive quantity is the same as to add a negative quantity having the same absolute value.*

Note. *Hence, there is no difficulty in finding the value of $a-b$ when b is greater than a; for $a-b$ can always be taken to be equivalent to $a+(-b)$, and the latter is equal to $-(b-a)$ when b is greater than a. Thus,* $3-8=3+(-8)=-(8-3)=-5$.

Cor. 2. From Cor. 1, it is evident that the sum of any number of quantities can be expressed by writing down the quantities one after the other with their respective signs. Thus, $a-b+c-d$ means the same as $a+(-b)+c+(-d)$.

Example 1. Find the value of $a-3b+2c-7d$, when $a=2$, $b=4$, $c=3$, $d=1$.

$$\begin{aligned} a-3b+2c-7d &= a+(-3b)+2c+(-7d) \\ &= 2+(-12)+6+(-7)=-10+6+(-7) \\ &= -4+(-7)=-11. \end{aligned}$$

Example 2. Find the value of $a^2b-b^2c+c^2d-d^2a-bc^2$, when $a=1$, $b=2$, $c=3$, $d=4$.

$$\begin{aligned} \text{The given exp.} &= (1^2\times 2)-(2^2\times 3)+(3^2\times 4)-(4^2\times 1)-(2\times 3^2) \\ &= 2-12+36-16-18=-10+36-16-18 \\ &= 26-16-18=10-18=-8. \end{aligned}$$

EXERCISE 7

1. Find the sum of 117 and -114.

2. Find the sum of 218 and -223.

3. Find the value of $x-y+z$, when $x=8$, $y=25$, $z=13$.

4. Find the sum of $3x$, $-6y$, $2z$, u and $14v$, when $x=2$, $y=5$, $z=1$, $u=3$, $v=-2$.

5. Find the value of $3m-5n+6q+r$, when $m=4$, $n=6$, $q=2$ $r=-8$.

6. Find the sum of $-a^2c$, bd^2, $-cb^2$ and $-a^2d^2$, when $a=2$, $b=5$, $c=3$, $d=6$.

7. Find the value of $2x^2y-3y^2x-5x^2y^2+x^3y^4$, when $x=y=2$.

8. Find the value of $a^3-3a^2b+3ab^2-b^3$, when $a=3$ and $b=5$.

9. Find the value of $m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5$, when $m=4$ and $n=6$.

10. Find the value of $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$, when $a=3$ and $b=2$.

31. When any number of quantities are added together, the result will be the same in whatever order the quantities may be taken.

Suppose, a man starting from a place travels 6 kilometres to the north and then travels back along the same path 8 kilometres to the south. Then his position at the end of the journey is 2 kilometres to the south of that place.

Again, if the man first travels 8 kilometres to the south and then travels 6 kilometres to the north, then also at the end of the journey he is still 2 kilometres to the south of the place.

Thus, we have $6+(-8)=(-8)+6$, each being equal to -2, or, more briefly, we have $6-8=-8+6$, and a similar result in every other case.

Hence, generally, $a-b=-b+a$.

Again, since $2-10+6=-8+6=-2$,

and also $-10+6+2=-4+2=-2$,

we have $2-10+6=-10+6+2$, and a similar result in every other case.

Hence, generally, $a-b+c=-b+c+a$.

Similarly, it may be shown that

$$\begin{aligned} a-b+c-d+e-f &= a+c+e-b-d-f \\ &= -b+e-d-f+c+a \\ &= \&c. \quad \&c. \quad \&c. \end{aligned}$$

32. When any number of quantities are added together, they can be divided into groups and the result expressed as the sum of these groups.

We have

$$\begin{aligned} 3-7-8+6-4+2 &= -4-8+6-4+2 = -12+6-4+2 \\ &= -6-4+2 = -10+2 = -8\,; \end{aligned}$$

$$(3-7)+(-8+6)+(-4+2) = -4+(-2)+(-2) = -8\,;$$

$$3+(-7-8+6)+(-4+2) = 3+(-9)+(-2) = -8\,;$$

$$3+(-7-8)+(6-4)+2 = 3+(-15)+2+2 = -8.$$

Thus, we have

$$3-7-8+6-4+2=(3-7)+(-8+6)+(-4+2)$$
$$=3+(-7-8+6)+(-4+2)$$
$$=3+(-7-8)+(6-4)+2,$$

and similar results in all other cases.

Hence, generally, the expression $a+b-c-d+e-f+g$ can be put in any one of the following forms :

(1) $(a+b)+(-c-d)+e+(-f+g)$,
(2) $a+(b-c)-d+(e-f+g)$,
(3) $(a+b-c)+(-d+e-f)+g$,
(4) $a+(b-c-d)+e+(-f+g)$,
(5) $(a+b-c-d)+(e-f+g)$,
&c &c. &c.

Cor. 1. Conversely, we have $(a+b)+(-c-d)+e+(-f+g)=a+b-c-d+e-f+g$. Hence, the following rule :

To add together two or more algebraical expressions write down the terms in succession with their proper signs.

Cor. 2. Since, $a-b+c-d+e-f=a+c+e-b-d-f$ [**Art. 31**] $=(a+c+e)+(-b-d-f)$, we have the following rule :

When any number of quantities are to be added some of which are positive and other negative, collect the positive terms in one group and the negative terms in another, and express the result as the sum of these two groups.

Thus, $3-7+8-9+5-6=(3+8+5)+(-7-9-6)=16+(-22)=-6$.

Example 1. Simplify $5a-3b+2c-4a+2b-7c$.

The given expression $=5a-4a-3b+2b+2c-7c$ [**Art. 31**]
$=(5a-4a)+(-3b+2b)+(2c-7c)$ [**Art. 32**]
$=a+(-b)+(-5c)=a-b-5c$.

Example 2. Simplify $3a^2b+5b^2c-6c^2a-10a^2b-7b^2c+8c^2a+4a^2b-b^2c+c^2a$.

The given expression

$$=3a^2b-10a^2b+4a^2b+5b^2c-7b^2c-b^2c-6c^2a+8c^2a+c^2a$$
$$=(3a^2b-10a^2b+4a^2b)+(5b^2c-7b^2c-b^2c)+(-6c^2a+8c^2a+c^2a)$$
$$=(-7a^2b+4a^2b)+(-2b^2c-b^2c)+(2c^2a+c^2a)$$
$$=(-3a^2b)+(-3b^2c)+(3c^2a)=-3a^2b-3b^2c+3c^2a.$$

Note. *In the process above, it must be noticed that when like terms are added together, the result is obtained by annexing the common letters to the sum of the numerical coefficients. For instance, we find that* $5b^2c-7b^2c-b^2c=-3b^2c$, *and evidently*, -3 *is the sum of the coefficients* 5, -7 *and* -1.

Example 3. Add together $3a-2b+c$ and $-5d+6e-f$, and find the numerical value of the sum, when $a=2$, $b=1$, $c=3$, $d=4$, $e=7$, $f=5$.

We have $(3a-2b+c)+(-5d+6e-f)$

$$=3a-2b+c-5d+6e-f=6-2+3-20+42-5$$
$$=(6+3+42)+(-2-20-5)=51+(-27)=24.$$

33. The ordinary rule for adding together compound expressions. Put the expressions under one another so that the different sets of like terms may stand in vertical columns and draw a line below the last expression; then add up each vertical column and put the result below it. The following examples will illustrate the method :

Example 1. Add together $3a-5b+7c-9d$, $-8c+5a-3d+7b$, $4d+2c-a$ and $2b-3c+6d$.

$$\begin{array}{lrrrrl}
\text{The first expression} = & 3a & -5b & +7c & -9d & \\
\text{The 2nd expression} = & 5a & +7b & -8c & -3d & \quad [\text{ Art. 31 }] \\
\text{The 3rd expression} = & -a & & +2c & +4d & \\
\text{The 4th expression} = & & 2b & -3c & +6d & \\
\hline
\therefore \quad \text{The sum} = & 7a & +4b & -2c & -2d &
\end{array}$$

Example 2. Find the numerical value of the sum of $20a^2b^3-25b^3c^4+d^7$, $-22a^2b^3+19b^3c^4-3d^7$ and $2a^2b^3+7b^3c^4+2d^7$, when $a=498$, $b=3$, $c=2$, $d=19$.

$$\begin{array}{lrrr}
\text{The first expression} = & 20a^2b^3 & -25b^3c^4 & +d^7 \\
\text{The 2nd expression} = & -22a^2b^3 & +19b^3c^4 & -3d^7 \\
\text{The 3rd expression} = & 2a^2b^3 & +7b^3c^4 & +2d^7 \\
\hline
\therefore \quad \text{The sum} = & & b^3c^4 &
\end{array}$$
$$=3^3\times2^4=27\times16=432.$$

EXERCISE 8

Simplify the following :

1. $2x+3y-z-3x-2y+z$. 2. $9m^2-7n^2+5p^2+8n^2-4p^2-8m^2$.

3. $8a^2-5a^2b-7a^2+5c^2-2a^2+6a^2b-4c^2$.

4. $3abc-5c^2+6mnp^2-abc+7c^2-9mnp^2-2c^2$.

5. $-7a^2b-5b^2c^2+10a^2b-3b^2c^2+3df-a^2b-b^2c^2-5df$.

6. $8x^4y-5xyz-17x^4y+20x^2y^2-2xyz-35x^2y^2+3x^4y-4xyz+5x^2y^2$.

7. $9a^2bc - 7b^2ca + 5c^2ab + 3b^2ca - 5a^2bc - c^2ab$.

8. $20x^3mn - 23m^3nx + 14n^3xm - 37x^3mn - 47n^3xm + 54m^3nx$
$- 8x^3mn + 13n^3xm - 15m^3nx + 20n^3xm$.

If $a=9$, $b=10$, $c=12$, $d=5$, $k=2$, $m=3$, $n=4$, $x=6$, $y=7$, $z=8$, find the numerical value of the sum of :

9. $-k+3m+5n$ and $5d-4x-6y$.

10. $5m-2y-7b-8c$ and $3d+x-10a$.

11. $3k^2$, $-5m^2+7n^2$, $-2x+5b-c$ and $10d-7a$.

12. $-2k+3m-4n$, $-d-5x+6y$ and $3z-5a-3b+5c$.

13. $-km+az$, $bc-4md+y$, $-n^2-d^2+ab$ and $6kn-5y-7x+kmn$.

14. k^3m-dnx, $by^2-ckm-x^2d$, $-bz+3a^2-2m^4d$ and $5n^2-7bdz+2ak^2-3b^2d$.

15. $3m^4b-5a^2x-4b^2z$, $-13k^2b+4z^2d-7d^2n$, $-5c^2n+8b^2y+9d^3$ and $5az^2-7b^2c-4x^3b+8adn$.

Add together :

16. $a-2b+5c$ and $-7a+3b-8c$.

17. $-3x+5y-9z$, $5x-3y+7z$ and $-2y+z$.

18. x^3+3x^2-5x+4, $2x^3-6x^2+7x-8$, $-x^3+7x^2-2x+9$ and $5x^2+2$.

19. $3a-2b+7c-8d$, $2c+6d-5a$, $3b+d-10c$, $c-4b+a$ and $-7d+5b$.

20. $x^2+2xy+3y^2-x+y+2$, $-5x^2+y^2+2x-5$, $-3xy-7y^2+3y+1$ and $6x^2+xy-x-4y+2$.

If $a=5$, $b=4$, $x=8$, $y=7$, find the numerical value of :

21. $(3x^3+5y^5-20a^2+49b^3)+(17a^2-27b^3-23x^3)+(-y^5+3b^3-3a^2)+(-23b^3-4y^5+7a^2+20x^3)$.

22. $(10a^2-26x^5y^4+30x^3b^5+17a^6y^7)+(35x^5y^4+16a^6y^7-304a^2-28x^3b^5)+(-8a^6y^7-9x^5y^4-7x^3b^5)+(5x^3b^5-25a^6y^7+289a^2)$.

23. $(2a^2-7b^2+9x^2-13y^2+15ab-21xy)+(5y^2+8b^2+17xy-6a^2-8ab-20x^2)+(13x^2-20ab+5a^2-16xy-10y^2-2b^2)+(13ab-2x^2+3b^2+23xy-a^2+18y^2)$.

24. $(29abx-39bxy+49xya-59yab)+(29bxy+49yab-19abx-39xya)+(2abx-12xya+6bxy+24yab)+(3xya+4bxy-13abx-14yab)$.

25. $(18a^2b^2-43b^2x^2+62x^2y^2-23abxy)+(39abxy+28b^2x^2-25a^2b^2-42x^2y^2)+(19b^2x^2+37a^2b^2-25abxy+35x^2y^2)+(9abxy-29a^2b^2-55x^2y^2-4b^2x^2)$.

II. Subtraction

34. Definition. Any quantity b is said to be subtracted from any other quantity a when a third quantity c is found such that the *sum* of b and c is equal to a. In other words, $c=a-b$, *when* c is such that $b+c=a$.

The quantity *from* which another quantity is subtracted is called the *minuend* and the *quantity subtracted* is called the *subtrahend*. The result is called the *difference* or the *remainder*. Thus, if $a-b=c$, a is the minuend, b the subtrahend and c the remainder.

35. To subtract a positive quantity is the same as to add a negative quantity having the same absolute value, and to subtract a negative quantity is the same as to add a positive quantity having the same absolute value.

Since, $3+4=7$, we have $7-3=4=7+(-3)$,
again, since $6+(-2)=4$, we have $4-6=-2=4+(-6)$.

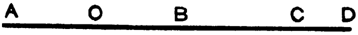
Hence, generally, $a-b=a+(-b)$; *i.e., to subtract a positive quantity is the same as to add a negative quantity having the same absolute value.* [See Art. 30, Cor. 1]

Since, $(-3)+5=2$, we have $2-(-3)=5$ [by definition] $=2+3$,
similarly, since $(-6)+(-4)=-10$,
we have $(-10)-(-6)=-4=(-10)+6$.

Thus, generally, since $(-b)+(a+b)=a$, we have $a-(-b)=a+b$, *i.e., to subtract a negative quantity is the same as to add a positive quantity having the same absolute value.*

Note. *One quantity a is said to be greater than another quantity b when $a-b$ is a positive quantity. Thus, -4 is greater than -5 for $(-4)-(-5)=-4+5=1$. Similarly, $-5>-7$, $-10>-20$; and so on. Hence, in the series 5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6, -7, -8, &c., each number is less than the one before it.*

36. Illustration. Suppose, AD is a railway line running from



west to east, and A, O, B, C, D are stations on it such that $AO=OB=20$ kilometres, $BC=30$ kilometres and $CD=10$ kilometres. Suppose, a man travels from O to C in two days.

Then evidently, (the distance travelled on the first day)+(the distance travelled on the second day) $=50$ kilometres ; and hence, by definition, 50 kilometres $-$(the distance travelled on the first day) $=$ the distance travelled on the second day.

Now, (i) if on the first day the man travels from O to B, *i.e.*, travels 20 kilometres towards the *east* of O, then on the second day he has to travel from B to C, a distance of 30 kilometres more towards the east; thus we have (50 kilometres) $-$ (20 kilometres) $=$ 30 kilometres.

(ii) If on the first day the man travels from O to A, *i.e.*, travels a distance of 20 kilometres towards the *west*, then on the second day he must travel from A to C, a distance of 70 kilometres *towards the east*; thus, we have (50 kilometres) $-$ (-20 kilometres) $=$ 70 kilometres.

(iii) Again, if on the first day the man travels from O to D, *i.e.*, a distance of 60 kilometres towards the *east*, then on the second day he must travel from D to C, *i.e.*, a distance of 10 kilometres *towards the west*; thus, we have (50 kilometres) $-$ (60 kilometres) $=$ -10 kilometres.

Hence, taking a kilometre as the unit of distance, we get the following results:

$$\left.\begin{aligned}50-\quad 20&=\ \ 30\\ 50-(-20)&=\ \ 70\\ 50-\quad 60&=-10\end{aligned}\right\}$$

Example 1. Find the value of $a-b+c$, when $a=5$, $b=-2$, $c=-3$.

$$\begin{aligned}a-b+c&=5-(-2)+(-3)\\&=5+2-3=4.\end{aligned}$$

Example 2. Find the value of $-a-(-b)+c$, when $a=-2$, $b=-3$, $c=-4$.

$$\begin{aligned}\text{The given expression}&=-a+b+c\\&=-(-2)+(-3)+(-4)\\&=2-3-4=-5.\end{aligned}$$

EXERCISE 9

If $a=3$, $b=-5$, $c=-6$, $d=-8$, find the values of:

1. $-a+b-c+d$. 2. $a+(-b)+c-d$. 3. $c-d-(-b)-a$.
4. $c-(-d)+b-a$. 5. $-(-a)+b-(-c)-d$.

If $m=-47$, $n=50$, $x=-154$, $y=-234$, find the values of:

6. $n-m-(-x)+y$. 7. $-(-m)+y-(-n)-x$.
8. $-(-x)+m-y-(-n)$. 9. $-(-y)-m-x-(-n)$.
10. $-(-n)-y-(-x)-m$.

37. To prove that $\boldsymbol{a-(b+c)=a-b-c}$,
and $\boldsymbol{a-(b-c)=a-b+c}$.

Since, $(b+c)+(a-b-c)=a$,

$\therefore$ by definition, $a-(b+c)=a-b-c$.

Again, since $(b-c)+(a-b+c)=a$,

$\therefore$ $a-(b-c)=a-b+c$.*

Cor. Thus, we arrive at the following rule for subtracting one algebraical expression from another: *Change the sign of every term of the subtrahend from + to − or from − to + as the case may be, and then write down those terms in succession after the minuend.* Thus, the result of subtracting $2a+3b-5c$ from $a-2b+c=a-2b+c-2a-3b+5c$ $=-a-5b+6c$.

Example 1. Subtract $-3a+2b-5c$ from $2a+b-8c$.

$$\begin{aligned}\text{The reqd. result}&=2a+b-8c+3a-2b+5c\\&=(2a+3a)+(b-2b)+(-8c+5c)\\&=5a+(-b)+(-3c)\\&=5a-b-3c.\end{aligned}$$

Example 2. Subtract $2a^2+3ab-5b^2$ from $-3a^2+2ab-4b^2$.

$$\begin{aligned}\text{The reqd. result}&=-3a^2+2ab-4b^2-2a^2-3ab+5b^2\\&=(-3a^2-2a^2)+(2ab-3ab)+(-4b^2+5b^2)\\&=-5a^2-ab+b^2.\end{aligned}$$

38. **The ordinary rule for subtracting one compound expression from another.** Put the subtrahend below the minuend in such a way that the different sets of like terms may stand in vertical columns and draw a line below the subtrahend; then supposing the sign of every term of the subtrahend to be changed, write down the sum of each vertical column underneath it.

Example 1. Subtract $-2x^2+3xy-y^2$ from $x^2-2xy+3y^2$.

$$\begin{array}{lll}&\text{The minuend}&=\quad x^2-2xy+3y^2\\&\text{The subtrahend}&=\underline{-2x^2+3xy-\ \ y^2}\\\therefore&\text{The remainder}&=\quad 3x^2-5xy+4y^2\end{array}$$

Note. *It must be noticed that the signs of the terms of the subtrahend are not actually altered in the process, but they are supposed to be altered and the operation of combining each pair of like terms is performed mentally.*

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*When a, b, c are all positive quantities and a is greater than b, and b is greater than c, the following proof is generally given of this result in most treatises on Algebra.

If we subtract b from a, we get $a-b$, but we thus subtract too much from a, for we have to subtract not b but a quantity which is less than b by c. Hence, we must add c to this result; thus, $a-(b-c)=a-b+c$.

Example 2. Subtract $a^2-3ab+5x^2-y^2$ from $3x^2+2y^2-7a^2$.

$$\begin{array}{lll} & \text{The minuend} & = 3x^2+2y^2-7a^2 \\ & \text{The subtrahend} & = 5x^2-\ y^2+\ a^2-3ab \\ \hline \therefore & \text{The remainder} & = -2x^2+3y^2-8a^2+3ab \end{array}$$

EXERCISE 10

Subtract :

1. $a-b+c$ from $3a+2b-c$. 2. $2a-5b+4c$ from $-a-2b+8c$.
3. $-x+y-z$ from $2x+3y-4z$.
4. $5m^2-6m+3$ from $7m^2-8m-1$.
5. $x^2-2y^2+3z^2$ from $3x^2-y^2+2z^2$.
6. $4y^2+4xy-2x^2$ from $2y^2-3xy+x^2$.
7. $-3a^2+2ab-7b^2$ from $a^2-5ab-8b^2$.
8. $-2bc+6c^2-8xy$ from $5bc-c^2+2xy$.
9. $2x^3-4x^2+7x+5$ from x^3-3x^2+6x+7.

What is to be added to :

10. $x+2y+z$ to make z ? 11. $-2x+5y-4z$ to make $x+y+z$?
12. $3m^2+5m-6$ to make m^2 ?
13. $a^3+3a^2b+3ab^2+b^3$ to make a^3+b^3 ?
14. $a^4-2a^2b^2+b^4$ to make a^4+b^4 ?
15. What is to be subtracted from $a^3-3a^2b+3ab^2-b^3$ to make a^3-b^3 ?

39. Removal and Insertion of Brackets.

(*a*) The laws for the **removal** of brackets are :

(i) If any number of terms be enclosed within a pair of brackets preceded by the sign +, the brackets may be struck out as of no value ;

(ii) If any number of terms be enclosed within a pair of brackets preceded by the sign −, the brackets may be removed provided that the sign of every term within the brackets be changed, namely, + to −, and − to +.

The reason is obvious, for any expression, included within brackets preceded by the sign +, has to be added to, whilst one, enclosed within brackets preceded by the sign − has to be subtracted from what goes before.

Thus, $a-b+(c-d+e)=a-b+c-d+e$,

whilst $a-b-(c-d+e)=a-b-c+d-e$.

(*b*) The laws of insertion of brackets are :

(i) Any number of terms in an expression may be enclosed within a pair of brackets, with the sign + prefixed ;

(ii) Any number of terms in an expression may be enclosed within a pair of brackets, with the sign − prefixed, if the sign of every term put within the brackets be altered.

Thus, $a-b+c-d+e-f=a-b-(-c+d-e+f)$.

Note. *We often find brackets within brackets as in the expression* $2a-[3b-\{4c-(5d-6e)\}]$; *here it is meant that the expression within the braces* { } *is to be subtracted from* $3b$ *and the result thus obtained is to be subtracted from* $2a$; *whilst the expression within the braces is to be found by subtracting the expression within the parentheses* ( ) *from* $4c$.

When an expression of this kind is to be cleared of brackets, it is best for a beginner to remove first the innermost pair, then the innermost of those that remain, and so on ; and lastly the outermost pair.

Example 1. Simplify $a-\{b-(c-d)\}$.

$a-\{b-(c-d)\}=a-\{b-c+d\}=a-b+c-d.$

Example 2. Simplify $a-[b-\{c-(d-e)\}-f]$.

$$a-[b-\{c-(d-e)\}-f]=a-[b-\{c-d+e\}-f]$$
$$=a-[b-c+d-e-f]=a-b+c-d+e+f.$$

Example 3. Simplify $a+[-b-\{c-(d-e-f)-g\}-h]$.

$$a+[-b-\{c-(d-\overline{e-f})-g\}-h]$$
$$=a+[-b-\{c-(d-e+f)-g\}-h]$$
$$=a+[-b-\{c-d+e-f-g\}-h]$$
$$=a+[-b-c+d-e+f+g-h]$$
$$=a-b-c+d-e+f+g-h.$$

Example 4. Simplify $2a-[3a+\{4b-(2a-b)+5a\}-7b]$.

The given expression $=2a-[3a+\{4b-2a+b+5a\}-7b]$

$$=2a-[3a+\{5b+3a\}-7b]$$
$$=2a-[3a+5b+3a-7b]=2a-[6a-2b]$$
$$=2a-6a+2b=-4a+2b.$$

Example 5. Simplify $a-[-b-\{c-(d-\overline{e-f})\}]$, first removing [ ], then { }, then ( ), and last of all the vinculum.

$$a-[-b-\{c-(d-\overline{e-f})\}]=a+b+\{c-(d-\overline{e-f})\}=a+b+c-(d-\overline{e-f})$$
$$=a+b+c-d+\overline{e-f}=a+b+c-d+e-f.$$

Note. *The expression within [ ] consists of two terms namely $-b$ and $-\{c-(d-\overline{e-f})\}$; hence, when this pair of brackets, which is preceded by the sign $-$, is removed, we get $b+\{c-(d-\overline{e-f})\}$. A similar reasoning applies to the removal of other brackets. It must be noticed carefully that* ***only one pair of brackets is to be removed at a time.***

Example 6. Simplify $[a-\{b-(c-d)\}]-[2a-\{3b+(2c-4d)\}]$.

We have $a-\{b-(c-d)\}=a-\{b-c+d\}=a-b+c-d$;

and $2a-\{3b+(2c-4d)\}=2a-\{3b+2c-4d\}$

$$=2a-3b-2c+4d.$$

Hence, the given expression

$$=[a-b+c-d]-[2a-3b-2c+4d]$$
$$=a-b+c-d-2a+3b+2c-4d$$
$$=-a+2b+3c-5d.$$

Example 7. Of the expression $a+b-c+d-e-f$ enclose the first three terms within a pair of brackets and the last three in another, each preceded by the sign $-$, and then put the last two terms of each of these bracketed expressions within an inner pair of brackets preceded by the sign $-$

According to the given directions,

$$a+b-c+d-e-f=-\{-a-b+c\}-\{-d+e+f\}$$
$$=-\{-a-(b-c)\}-\{-d-(-e-f)\}.$$

EXERCISE 11

Simplify :

1. $2a-3b-(4a-6b)+(-2a+5b)$.
2. $x+(-y+4x)-(-2x+3y)$.
3. $-(5x-y)+(-3x+y)-(2y-6x)$.
4. $3a-6a+(2b-a)$.
5. $-a-\{2b-(6a+4b)\}$.
6. $2a-\{5b-7\overline{b-2a}\}$.
7. $3-\{5-(6-\overline{7-9})\}$.
8. $-2-[-3-\{-4-(-5-6)\}]$.
9. $-a-[-3b-\{-2a-(-a-4b)\}]$.
10. $a-[2b-\{3c-(a-\overline{2b-3c})\}]$.
11. $3x-[5y-\{10z-(5x-\overline{10y-3z})\}]$.
12. $-a-[-b-\{-c-(-\overline{a-b-c})\}]$.

Simplify the following expressions removing the brackets in the reverse order, *i.e.*, the outermost first and the innermost last :

13. $2x-[5y-\{9x-(10y-4x)\}]$.
14. $-5a-[3b-\{6a-(5b-7a)\}]$.
15. $-7m-[3n-\{8m-(4n-10m)\}]$.

16. $-2a-[-4b-\{-6c-(-8a-\overline{-10b-12c})\}]$.

17. $-3x-[-5y-\{-7z-(-9x-\overline{-11y-13z})\}]$.

18. $-2x-[-4y-\{-6z-(-3x-\overline{-5y-7z})\}]$.

19. $-x-[-3y+\{-5z-(-2x+\overline{-4y-6z})\}]$.

20. $-2a+[-5b-\{-8c+(-3a-\overline{-6b+9c})\}]$.

21. $-x+[-5y-\{-9z+(-3x-\overline{-7y+11z})\}]$.

Simplify :

22. $\{2a-(3b-5c)\}-[a-\{2b-(c-4a)\}-7c]$.

23. $[x-\{y-(z-x)\}-(y-z)]-[z-\{x-(y-z)\}]$.

24. $[2a-(b-c)-\{3b-(2a-c)\}-\{-2a+(c-4b)\}]$
$-[-3b-(2a-4c)+\{6c-(2b-3a)\}-\{-5c+(6a-7b)\}]$.

In the expression $a-b-c+d-m+n-x+y-z$:

25. Include the 2nd, 3rd and 4th terms in a pair of brackets preceded by the sign $-$, and the 5th, 6th and 7th in a pair of brackets preceded by the sign $+$.

26. Include all the terms after the 1st in a pair of brackets preceded by the sign $-$, and of the expression thus enclosed put the last four terms within a pair of brackets preceded by the sign $+$.

27. Enclose the first five terms within a pair of brackets preceded by no sign and the last four within a pair of brackets preceded by the sign $-$, and then put the last three terms of each of these bracketed expressions within a pair of brackets preceded by the sign $-$.

28. Enclose every three terms from the first in a pair of brackets preceded by the sign $-$, and then put the last two terms of each of these bracketed expressions within a pair of brackets preceded by the sign $-$.

III. Multiplication

40. Definition. One number is said to be multiplied by another when we do to the former what is done to unity to obtain the latter.

Thus, since $4=1+1+1+1$, we must have

$$4\times x \text{ or } 4x=x+x+x+x.$$

Similarly,

$$\left.\begin{aligned} 4\times5&=5+5+5+5&&=20\\ 3\times6&=6+6+6&&=18\\ 5\times3&=3+3+3+3+3&&=15\end{aligned}\right\}\quad\cdots\quad\cdots\quad\text{I}$$

$$\left.\begin{aligned} 3\times(-5)&=(-5)+(-5)+(-5)&&=-15\\ 4\times(-3)&=(-3)+(-3)+(-3)+(-3)&&=-12\\ 5\times(-4)&=(-4)+(-4)+(-4)+(-4)+(-4)&&=-20\end{aligned}\right\}\quad\cdots\quad\cdots\quad\text{II}$$

Again, since $-4=-1-1-1-1$, we must have

$$(-4)\times x=-x-x-x-x.$$

Similarly,

$$\left.\begin{array}{l} (-4)\times 5=-5-5-5-5 \qquad\quad =-20\\ (-3)\times 6=-6-6-6 \qquad\qquad\;\; =-18\\ (-5)\times 3=-3-3-3-3-3=-15 \end{array}\right\} \qquad \cdots \qquad \text{III}$$

Also,

$$\left.\begin{array}{ll} (-3)\times(-5)=-(-5)-(-5)-(-5) & \\ \qquad\qquad\;\; =5+5+5 & =\;15\\ (-4)\times(-3)=-(-3)-(-3)-(-3)-(-3) & \\ \qquad\qquad\;\; =3+3+3+3 & =\;12\\ (-5)\times(-4)=-(-4)-(-4)-(-4)-(-4)-(-4) & \\ \qquad\qquad\;\; =4+4+4+4+4 & =\;20 \end{array}\right\} \qquad \cdots \qquad \text{IV}$$

The number multiplied is called the **multiplicand** and the **number** by which it is multiplied is called the **multiplier**; the result is called **the product.**

EXERCISE 12

From the definition of multiplication deduce the result:

1. When 5 is multiplied by 3.
2. When 6 is multiplied by 3.
3. When 9 is multiplied by 4.
4. When -8 is multiplied by 4.
5. When -15 is multiplied by 3.
6. When -13 is multiplied by 6.
7. When 8 is multiplied by -3.
8. When 7 is multiplied by -5.
9. When 15 is multiplied by -3.
10. When -9 is multiplied by -4.
11. When -12 is multiplied by -5.
12. When -16 is multiplied by -4.

41. The Law of Signs. From the last article it is clear that if a and b are two whole numbers, we have

$$\left.\begin{array}{l} (+a)\times(+b)=+(ab)\\ (+a)\times(-b)=-(ab)\\ (-a)\times(+b)=-(ab)\\ (-a)\times(-b)=+(ab) \end{array}\right\}$$

Thus, the product of two whole numbers is positive or negative according as the multiplicand and the multiplier have like or unlike signs.

The same thing can be found when the numbers are fractional. For instance, since, $-\frac{2}{3}=-\frac{1}{3}-\frac{1}{3}$, *i.e.*, since, $-\frac{2}{3}$ is obtained by subtracting a third part of unity, twice, to multiply any number x by $-\frac{2}{3}$ we must subtract a third part of x twice.

$$\text{Hence,}\quad (-\tfrac{2}{3})\times x=-\frac{x}{3}-\frac{x}{3}=-\frac{2x}{3}.$$

$$\text{Similarly, } (-\tfrac{2}{3})\times\tfrac{4}{5}=-\tfrac{4}{15}-\tfrac{4}{15}=-\tfrac{8}{15},$$

$$(-\tfrac{2}{3})\times(-\tfrac{4}{5})=-(-\tfrac{4}{15})-(-\tfrac{4}{15})$$

$$=\tfrac{4}{15}+\tfrac{4}{15}=\tfrac{8}{15}\text{ ; and so on.}$$

Hence, we can enunciate the *Law of Signs* in a more general way, thus: The sign of the product of any two quantities is positive or negative according as the multiplicand and the multiplier have like or unlike signs. Or, more briefly, thus: *Like signs produce +, and unlike signs −.*

Cor. Since, $(-x)\times(-x)=x^2$ and also $(+x)\times(+x)=x^2$ we have $\sqrt{x^2}=\pm x$. Thus, every positive algebraical quantity has got two square roots which are equal in absolute value but opposite in sign.

Example. Find the value of $(a^2b-cd)(c^2-d^2)$, when $a=-2$, $b=-3$, $c=-4$, $d=5$.

Since, $a^2b=(-2)^2\times(-3)=4\times(-3)=-12$,

and $cd=(-4)\times 5=-20$,

$\therefore\quad a^2b-cd=-12-(-20)=-12+20=8.\qquad\cdots$ (A)

Also, since $c^2=(-4)^2=16$,

and $d^2=(5)^2=25$, $\therefore\ c^2-d^2=16-25=-9.\qquad\cdots$ (B)

Hence, from (A) and (B), we have

$$(a^2b-cd)(c^2-d^2)=8\times(-9)=-72.$$

EXERCISE 13

Find the value of:

1. $ab-cd$, when $a=-2$, $b=-3$, $c=-8$, $d=6$.
2. $(x^2-y^2)b-axy$, when $a=1$, $b=-3$, $x=4$, $y=-5$.
3. $3x^2y-3xy^2+xyz$, when $x=-1$, $y=-2$, $z=-7$.
4. $(-a)b^2-cd^2+b(-c)^2$, when $a=5$, $b=-7$, $c=4$, $d=-3$.
5. $-x^2(-c)+b^2(-y)+4a^3$, when $a=-2$, $b=-3$, $c=-1$, $x=5$, $y=6$.

6. $a^2(b-c)+b^2(c-a)+c^2(a-b)$, when $a=-2$, $b=-5$, $c=-7$.

7. $x^3(y-z)+y^3(z-x)+z^3(x-y)$, when $x=-3$, $y=8$, $z=-5$.

8. $p^3(q^2-r^2)+q^3(r^2-p^2)+r^3(p^2-q^2)$, when $p=-3$, $q=-5$, $r=-7$.

9. $a^3+b^3+c^3-3abc$, when $a=-12$, $b=-13$, $c=-15$.

10. Show that $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, when $a=3$, $b=-5$.

42. To prove that $a\times b=b\times a$, *i.e.*, b multiplied by a gives the same result as a multiplied by b.

(i) First let a and b be any two positive integers.

Place b units in a horizontal row and write down a such rows in such a manner that units in similar positions in the different rows may be in the same vertical column; thus:

$$\begin{array}{cccccccccl} 1 & 1 & 1 & 1 & 1 & . & . & . & . & b \text{ times} \\ 1 & 1 & 1 & 1 & 1 & . & . & . & . & b \text{ times} \\ 1 & 1 & 1 & 1 & 1 & . & . & . & . & b \text{ times} \\ . & . & . & . & . & . & . & . & & \\ . & . & . & . & . & . & . & . & & \end{array}$$

to a rows.

This being done, evidently it may also be said that we have written down b columns, each containing a units.

Now let us count up the total number of units thus written down.

Since, we have got a rows each containing b units, the total number of units $=$ (the number in the 1st row) $+$ (the number in the 2nd row) $+$ (the number in the 3rd row) $+\cdots\cdots\cdots+$ (the number in the ath row) $=b+b+b+\cdots\cdots\cdots$ to a terms $=a\times b$. ... ... (1)

Also, since we have got b columns each containing a units, the total number of units $=$ (the number in the 1st column) $+$ (the number in the 2nd column) $+$ (the number in the 3rd column) $+\cdots+$ (the number in the bth column) $=a+a+a+\cdots$ to b terms $=b\times a$. ... ... (2)

Hence from (1) and (2), we have $a\times b=b\times a$,*

i.e., b taken a times $=$ a taken b times.

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* Since $ab=ba$, it does not matter much whether we read ab as a times b or b times a (*i.e.*, as b multiplied by a or a multiplied by b); but *until the proposition of the present article has been proved* it seems expedient to stick to one and the same mode of interpreting it. If a beginner is taught to read $7a$ as '7 times a' whilst 7×4 as '4 times 7' he is but unconsciously led to think that such expressions as ba and ab mean the same, but that consequently no amount of reasoning is necessary to establish the above proposition. As a safeguard against this evil, I have hitherto throughout taken $a\times b$ to mean 'a times b' or 'b multiplied by a'.

(ii) Next let a and b be two positive fractions: suppose, $a=\frac{m}{n}$ and $b=\frac{p}{q}$, where m, n, p, q are positive integers.

$$\text{Then,}\quad a\times b=\frac{m}{n}\times\frac{p}{q}=m\times\left\{\left(\frac{p}{q}\right)\div n\right\}=m\times\frac{p}{nq}=\frac{mp}{nq} \qquad \ldots \quad \text{(I)}$$

$$\text{and}\quad b\times a=\frac{p}{q}\times\frac{m}{n}=p\times\left\{\left(\frac{m}{n}\right)\div q\right\}=p\times\frac{m}{qn}=\frac{pm}{qn}. \qquad \ldots \quad \text{(II)}$$

But m and p are positive integers, therefore, $mp=pm$, and similarly, $nq=qn$.

Hence, from (I) and (II), we have $a\times b=b\times a$.*

Thus, it is established that for *all positive values* of a and b we must have $a\times b=b\times a$. ... ... (A)

Cor. 1. From Art. 41, we have $x\times(-y)=-(xy)$,

and $(-y)\times x=-(yx)$; but $xy=yx$,

$\therefore$ $x\times(-y)=(-y)\times x$. ... ... (B)

Cor. 2. From Art. 41, $(-x)\times(-y)=+xy$,

and $(-y)\times(-x)=+yx$; but $xy=yx$,

$\therefore$ $(-x)\times(-y)=(-y)\times(-x)$. ... (C)

Hence, from (A), (B) and (C), we conclude that for *all values* of a and b, $a\times b=b\times a$.

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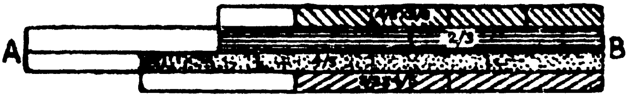
* We can illustrate $a\times b=b\times a$ when b and a are fractions as follows:

Let us prove that $\frac{2}{3}\times\frac{4}{5}=\frac{4}{5}\times\frac{2}{3}$.

$\frac{2}{3}\times\frac{4}{5}$ means that we have to divide $\frac{4}{5}$ of a thing into 3 equal parts and take 2 of those parts, whilst $\frac{4}{5}\times\frac{2}{3}$ means that we have to divide $\frac{2}{3}$ of a thing into 5 equal parts and take 4 of those parts.



Take a line AB 15 centimetres long, then $\frac{4}{5}$ of the line will be 12 centimetres, and evidently $\frac{2}{3}$ of 12 centimetres = 8 centimetres; thus, $\frac{2}{3}\times\frac{4}{5}$ of the line = 8 centimetres.

Again, $\because$ $\frac{2}{3}$ of the line is 10 centimetres, and $\frac{4}{5}$ of 10 centimetres = 8 centimetres, $\therefore$ $\frac{4}{5}\times\frac{2}{3}$ of the line also = 8 centimetres.

Hence, we have $\frac{2}{3}\times\frac{4}{5}=\frac{4}{5}\times\frac{2}{3}$. Similarly, any other case may be illustrated.

EXERCISE 14

Prove that :

1. $4\times5=5\times4$.
2. $6\times3=3\times6$.
3. $7\times5=5\times7$.
4. $4\times8=8\times4$.
5. $9\times5=5\times9$.

43. To prove that $(ab)\times c=a\times(bc)$, or, $=b\times(ac)$, *i.e.*, to multiply c by the product of a and b is the same as to multiply c first by either of them and then that result by the other.

Place b brackets in a horizontal row each containing c units and write down a such rows in such a manner that the brackets in similar positions in the different rows may be in the same vertical column, thus :

$$\begin{array}{cccl} [c] & [c] & [c] & [c]\ldots b\text{ times} \\ [c] & [c] & [c] & [c]\ldots b\text{ times} \\ [c] & [c] & [c] & [c]\ldots b\text{ times} \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \end{array}$$

to a rows.

This being done, it may also be said that we have written down b columns each containing a brackets.

As we have got together $a\times b$ brackets and as each bracket contains c units, the total number of units $=(ab)\times c$. ... ... (α)

Again, since we have got b brackets in a row each containing c units, the number of units in a row $=bc$, and as there are a rows altogether, therefore, the total number of units $=a\times(bc)$ ... (β)

Again, since we have got a brackets in a column each containing c units, the number of units in a column $=ac$, and as there are b columns altogether, therefore the total number of units $=b\times(ac)$. ... (γ)

Hence, from (α), (β) and (γ), we have

$$(ab)\times c=a\times(bc)=b\times(ac).$$

Cor. From the results of the last article and this, we deduce that $abc=bca=cab$. For, by the present article $abc=a\times(bc)$, and by the last article $a\times(bc)=(bc)\times a=bca$; hence, we have $abc=bca$, and similarly, $bca=cab$. Thus, we are led to conclude that *the value of a product is the same in whatever order the factors may be taken.**

Note 1. *Although the factors of a product can be taken in any order, it is always found convenient to place first the factor expressed in figures, and to put after*

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* The validity of the conclusion has been established only for three factors. A general proof, however, has not been attempted as being too tedious for the class of students for whom the book is meant.

the factors expressed in letters in the alphabetical order of those letters. **Thus,** $c^2\times d\times 7\times b\times a^4$ *is written* $7a^4bc^2d$

Note 2. *We are now in a position to modify a little the definition of* **Coefficient** *given in Art. 15. In an algebraical product one or more of the factors may be called the coefficient of the remaining factors.*

For instance, in 7abcd we may call 7ac as the coefficient of bd for 7abcd can be written as 7acbd and therefore by the definition alluded to, 7ac is the coefficient of bd.

44. To prove that $a^m\times a^n=a^{m+n}$, where m and n are any two positive integers.

N. B. From Art. 42, we know that the quantity on either side of × may be regarded as the multiplier and that on the other as the multiplicand. Hence, we need not any longer observe the restriction we have hitherto placed upon the meaning of $a\times b$. [See foot-note, pages 35, 36.]

Since, $a^2=aa$,

and $a^3=aaa$,

$$\begin{aligned}\therefore\quad a^2\times a^3&=(aa)\times(aaa)\\&=a\times a\times a\times a\times a\qquad\text{[Art. 43]}\\&=a^5=a^{2+3}.\end{aligned}$$

Again, since $a^4=aaaa$,

and $a^6=aaaaaa$,

$$\begin{aligned}\therefore\quad a^4\times a^6&=(aaaa)\times(aaaaaa)\\&=a\times a\times a\times a\times a\times a\times a\times a\times a\times a\qquad\text{[Art. 43]}\\&=a^{10}=a^{4+6}\end{aligned}$$

Generally, since $a^m=aaaa\ \ldots$ to m factors,

and $a^n=aaaaa\ \ldots$ to n factors,

$$\begin{aligned}\therefore\quad a^m\times a^n&=(aaaa\ \ldots\text{ to } m \text{ factors})\\&\quad\times(aaaaa\ \ldots\text{ to } n \text{ factors})\\&=aaaaaaaaa\ \ldots\text{ to } (m+n) \text{ factors}\\&=a^{m+n}.\end{aligned}$$

Cor. 1. $a^m\times a^n\times a^p=a^{m+n+p}$, when m, n and p are positive integers.

For $a^m\times a^n=a^{m+n}$; $\therefore\ a^m\times a^n\times a^p=a^{m+n}\times a^p=a^{(m+n)+p}=a^{m+n+p}$.

Cor. 2. $\left(a^m\right)^n=a^{mn}$, where m and n are positive integers.

$$\begin{aligned}\text{For}\quad\left(a^m\right)^n&=a^m\times a^m\times a^m\times\cdots\text{ to } n \text{ factors}\\&=a^{m+m+m\cdots\cdots\text{ to } n \text{ terms}}\\&=a^{mn}.\end{aligned}$$

45. Applications of the principles established in the preceding articles.

Example 1. Show that $(-ab)^2=a^2b^2$.

$$\begin{aligned}(-ab)^2&=(-ab)\times(-ab)\\&=(ab)\times(ab) && [\text{Art. 41}]\\&=a\times b\times a\times b && [\text{Art. 43}]\\&=a\times a\times b\times b && [\text{Cor., Art. 43}]\\&=(aa)\times(bb) && [\text{Art. 43}]\\&=a^2b^2. && [\text{Art. 44}]\end{aligned}$$

Example 2. Multiply $-5a^3b^2$ by $4a^5b^4$.

$$\begin{aligned}(-5a^3b^2)\times(4a^5b^4)&=-\{(5a^3b^2)\times(4a^5b^4)\} && [\text{Art. 41}]\\&=-\{5\times a^3\times b^2\times 4\times a^5\times b^4\} && [\text{Art. 43}]\\&=-\{5\times 4\times a^3\times a^5\times b^2\times b^4\} && [\text{Cor., Art. 43}]\\&=-\{20\times(a^3a^5)\times(b^2b^4)\} && [\text{Art. 43}]\\&=-20a^8b^6. && [\text{Art. 44}]\end{aligned}$$

Example 3. Simplify $(-2x^5y^4z)\times(4x^2y^7z^2)\times(-6xy^3z^4)$.

We have $(-2x^5y^4z)\times(4x^2y^7z^2)$

$$\begin{aligned}&=-\{(2x^5y^4z)\times(4x^2y^7z^2)\}\\&=-\{2\times x^5\times y^4\times z\times 4\times x^2\times y^7\times z^2\}\\&=-\{2\times 4\times x^5\times x^2\times y^4\times y^7\times z\times z^2\}\\&=-\{8\times(x^5x^2)\times(y^4y^7)\times(zz^2)\}=-8x^7y^{11}z^3.\end{aligned}$$

Hence, the given expression

$$\begin{aligned}&=(-8x^7y^{11}z^3)\times(-6xy^3z^4)=(8x^7y^{11}z^3)\times(6xy^3z^4)\\&=8\times x^7\times y^{11}\times z^3\times 6\times x\times y^3\times z^4\\&=8\times 6\times x^7\times x\times y^{11}\times y^3\times z^3\times z^4\\&=48\times(x^7x)\times(y^{11}y^3)\times(z^3z^4)=48x^8y^{14}z^7\end{aligned}$$

EXERCISE 15

Show that :

1. $(-a)\times 6b=-6ab$.
2. $(4a)\times(-2b)=-8ab$.
3. $-7x^7\times 8x^8=-56x^{15}$.
4. $(-2b)\times(-10a)=20ab$.
5. $(-7c)\times(-3ab)=21abc$.
6. $10\times 35=25\times 14$.
7. $15\times 75=5^3\times 3^2$.
8. $(-a)^3=-a^3$.
9. $(-ab)^3=-a^3b^3$.
10. $(a^4b^2)^3=a^{12}b^6$.
11. $(-a^3b^5)^2=a^6b^{10}$.
12. $(-x)^5=-x^5$.
13. $(-4x^2y^4)^2=16x^4y^8$.

Multiply :

14. $2x^2y$ by $-3x^5y^4$.
15. $-7a^2b^3c$ by $-3abc^2$.
16. $-5x^{12}y^3$ by $-8x^5y^{13}$.
17. $-12x^3y^3z^2$ by $13x^7y^6z^4$.
18. $-14xy^5z^8$ by $-10x^5y^2z^{12}$.

Simplify :

19. $(-x)^3 \times (-2xy^2)^2 \times (x^2y)^3$.
20. $(-2a^2) \times (7a^4b^7) \times (5a^9b^5)$.
21. $(-6x^5y^2z) \times (2z^4x^3y^5) \times (-4y^3z^2x^8)$.
22. $(-3x^2y) \times (4zy^2x) \times (-x^3z^5y^4) \times (2zxy)$.

46. Products of monomial expressions can be always found by the method illustrated in the last article ; it is necessary, however, when dealing with more complicated cases of multiplication, that such products should be found mentally. Hence, the student must get thoroughly accustomed to this kind of mental work, for which an exercise is added below.

Example 1. Write down the product of $3x^2$ and $-5xy$.

$(3x^2) \times (-5xy) = -15x^3y$.

Example 2. Write down the product of $-5a^2b$ and $-8ab^2$.

$(-5a^2b) \times (-8ab^2) = 40a^3b^3$.

EXERCISE 16

Write down the product of :

1. $-2x^3$ and $5x^4$.
2. $5a^3b$ and $-4ab^5$.
3. $-3m^2n^5$ and $-7n^3m^5$.
4. $3x^3y^5$ and $-6xy^2$.
5. $-a^3b^2$ and $-3a^4b^3$.
6. $5mn^6$ and $-8m^7n$.
7. $-10xyz^2$ and $-5xy^2z$.
8. $4x^3y^3z$ and $-6xyz^3$.
9. $-6x^2y^3z^4$ and $-8x^3y^2z$.
10. $-5a^3b^5c^7$ and $-5a^2b^4c^6$.
11. $3x^2yz^4$ and $-8xy^2z$.
12. $-4abxy$ and $-8a^2xby^2$.
13. $-7a^2b^2z^3$ and $-5abz$.
14. $5a^4x^2y$ and $-12x^5y^4a^2$.
15. $-14xy^4$ and $-5x^4yz$.
16. $2abc^5$ and $-9a^7b^5c$.
17. $-7a^3x^5y$ and $-9x^3ya^6$.
18. $-8x^6y^2z^5$ and $-20y^5z^2x^3$.
19. $-13a^8b^{13}c^{15}$ and $-5bc^5a^2$.
20. $-7a^7x^6y^6z^2$ and $-16z^5x^2a^6y^3$.

47. To prove that $a(b+c)$[*] $= ab+ac$.

Whatever b and c may be if a be a *positive integer*, we have

$$a(b+c) = (b+c)+(b+c)+(b+c)+\cdots\cdots \text{ to } a \text{ terms}$$

---

[*] Every binomial expression can be put in the form $b+c$. For instance, the expression $2x^2-3y^2$, which can also be written as $(2x^2)+(-3y^2)$ is of the form $b+c$, $2x^2$ being regarded as b and $-3y^2$ as c.

$$=(b+b+b+\cdots \text{ to } a \text{ terms})$$
$$+(c+c+c+\cdots\cdots \text{ to } a \text{ terms})$$
$$=ab+ac. \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

Hence, conversely, $\frac{ab+ac}{a}=b+c=\frac{ab}{a}+\frac{ac}{a}$; that is, if p and q be any two quantities and r a *positive integer*, then $\frac{p+q}{r}=\frac{p}{r}+\frac{q}{r}$. $\cdots$ (A)

Next suppose, a is a positive fraction, *i.e.*, suppose, $a=\frac{m}{n}$, where m and n are positive integers.

Then, $\frac{m}{n}(b+c)=m\times\frac{b+c}{n}$ [by the definition of multiplication]

$$=\frac{m(b+c)}{n}$$
$$=\frac{mb+mc}{n} \quad [\text{ by } (1)\,]$$
$$=\frac{mb}{n}+\frac{mc}{n} \quad [\text{ by } (A)\,]$$
$$=\frac{m}{n}b+\frac{m}{n}c. \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

Hence, from (1) and (2), for all *positive values of a*, we have

$$a(b+c)=ab+ac. \quad \cdots \quad \cdots \quad \cdots \quad (3)$$

Next suppose, a is any negative quantity, *i.e.*, suppose, $a=-x$, where x is any positive quantity.

Then, $(-x).(b+c)=-[x(b+c)]$

$$=-(xb+xc) \quad [\text{ by } (3)\,]$$
$$=-xb-xc=(-x).b+(-x).c\,;$$

thus, for any *negative* value of a also, we have

$$a(b+c)=ab+ac. \quad \cdots \quad \cdots \quad \cdots \quad (4)$$

Hence, from (3) and (4), for *all values* of a, b and c, we have

$$a(b+c)=ab+ac.$$

Cor. 1. Conversely, $ab+ac=a(b+c)$.

Note. Here a is a factor common to both the terms and is called a common factor of the two terms. Similar results hold for three or more terms having common factor.

Similarly, $xya^2+xyb^2=xy(a^2+b^2)$.

Cor. 2. Since, $b-c=b+(-c)$, we have

$$a(b-c)=a[b+(-c)]=ab+a(-c)=ab-ac.$$

Conversely, $ab-ac=a(b-c)$. Hence, $2ax-2ay=2a(x-y)$.

Cor. 3. $a(b+c+d)=a\{b+(c+d)\}=ab+a(c+d)=ab+ac+ad.$

Similarly, $a(b+c+d+e+f+\cdots)=ab+ac+ad+ae+af+\cdots$

Thus, when any multinomial expression is multiplied by a monomial, the result is the sum of the products obtained by multiplying the different terms of the multinomial by the monomial.

Conversely, $ab+ac+ad+ae+\cdots=a(b+c+d+e+\cdots)$.

Example 1. Multiply $2ab-3b^2$ by $5ab$.

$$\begin{aligned}5ab(2ab-3b^2)&=5ab\{2ab+(-3b^2)\}\\&=5ab\times 2ab+5ab\times(-3b^2)\\&=10a^2b^2-15ab^3.\end{aligned}$$

Example 2. Multiply $x^4-3x^3+5x^2-6x+4$ by $-6x^2$.

$$\begin{aligned}&(-6x^2)(x^4-3x^3+5x^2-6x+4)\\&=(-6x^2)\{x^4+(-3x^3)+5x^2+(-6x)+4\}\\&=(-6x^2).x^4+(-6x^2)(-3x^3)+(-6x^2).5x^2\\&\qquad+(-6x^2)(-6x)+(-6x^2).4\\&=-6x^6+18x^5-30x^4+36x^3-24x^2.\end{aligned}$$

N. B. The beginner is particularly recommended to work out at first each example in the method shown above, but after some practice he can safely do away with the intermediate steps and write down the result at once in the manner exemplified below.

Example 3. Write down the product of

$$-4a^4+5a^3b-6a^2b^2-8ab^3+9b^4 \text{ and } -3a^2b^2.$$

$$\begin{array}{l}-4a^4+5a^3b-6a^2b^2-8ab^3+9b^4\\-3a^2b^2\\\hline 12a^6b^2-15a^5b^3+18a^4b^4+24a^3b^5-27a^2b^6\end{array}$$

Example 4. Simplify $2x^2(3x-2)+2x(2x+3)-6(x-3)$.

We have
$$\begin{aligned}2x^2(3x-2)&=6x^3-4x^2,\\2x(2x+3)&=4x^2+6x,\\6(x-3)&=6x-18.\end{aligned}$$

Therefore, the given expression

$$\begin{aligned}&=(6x^3-4x^2)+(4x^2+6x)-(6x-18)\\&=6x^3-4x^2+4x^2+6x-6x+18=6x^3+18.\end{aligned}$$

Example 5. Simplify $3a(2a-5)-3a(a-6)$.

Putting x for $2a-5$ and y for $a-6$, we have

$$3a(2a-5)-3a(a-6)=3ax-3ay=3a(x-y)$$
$$=3a\{(2a-5)-(a-6)\}=3a(a+1)=3a^2+3a.$$

EXERCISE 17

Multiply :

1. $2x-y$ by $-x$.
2. $a-2b+3c$ by $-5a$.
3. $2x-3y$ by $4xy$.
4. $2a^2-3b^2-c^2$ by abc.
5. $x^2y-2xy^2-y^3$ by $-3xy$.
6. $3a^2b^2-ab^2-5a^3+a^2b$ by $7b^2$.

Write down the product of :

7. $3a^2x-4ax^2+5ax$ and $-2a^2$.
8. $-2m^3+3m^2n-5mn^2$ and $4mn$.
9. $a^2bc-b^2ca+c^2ab$ and $-abc$.
10. $x^2+y^2+z^2-yz-zx-xy$ and xyz.
11. $-2c^3d+3d^3c-5cd^3-4c^2d^2$ and $-6c^2d^4$.
12. $8a^4-6a^3b+5a^2b^2-4ab^3$ and $-2a^3b^3$.

Simplify :

13. $7x^3(x-2)-2x^2(x-3)-8x^2(1-2x)$.
14. $x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)$.
15. $9x^3(x^3-2y^2)+5y^2(3x^3+y^2)+3y^2(x^3-10y^2)$.
16. $x^3(x^3+2x^2+2x)-2x^2(x^3+2x^2+2x)+2x(x^3+2x^2+2x)$.
17. $a^6b^3(a^6b^3-2a^4b^2+2a^2b)+2a^4b^2(a^6b^3-2a^4b^2+2a^2b)$
$+2a^2b(a^6b^3-2a^4b^2+2a^2b)$.
18. $2a^9b^6(2a^9b^6+6a^6b^4+9a^3b^2)-6a^6b^4(2a^9b^6+6a^6b^4+9a^3b^2)$
$+9a^3b^2(2a^9b^6+6a^6b^4+9a^3b^2)$.
19. $a^2(2x-3y)+a^2(3x+4y)-a^2(5x-2y)$.
20. If $a=x^2-yz$, $b=y^2-zx$ and $c=z^2-xy$,

find the values of (i) $ax+by+cz$; (ii) $cx+ay+bz$.

IV. Division

48. Definition. One quantity a is said to be divided by another quantity b, when a third quantity c is found such that $c\times b=a$. In other words, $a\div b=c$, when $a=b\times c$.

Thus, when $x=y\times z$, we have $x\div y=z$, and $x\div z=y$.

When one quantity is divided by another, the former is called the *dividend* and the latter the *divisor* ; the result is called the *quotient*.

49. Fundamental Propositions.

(i) To prove that $a \div b \times b = a$.

If we denote $a \div b$ by x, we must have, by definition,

$$x \times b = a.$$

Hence, $\quad a \div b \times b = x \times b = a.$

(ii) To prove that $a \div b \div c = a \div bc$.

We have
$$\begin{aligned}(a \div b \div c) \times bc &= \{(a \div b) \div c\} \times c \times b \\ &= [\{(a \div b) \div c\} \times c] \times b \\ &= (a \div b) \times b \qquad \text{[by the last result]} \\ &= a.\end{aligned}$$

Hence, by definition, $a \div b \div c = a \div bc$.

That is, *to divide any quantity successively by two others is the same as to divide it at once by their product.*

Cor. Hence, $a \div b \div c = a \div c \div b$, for each of them $= a \div (bc)$.

(iii) To prove that $a \div b = a \times \frac{1}{b}$.

We have $\quad \frac{1}{b} \times b = 1 \div b \times b = 1. \quad$ [by (i)]

Hence, $\quad a \times \frac{1}{b} \times b = a \times \left(\frac{1}{b} \times b\right) \quad$ [Art. 43]

$$= a \times 1 = a\,;$$

i.e., $\quad \left(a \times \frac{1}{b}\right) \times b = a.$

Therefore, by definition, $a \div b = a \times \frac{1}{b}$.

Thus, *to divide any quantity by another is the same as to multiply the former by the reciprocal of the latter.*

Cor. $a \div b \times c = a \times c \div b$.

For $\quad a \div b \times c = a \times \frac{1}{b} \times c = a \times c \times \frac{1}{b}, \quad$ [Cor., Art. 43]

and this latter $= a \times c \div b$.

50. Law of signs.

Since, $\quad a \times (-b) = -ab,$

$$\therefore \text{ by definition, } \left.\begin{aligned}(-ab) \div a &= -b \\ \text{and } (-ab) \div (-b) &= a\end{aligned}\right\} \quad \ldots \quad \text{I}$$

Again, since $(-a)\times(-b)=ab$,

$$\left.\begin{aligned} \therefore\quad & ab\div(-a)=-b \\ \text{and}\quad & ab\div(-b)=-a \end{aligned}\right\} \quad \cdots \text{ II}$$

It is evident also that

$$\left.\begin{aligned} & ab\div a=\ \ b \\ \text{and}\quad & ab\div b=\ \ a \end{aligned}\right\} \quad \cdots \text{ III}$$

Hence, from I, II and III, we have the following law of signs in division :

When the dividend and the divisor have the same sign, the quotient is positive, and when they have different signs, the quotient is negative. In other words, *like signs produce* +, *and unlike signs* −.

51. Division of one monomial expression by another.

Let us examine a few particular cases :

(i) Since, $3a^2b\times5a^3b^2c=15a^5b^3c$, we must have
$(15a^5b^3c)\div(5a^3b^2c)=3a^2b$.

$$\left.\begin{aligned} &\text{Thus, if the dividend}=15a^5b^3c \\ &\qquad\qquad\qquad\qquad=3\times5\times a^3\times a^2\times b^2\times b\times c, \\ &\quad\text{and the divisor}=5a^3b^2c, \\ &\text{we have the quotient}=3a^2b. \end{aligned}\right\} \quad \cdots \text{ I}$$

(ii) Since, $(-2a^{10}b^2cd)\times(-3a^5c^2)=6a^{15}b^2c^3d$,
we must have $6a^{15}b^2c^3d\div(-2a^{10}b^2cd)=-3a^5c^2$.

$$\left.\begin{aligned} &\text{Thus, if the dividend}=6a^{15}b^2c^3d \\ &\qquad\qquad\qquad\qquad=2\times3\times a^{10}\times a^5\times b^2\times c\times c^2\times d, \\ &\quad\text{and the divisor}=-2a^{10}b^2cd, \\ &\text{we have the quotient}=-3a^5c^2. \end{aligned}\right\} \quad \cdots \text{ II}$$

(iii) Since, $(-5a^8b^5c^2d)\times(4b^3c^4)=-20a^8b^8c^6d$,
we must have $(-20a^8b^8c^6d)\div(-5a^8b^5c^2d)=4b^3c^4$.

$$\left.\begin{aligned} &\text{Thus, if the dividend}=-20a^8b^8c^6d \\ &\qquad\qquad\qquad\qquad=(-5)\times4\times a^8\times b^5\times b^3\times c^2\times c^4\times d, \\ &\quad\text{and the divisor}=-5a^8b^5c^2d, \\ &\text{we have the quotient}=4b^3c^4. \end{aligned}\right\} \quad \cdots \text{ III}$$

Hence, from I, II and III, we are led to deduce the following rule for dividing one monomial expression by another :

Take away from the dividend all those factors which make up the divisor and to the remaining factors prefix the sign +, *or no sign, if the two expressions have the same sign, and the sign* −, *if they have different signs.*

Note. *We have* $a^{12}\div a^7=(a^5\times a^7)\div a^7=a^5\ [=a^{12-7}]$.

Similarly, $a^{20} \div a^{9} = a^{11}$, $a^{21} \div a^{14} = a^{7}$, *and so on. Hence, generally* $a^{m} \div a^{n} = a^{m-n}$, *where* m *and* n *are positive integers and* $m > n$.

Example 1. Divide $18m^3n^2p$ by $-6m^2n^2p$.

The dividend $= 18m^3n^2p$

$= 6 \times 3 \times m^2 \times m \times n^2 \times p.$

The divisor $= -6m^2n^2p$.

$\therefore$ the quotient $= -3m$.

Example 2. Divide $-24a^7b^3c$ by $-6a^4bc$.

The dividend $= -24a^7b^3c$

$= (-6) \times 4 \times a^4 \times a^3 \times b \times b^2 \times c.$

The divisor $= -6a^4bc$.

$\therefore$ the quotient $= 4a^3b^2$.

EXERCISE 18

Divide :

1. $16x^4$ by $-4x$.
2. $-18x^6$ by $6x^3$.
3. $-20a^7x^5$ by $-5a^3x^2$.
4. $36x^{10}y^9$ by $12x^5y^4$.
5. $-14a^4b^3c$ by $-7a^2bc$.
6. $-20p^{12}q^8r^3$ by $10p^{10}q^6r^2$.
7. $-70x^{16}y^6z$ by $-14x^{10}y^5$.
8. $64a^{12}b^7c^5$ by $-8a^9b^7c^3$.
9. $-81m^{13}n^{14}p^5$ by $27m^8n^6p^4$.
10. $-69a^7b^4c^9$ by $-23a^5b^4c^7$.
11. $25x^{20}y^3z^8$ by $-5x^{16}yz^8$.
12. $-42a^{23}x^{23}y^9z^3$ by $-14a^{17}x^{18}y^5z$.
13. a^{101} by a^{67}.
14. $28x^{205}$ by $-4x^{157}$.
15. $56m^{307}$ by $-8m^{269}$.
16. $-91a^{188}b^{209}$ by $13a^{97}b^{89}$.

52. Division of a multinomial by a monomial.

From Cor. 3, Art. 47, we have

$$a(b+c+d+e+f+\cdots) = ab+ac+ad+ae+af+\cdots\cdots$$

Hence, $(ab+ac+ad+ae+\cdots) \div a = b+c+d+e+\cdots\cdots$

$$= (ab \div a)+(ac \div a)+(ad \div a)+(ae \div a)+\cdots\cdots$$

Thus, *to divide a multinomial expression by a monomial we have to divide each term of the dividend by the divisor and take the sum of those partial quotients for the complete quotient.*

Example 1. Divide $4a^3x^2 - 6a^2x^3 + 10ax^4$ by $-2ax$.

The required quotient $= \dfrac{4a^3x^2 - 6a^2x^3 + 10ax^4}{-2ax}$

$$= \frac{4a^3x^2}{-2ax} + \frac{-6a^2x^3}{-2ax} + \frac{10ax^4}{-2ax}$$
$$= -2a^2x + 3ax^2 - 5x^3$$

Example 2. Divide $9x^5 - 4x^4a - 2x^3a^2$ by $3x^3$.

The required quotient $= \frac{9x^5 - 4x^4a - 2x^3a^2}{3x^3} = \frac{9x^5}{3x^3} + \frac{-4x^4a}{3x^3} + \frac{-2x^3a^2}{3x^3}$

$$= 3x^2 - \tfrac{4}{3}xa - \tfrac{2}{3}a^2.$$

Note. *After a little practice the student can safely do away with the (intermediate) step in each case and write down the quotient at once.*

EXERCISE 19

Divide :

1. $3a^3b^2 - 2a^2b^3$ by a^2b^2.
2. $2a^3b - 3ab^3$ by $-ab$.
3. $6a^4b^2 - 9a^2b^4$ by $3a^2b^2$.
4. $12x^4y^2 - 9x^5y$ by $-3x^3y$.
5. $14x^7y^5 - 21x^5y^7$ by $-7x^5y^5$.
6. $4mn^3 - 12m^2n^2 + 16m^3n$ by $4mn$.
7. $-3a^3x^4 + 6a^2x^5 - 9a^4x^3$ by $-3a^2x^3$.
8. $12x^5 - 8x^3a^2 + 20ax^4$ by $-4x^3$.
9. $10m^5n^4 - 15m^7n^2 - 20m^8n^6$ by $5m^3n^2$.
10. $8p^4q^2 - 5p^3q^3 - 3p^2q^4$ by $-8p^2q^2$.
11. $-14x^8y^5 + 21x^{10}y^3 - 28x^7y^6$ by $7x^7y^3$.
12. $15a^4x^8 - 30a^7x^5 - 45a^6x^6$ by $20a^4x^5$.
13. $-60x^4a^5 - 75x^3a^6 + 80x^5a^4$ by $-20x^3a^4$.
14. $125m^6n^4p^2 - 175m^4n^6p^2 - 200m^2n^2p^8$ by $25m^2n^2p^2$.
15. $-a^2b^4c^4x^4y^4z^2 + 2a^4b^2c^4x^2y^4z^4 - 3a^4b^4c^2x^4y^2z^4$
    by $-a^2b^2c^2x^2y^2z^2$.

MISCELLANEOUS EXERCISES I

I

1. What number will represent an interval of 5 hours (i) if the unit of time be half an hour ; (ii) if the unit of time be 10 hours ?

2. If x stands for 17 and y for 25, what does $x \sim y$ denote ?

3. Define "Coefficient". Distinguish between a ***numerical*** coefficient and a ***literal*** coefficient.

What are the coefficients of x^3 in $15x^3$, $2ax^3$, $7ab^2x^3$ and $16m^2pqx^3$?

4. Distinguish between $\sqrt{ab}$ and $\sqrt{a}b$. Find the value of $\sqrt{ab} \sim \sqrt{a}b$, when $a=9$, $b=4$.

5. If a distance of half a kilometre to the north of a place be represented by 40, what will represent a distance of 6¼ metres to the south of it ?

6. State the result when a negative quantity is added to a positive quantity. Hence deduce that $+(-b)=-b$.

7. Define *subtraction*. Hence deduce that $4-6=-2$ and that $5-(-3)=8$.

8. Arrange the following numbers in descending order of magnitude : $2, 5, -3, 7, -8, -1, 9, -4, -12$.

II

1. If $a=4$, $b=5$, find the values of :

(i) $ab-a\times b$; (ii) $45-ab$; (iii) $74-7a$; (iv) $85-8b$.

2. What does a^n mean ? Distinguish between a^n and n^a. Find the value of $a^4-4a^3b+6a^2b^2-4ab^3+b^4$, when $a=7$, $b=5$.

3. What is the relation between a and each of the following : $\sqrt[3]{a}$, $\sqrt[5]{a}$, $\sqrt[8]{a}$ and $\sqrt[n]{a}$?

Find the value of $\sqrt{a^2-3d}\times\sqrt[3]{b^3-c^3-2e}$, when $a=8$, $b=7$, $c=6$, $d=5$ and $e=1$.

4. What is meant by the *absolute value* of a positive or a negative quantity ? Illustrate this by an example.

5. Add together $3x^2y$, $-8x^2y$, $-19x^2y$ and $17x^2y$; and find the numerical value of the sum, when $x=4$, $y=5$.

6. Write down the sum of $16x^4$, $-8xy^3$, $24x^2y^2$, y^4 and $-32x^3y$ and find its numerical value, when $x=4$, $y=5$.

7. Subtract $4a-13b-25c$ from $17b-12c-19a$.

8. Simplify $3x-[4y+\{2z-(x-5y+3z)\}]-(3x-7y)$.

III

1. Express algebraically the following statements :

(i) The result of multiplying the sum of a and b by c is the same as the result of dividing x by the product of y and z.

(ii) The square of the sum of x and y is the same as the result of adding together the square of x, the square of y, and twice the product of x and y.

(iii) If the cube root of the result of subtracting n from m be divided by the product of the cube of m and n, we get a quantity which is less than the sum of the square roots of x and y.

(iv) Since, a is greater than b, therefore, three times a is greater than three times b.

2. A, B, C, D, E, F, G are a number of successive points on a straight line such that the distances AB, BC, CD, DE, EF, FG are respectively 3, 4, 6, 8, 5 and 7 centimetres. If DC be represented by 3, what number will represent DB, DE, DF, DA and DG respectively ?

3. State the result when one negative quantity is added to another. Find the value of the sum of $-a^3$, $-3a^2b$, $-3ab^2$, $-b^3$, when $a=6$, $b=4$.

4. Show by a numerical example that when any number of quantities are added together, the result is the same in whatever order the quantities may be taken.

5. If $a=16$, $b=10$, $c=5$, $d=1$, find the value of

$$(a-b)(5\sqrt{a}-b)+\sqrt{(a-b)(c+d)}.$$

6. If $a=\frac{1}{2}$, $b=\frac{2}{3}$, prove that

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$$

7. Add together $3a^2+4bc-x^2+10$, $2x^2-5a^2-15+6bc$ and $21-9bc-4a^2-10x^2$.

8. Simplify $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.

IV

1. If $a=9$, find the value of :

(i) $\sqrt{49}-\sqrt{4}a$; (ii) $\sqrt{49}-\sqrt{4a}$.

2. Show by a numerical example that when any number of quantities are added together, they can be divided into groups and the result expressed as the sum of these groups.

3. If $a=2$, $b=3$, $c=4$, find the value of

$$\frac{a-b+c}{a+b-c}+\frac{b-c+a}{b+c-a}+\frac{c-a+b}{c+a-b}.$$

4. Define an *Algebraical Expression*. Distinguish between a *simple* expression and a *compound* expression.

Is $42abx^2$ a simple or a compound expression ? Give the names with illustrations of the different classes of compound expressions.

5. If $x=2$, $y=3$, $a=6$, $b=5$, find the value of

$$\sqrt[3]{b(x+y)^3}+\sqrt[3]{(x+a)(b-2x)}+\sqrt[3]{x(b-y)^2}.$$

6. A certain sum is divided between A, B and C ; B receives a pounds more than A, and C receives b pounds more than B ; if A receives x pounds, find an expression for the whole sum divided.

7. Add together $a^2-3ab-\frac{1}{2}\frac{1}{4}b^2$, $2b^2-\frac{2}{3}b^3+c^2$, $ab-\frac{1}{3}b^2+b^3$ and $2ab-\frac{1}{3}b^2$.

8. Reduce to its simplest form

$$\{2x^2-(y^2-xy)\}-\{y^2-(4x^2-y^2)\}+\{2y^2-(3xy-x^2)\}.$$

V

1. What is meant by the *dimensions* and *degree* of a product? What is a *Homogeneous Expression*? Write down two trinomial homogeneous expressions, one of six dimensions and the other of seven.

2. If you were asked to find the value of the expression $a\times b-c+d\times e+f+gh$, how would you proceed?

3. Define *factor*. What are the *simple factors* of $2ab(a+b)$?

4. If $a=4$ and $x=2$, find the numerical value of

$$\frac{2ax^2}{(a-x)^2}-\frac{6\sqrt[3]{ax}}{a^3\cdot\sqrt{2a+4x}}-\frac{29x^2}{64a}.$$

5. Find the value of $(x^3-7x^2+6x+5)+(-3x+2x^3+4+5x^2)$ $+(-11-4x^3+2x-7x^2)+(9x^2+2+5x^3-4x)$, when $x=5$.

6. Prove that $a-(b-c)=a-b+c$. How is this generally proved when a, b, c are all positive quantities and a is greater than b, and b is greater than c?

7. Simplify $2x-[(3x-9y)-\{(2x-3y)-(x+5y)\}]$.

8. When is one number said to be multiplied by another? From the definition deduce the result when -8 is multiplied by -4.

VI

1. Define the *power* of a number, and the *index* of the power; and illustrate them by a numerical example.

2. If $a=16$, $b=10$, $x=5$, $y=1$, find the numerical value of

$$(a-y)\sqrt{24bx+x^2}+\sqrt{(a-x)(b+y)}.$$

3. Show that $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$,

(i) when $a=3$, $b=4$, $c=5$;

(ii) when $a=\frac{2}{3}$, $b=\frac{5}{3}$, $c=\frac{7}{3}$.

4. State the propositions from which the following result may be deduced

$$a-b+c-d+e-f=(a+c+e)+(-b-d-f).$$

5. Illustrate clearly by an example that $40-(-15)=55$.

6. Find the numerical value of the sum of $7x^2-25\sqrt{yz}+z^4$, $19\sqrt{yz}-3z^4-12x^2$ and $2z^4+5x^2+7\sqrt{yz}$, when $x=17$, $y=16$, $z=15$.

7. State the operations indicated by the expression
$$5a-[4b-\{3c-(2d-7e)\}].$$

8. Find the value of
$$[(a^3+b^3+c^3+d^3)\{a+b-(c-a)\}+a^2b+c^2d]\{a^2-(b^2+c^2)+d^2\}$$
when $a=4,\ b=3,\ c=2,\ d=1$.

VII

1. Distinguish between :

(i) $a+bc$ and $a+b\times c$; (ii) a^4 and $4a$; (iii) $3\sqrt{a}$ and $\sqrt[3]{a}$

(iv) $\sqrt{a+b}$ and $\sqrt{a}+b$; (v) $\sqrt{ab}$ and $\sqrt{a}b$.

2. If $a=1,\ b=2,\ c=3,\ d=0$, find the value of :

(i) $\dfrac{a^2b+b^2c+c^2d+d^2a}{(a+b)(c+d)-\{(a-d)+(c-b)\}}$;

(ii) $\sqrt[3]{b-a^3}+\sqrt[3]{4(c-a)}-\sqrt[4]{3(8a+5b+3c-2d)}$.

3. Show that the expressions
$$(a+b+c)^3+a^3+b^3+c^3,\qquad (a+b)^3+(b+c)^3+(c+a)^3+6abc \text{ and}$$
$2a^3+3b^2(a+c)+2b^3+3c^2(a+b)+2c^3+3a^2(b+c)+6abc$ are equal to one another,

(i) when $a=2,\ b=3,\ c=4$;

(ii) when $a=7,\ b=4,\ c=1$.

4. Simplify : (i) $1-[1-\{1-(1-\overline{1+x})\}]$;

(ii) $3a-(b-2c)-\{a+c-(3a-b-2c)\}-(2a-3b+4c)$.

5. Express algebraically the following statements :

(i) That the product of the sum of the two numbers multiplied by their difference is equal to the difference of the squares of the numbers.

(ii) That the square of the sum of two numbers exceeds the sum of their squares by twice their product.

6. Find the value of
$$17a-5b-[7a-3b-\{4(a-b)-(2a+3b)\}],\text{ when } a=39,\ b=52.$$

7. If $V=5a+4b-6c,\qquad X=-3a-9b+7c,$
$Y=20a+7b-5c,\qquad Z=13a-5b+9c,$
calculate the value of $V-(X+Y)+Z$. [Mad. U. Matric., 1883]

8. From the sum of $a-\frac{1}{3}b+\frac{1}{2}c-\frac{1}{3}d$, $-\frac{1}{3}c+\frac{1}{3}a-\frac{1}{2}b+d$, $\frac{1}{2}d-\frac{1}{3}b+c-a$, $\frac{1}{3}a-\frac{2}{3}d+b-\frac{5}{6}c$ and $8a-6b+3c-4d$ subtract $\frac{13}{3}a-\frac{151}{60}b+\frac{2}{3}c-\frac{13}{2}d$.

VIII

1. Prove that $a \times b = b \times a$, when a and b are any two positive integers.

2. If M stands for $a(m+n)$ and N stands for $b(m-n)$, find the values of $\frac{M}{a} + \frac{N}{b}$ and $\frac{M}{a} - \frac{N}{b}$.

3. In the identity* $c(a+b) = ca+cb$ substitute :
   (i) $m+n$ for c and find the value of the product $(m+n)(a+b)$;
   (ii) $a+b$ for c and evaluate $(a+b)^2$.

4. Simplify : (i) $x(y-z)+y(z-x)+z(x-y)$;
   (ii) $\frac{y-z}{yz}+\frac{z-x}{zx}+\frac{x-y}{xy}$.

5. Prove that
   (i) $a^m \div a^n = a^{m-n}$, where m and n are positive integers and $m > n$;
and (ii) $a \div b \div c = a \div c \div b = a \div bc$.

6. If $a = 3xy - yz - zx$, $b = 3yz - xy - zx$ and $c = 3zx - xy - yz$, find the value of $\frac{a+b+c}{xyz}$.

7. Multiply $\frac{3}{8}a^5b^{10}c^{15}x^8y^6z^4 + \frac{15}{4}a^{10}b^{15}c^5x^6y^4z^2 + \frac{5}{12}a^{15}b^5c^{10}x^4y^2$ by $24a^3b^5c^7x^2y^4z^6$.

8. Divide $\frac{3}{2}a^{10}b^{15}c^{20}x^{12}y^{10}z^8 + \frac{15}{4}a^{15}b^{20}c^{10}x^{10}y^8z^{12} + \frac{9}{2}a^{20}b^{10}c^{15}x^8y^{12}z^{10}$ by $\frac{3}{2}a^{10}b^{10}c^{10}x^8y^8z^8$.

CHAPTER IV

SIMPLE FORMULÆ AND THEIR APPLICATION

53. Definition. Any general result expressed in symbols is called a **formula.** In other words, a formula is the most general expression for any theorem respecting numerical quantities.

54. Formula. $(a+b)^2 = a^2+2ab+b^2$.

$$[\,(a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b)$$
$$= a^2+2ab+b^2.\,]$$

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*If in an equality both sides are equal for any and every value of the symbolical quantities involved, or when one side of the equality can be simplified into the other, the equality is said to be an *identity*. (Also see Art. 62.)

That is, *the square of the sum of any two quantities is equal to the sum of their squares plus twice their product.*

Cor. $a^2+b^2=(a^2+2ab+b^2)-2ab=(a+b)^2-2ab.$

Example 1. Find the square of $2x+3y$.

$(2x+3y)^2=(2x)^2+2(2x)(3y)+(3y)^2=4x^2+12xy+9y^2.$

Example 2. Find the square of $5x+4$.

$(5x+4)^2=(5x)^2+2(5x).4+4^2=25x^2+40x+16.$

Example 3. Find the square of $4a^3+7b^4$.

$(4a^3+7b^4)^2=(4a^3)^2+2(4a^3)(7b^4)+(7b^4)^2=16a^6+56a^3b^4+49b^8.$

Example 4. Find the square of $\frac{1}{2}x+\frac{2}{3}y$.

$$\begin{aligned}(\tfrac{1}{2}x+\tfrac{2}{3}y)^2&=(\tfrac{1}{2}x)^2+2(\tfrac{1}{2}x)(\tfrac{2}{3}y)+(\tfrac{2}{3}y)^2\\&=\tfrac{1}{4}x^2+\tfrac{2}{3}xy+\tfrac{4}{9}y^2.\end{aligned}$$

Example 5. Find the square of (i) $\left(\frac{x}{y}+\frac{y}{x}\right)$, (ii) $\left(\frac{1}{x}+\frac{1}{y}\right)$.

(i) $\left(\frac{x}{y}+\frac{y}{x}\right)^2=\left(\frac{x}{y}\right)^2+2\cdot\left(\frac{x}{y}\right)\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2=\frac{x^2}{y^2}+2+\frac{y^2}{x^2}.$

(ii) $\left(\frac{1}{x}+\frac{1}{y}\right)^2=\left(\frac{1}{x}\right)^2+2\cdot\frac{1}{x}\cdot\frac{1}{y}+\left(\frac{1}{y}\right)^2=\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}.$

Example 6. Find the square of 8012.

$$\begin{aligned}8012^2=(8000+12)^2&=8000^2+2.8000.12+12^2\\&=64000000+192000+144=64192144.\end{aligned}$$

Example 7. Express $16x^2+40xy+25y^2$ as a perfect square.

$$\begin{aligned}16x^2+40xy+25y^2&=(4x)^2+2.4x.5y+(5y)^2\\&=(4x+5y)^2.\end{aligned}$$

Example 8. Find the value of $x^2+\frac{1}{x^2}$, when $x+\frac{1}{x}=3$.

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2.x.\frac{1}{x}=3^2-2=7.$$

Example 9. Find the value of $a^4+\frac{1}{a^4}$, when $a+\frac{1}{a}=4$.

$$a^2+\frac{1}{a^2}=\left(a+\frac{1}{a}\right)^2-2.a.\frac{1}{a}=4^2-2=14.$$

$$a^4+\frac{1}{a^4}=\left(a^2+\frac{1}{a^2}\right)^2-2.a^2\cdot\frac{1}{a^2}=14^2-2=196-2=194.$$

Example 10. Find the square of $a+b+c$.

$(a+b+c)^2=\{a+(b+c)\}^2$, [regarding $b+c$ as one term]

$$=a^2+2a(b+c)+(b+c)^2$$
$$=a^2+2ab+2ac+b^2+2bc+c^2$$
$$=a^2+b^2+c^2+2ab+2ac+2bc.^*$$

Example 11. Find the square of $a+b+c+d$.

$(a+b+c+d)^2=\{(a+b)+(c+d)\}^2$, [regarding $a+b$ as one term and $c+d$ as another]

$$=(a+b)^2+2(a+b)(c+d)+(c+d)^2$$
$$=(a^2+2ab+b^2)+2(ac+ad+bc+bd)+(c^2+2cd+d^2)$$
$$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

Example 12. Simplify

$$(a+b-c)^2+2(a+b-c)(a-b+c)+(a-b+c)^2.$$

Putting x for $(a+b-c)$ and y for $(a-b+c)$, we have the given expression $=x^2+2xy+y^2=(x+y)^2$

$$=\{(a+b-c)+(a-b+c)\}^2$$
$$=(2a)^2=4a^2.$$

Example 13. Find the value of $9x^2+30xy+25y^2$, when $x=15$ and $y=-9$.

The given expression $=(3x)^2+2(3x)(5y)+(5y)^2=(3x+5y)^2$.

But $3x+5y=3\times15+5\times(-9)=45-45=0$.

$\therefore$ the given expression $=0$.

Example 14. Simplify $986\times986+986.28+196$.

The given expression $=(986)^2+2.986.14+(14)^2$

[$\because$ $196=14^2$, $28=2.14$]

$$=(986+14)^2=1000^2=1000000.$$

EXERCISE 20

Find the square of each of the following expressions :

1. $x+4$.
2. $3a+2$.
3. $x+2y$.
4. $2x+7y$.
5. $3a+4b$.
6. $5a+7b$.
7. $ay+3bx$.
8. a^2+2bc.
9. $3x^2+2y^2$.

---

*This can be written as $a^2+b^2+c^2+2(ab+ac+bc)$.

Therefore, (i) $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$.

(ii) $2(ab+ac+bc)=(a+b+c)^2-(a^2+b^2+c^2)$;

$\therefore$ $(ab+ac+bc)=\frac{1}{2}\{(a+b+c)^2-(a^2+b^2+c^2)\}$.

10. $4x^2+y^2$.   11. $\frac{1}{2}x+\frac{2}{3}y$.   12. $\left(\frac{1}{a}+\frac{1}{b}\right)$.

13. $\left(\frac{a}{b}+\frac{b}{a}\right)$.   14. 907.   15. $x+\frac{1}{2x}$.

16. $a+2b+3c$.   17. $ab+bc+ca$.   18. $2p+3q+4r$.

19. $x^2+y^2+z^2$.   20. $2x+3y+4z$.   21. $x^2+y^3+z^4$.

22. $x+y+2a+3b$.   23. $3a+4b+c+2d$.   24. $2a+x+4y+3z$.

25. $4m+3n+3p+2q$.

Express each of the following expressions as a perfect square :

26. x^2+4x+4.   27. $9a^2+24ab+16b^2$.

28. $25x^2+5xy+\frac{y^2}{4}$.   29. $x^2+2+\frac{1}{x^2}$.

30. $\frac{16a^2}{9b^2}+2+\frac{9b^2}{16a^2}$.

Simplify :

31. $(x+y)^2+2(x+y)(x-y)+(x-y)^2$.

32. $(x-y+z)^2+(y+z-x)^2+2(x-y+z)(y+z-x)$.

33. $(2a-3b+4c)^2+(2a+3b-4c)^2+2(2a-3b+4c)(2a+3b-4c)$.

34. $(5a-7b)^2+2(5a-7b)(9b-4a)+(9b-4a)^2$.

35. $(2x-5y-3z)^2+(6y+3z-x)^2+2(2x-5y-3z)(6y+3z-x)$.

36. $45^2+2.45.5+25$.   37. $992\times992+8\times8+2\times8\times992$.

38. $3{\cdot}88\times3{\cdot}88+2{\cdot}4\times3{\cdot}88+1{\cdot}44$.

Find the value of :

39. $9x^2+12x+4$, when $x=-1$.   40. $16x^2+64x+64$, when $x=-2$.

41. $25m^2+40mn+16n^2$, when $m=-18$ and $n=23$.

42. $49a^2+56ab+16b^2$, when $a=-7$ and $b=13$.

43. $64a^2+16ac+c^2$, when $a=6$ and $c=-49$.

44. $81x^2+18xz+z^2$, when $x=7$ and $z=-67$.

45. $36p^2+132pq+121q^2$, when $p=12$ and $q=-7$.

46. If $m+\frac{1}{m}=4$, show that $m^2+\left(\frac{1}{m}\right)^2=14$.

47. If $x+\frac{1}{x}=a$, find the value of $x^2+\frac{1}{x^2}$.

48. If $x+\frac{1}{x}=\sqrt{2}$, show that $x^2+\frac{1}{x^2}=0$ and $x^4+\frac{1}{x^4}=-2$.

49. Find the value of $\frac{a^2}{b^2}+\frac{b^2}{a^2}$, when $\frac{b}{a}+\frac{a}{b}=4$.

50. If $x+\frac{1}{2x}=2$, find the value of $x^2+\left(\frac{1}{2x}\right)^2$.

55. Formula $(a-b)^2=a^2-2ab+b^2$.

$$[\,(a-b)^2=(a-b)(a-b)=a(a-b)-b(a-b)$$
$$=a^2-2ab+b^2.]$$

That is, *the square of the difference of any two quantities is equal to the sum of their squares minus twice their product.*

Note. *This formula is virtually included in the formula of the last article*, for, $(a-b)^2=\{a+(-b)\}^2=a^2+2a(-b)+(-b)^2=a^2-2ab+b^2$.

Cor. 1. $a^2+b^2=(a^2-2ab+b^2)+2ab=(a-b)^2+2ab$.

Cor. 2. Since $(a+b)^2=a^2+2ab+b^2$, and $(a-b)^2=a^2-2ab+b^2$, evidently we have $(a+b)^2=(a-b)^2+4ab$ and $(a-b)^2=(a+b)^2-4ab$.

Example 1. Find the square of $3a-4b$.

$(3a-4b)^2=(3a)^2-2(3a)(4b)+(4b)^2=9a^2-24ab+16b^2$.

Example 2. Express $25y^2-60yz+36z^2$ as a perfect square.

$25y^2-60yz+36z^2=(5y)^2-2.5y.6z+(6z)^2=(5y-6z)^2$.

Example 3. Find the square of 898.

$$898^2=(900-2)^2=900^2-2.900.2+2^2$$
$$=810000-3600+4=806404.$$

Example 4. Find the square of $(\frac{3}{5}x-\frac{5}{6}y)^2$.

$$(\tfrac{3}{5}x-\tfrac{5}{6}y)^2=(\tfrac{3}{5}x)^2-2.\tfrac{3}{5}x.\tfrac{5}{6}y+(\tfrac{5}{6}y)^2$$
$$=\tfrac{9}{25}x^2-xy+\tfrac{25}{36}y^2.$$

Example 5. Find the square of $-4ab-5bc$.

$$(-4ab-5bc)^2=\{(-4ab)-5bc\}^2$$
$$=(-4ab)^2-2.(-4ab)(5bc)+(5bc)^2$$
$$=16a^2b^2+40ab^2c+25b^2c^2.$$

Example 6. Find the square of $x-y-z$.

$$(x-y-z)^2=\{x-(y+z)\}^2=x^2-2x(y+z)+(y+z)^2$$
$$=x^2-2xy-2xz+y^2+2yz+z^2$$
$$=x^2+y^2+z^2-2xy-2xz+2yz.$$

Example 7. Find the square of $2x-3y-4z$.

$$\begin{aligned}(2x-3y-4z)^2 &= \{2x-(3y+4z)\}^2 = (2x)^2-2(2x)(3y+4z)+(3y+4z)^2\\ &= 4x^2-2(6xy+8xz)+\{(3y)^2+2(3y)(4z)+(4z)^2\}\\ &= 4x^2-12xy-16xz+9y^2+24yz+16z^2\\ &= 4x^2+9y^2+16z^2-12xy-16xz+24yz.\end{aligned}$$

Example 8. Find the square of $a-b-c+d$.

$$\begin{aligned}(a-b-c+d)^2 &= \{(a-b)-(c-d)\}^2\\ &= (a-b)^2-2(a-b)(c-d)+(c-d)^2\\ &= (a^2-2ab+b^2)-2(ac-ad-bc+bd)+(c^2-2cd+d^2)\\ &= a^2-2ab+b^2-2ac+2ad+2bc-2bd+c^2-2cd+d^2\\ &= a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd.\end{aligned}$$

Example 9. Simplify

$$(ax-by+cz)^2+(ax-by-cz)^2-2(ax-by+cz)(ax-by-cz).$$

Putting m for $(ax-by+cz)$ and n for $(ax-by-cz)$, we have the given expression

$$\begin{aligned}&= m^2+n^2-2mn=(m-n)^2\\ &= \{(ax-by+cz)-(ax-by-cz)\}^2\\ &= (2cz)^2=4c^2z^2.\end{aligned}$$

Example 10. Find the value of $9a^2-48ab+64b^2$, when $a=15$ and $b=6$.

The given expression $=(3a)^2-2(3a)(8b)+(8b)^2=(3a-8b)^2$

$$=(45-48)^2=(-3)^2=9.$$

Example 11. Simplify :

$$48325\times48325-48313\times96650+48313\times48313.$$

Putting x for 48325 and y for 48313, we get the given expression

$$\begin{aligned}&= x\times x-y.(2x)+y\times y \quad [\because 96650=2.48325=2x]\\ &= x^2-2xy+y^2=(x-y)^2.\end{aligned}$$

$\therefore$ the given expression $=(48325-48313)^2=12^2=144.$

Example 12. Find the value of

(i) $\left(x+\dfrac{1}{x}\right)^2$, (ii) $x^2+\dfrac{1}{x^2}$ and (iii) $x^4+\dfrac{1}{x^4}$, when $x-\dfrac{1}{x}=2$.

(i) $\left(x+\dfrac{1}{x}\right)^2=\left(x-\dfrac{1}{x}\right)^2+4.x.\dfrac{1}{x}=2^2+4=8.$

(ii) $x^2+\dfrac{1}{x^2}=\left(x-\dfrac{1}{x}\right)^2+2.x.\dfrac{1}{x}=2^2+2=6.$

(iii) $x^4+\dfrac{1}{x^4}=\left(x^2+\dfrac{1}{x^2}\right)^2-2.x^2.\dfrac{1}{x^2}=6^2-2=34.$

Example 13. Find the value of $a^2+b^2+c^2-ab-ac-bc$, when $a=x+y$, $b=x-y$ and $c=x+2y$. [C. U. 1914 ; D. B. 1931]

$$\begin{aligned} a^2+b^2+c^2-ab-ac-bc &= \tfrac{1}{2}[2a^2+2b^2+2c^2-2ab-2ac-2bc] \\ &= \tfrac{1}{2}[a^2-2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2] \\ &= \tfrac{1}{2}[(a-b)^2+(a-c)^2+(b-c)^2] \\ &= \tfrac{1}{2}[(x+y-x+y)^2+(x+y-x-2y)^2 \\ &\qquad +(x-y-x-2y)^2] \\ &= \tfrac{1}{2}[(2y)^2+(-y)^2+(-3y)^2] \\ &= \tfrac{1}{2}[4y^2+y^2+9y^2]=\tfrac{1}{2}[14y^2]=7y^2. \end{aligned}$$

EXERCISE 21

Find the square of each of the following expressions :

1. $x-3$.
2. $2x-5$.
3. $3x-5y$.
4. $ax-by$.
5. $8m-3n$.
6. $pm-qn$.
7. p^2-mn.
8. x^2y-xy^2.
9. x^2-2xz.
10. $3a^2-5b^2$.
11. $-xyz-abc$.
12. x^2yz-y^2zx.
13. $a^2x^4-b^2y^4$.
14. $\frac{3}{7}x-\frac{7}{12}y$.
15. $\frac{1}{2x}-\frac{1}{y}$.
16. $\frac{2x}{y}-\frac{y}{2x}$.
17. $2x-\frac{1}{4y}$.
18. $a-2b-2c$.
19. $5x-3y-6z$.
20. $3m-4n-5q$.
21. $a^2-3b^2-5c^2$.
22. $x-y-a-b$.
23. $a-2x-3b-4y$.
24. $90-1$.
25. $120-3$.
26. $500-2$.
27. 993.

Express each of the following expressions as a perfect square :

28. $16a^2-8ab+b^2$.
29. $4x^2-20xy+25y^2$.
30. $4x^2-2+\frac{1}{4x^2}$.
31. $\frac{1}{a^2}+\frac{1}{4b^2}-\frac{1}{ab}$.
32. $9x^2+\frac{1}{121x^2}-\frac{6}{11}$
33. $\cdot645\times\cdot645+1\cdot355\times1\cdot355-1\cdot29\times1\cdot355$.

Simplify :

34. $(a+3b)^2-2(a+3b)(a-3b)+(a-3b)^2$.
35. $(2a-4b+5c)^2+(2a+4b+5c)^2-2(2a-4b+5c)(2a+4b+5c)$.
36. $(3a+5b+7c)^2+(7c-4a+5b)^2-2(3a+5b+7c)(7c-4a+5b)$.
37. $(2x^2-y^2-5z^2)^2-2(2x^2-y^2-5z^2)(6z^2+2x^2-y^2)$
 $+(6z^2+2x^2-y^2)^2$.
38. $(ab-bc+ca)^2+(ab+4bc+2ca)^2-2(ab-bc+ca)(ab+4bc+2ca)$.

39. $78946 \times 78946 + 78941 \times 78941 - 2.78946 \times 78941$.

40. $\dfrac{24567 \times 24567 - 2.24567 \times 24560 + 24560 \times 24560}{24567 - 24560}$.

Find the value of :

41. $a^2b^2 - 12abc + 36c^2$, when $a=4$, $b=7$ and $c=5$.

42. $x^2y^2 - 24xyz + 144z^2$, when $x=7$, $y=9$ and $z=6$.

43. $25(x+y)^2 + z^2 - 10z(x+y)$, when $x=47$, $y=-22$ and $z=129$.

44. $9c^2 - 42c(a+b) + 49(a+b)^2$, when $a=-37$, $b=57$ and $c=45$.

45. $64(7p-5q)^2 - 96(7p-5q)r + 36r^2$, when $p=28$, $q=32$ and $r=46$.

46. If $c - \dfrac{1}{c} = 4$, show that $c^2 + \left(\dfrac{1}{c}\right)^2 = 18$.

47. If $x - \dfrac{1}{x} = c$, show that $x^2 + \dfrac{1}{x^2} = c^2 + 2$.

48. If $x - \dfrac{1}{x} = \sqrt{2}$, show that $x^2 + \dfrac{1}{x^2} = 4$.

49. If $x - \dfrac{1}{x} = 3$, show that

(i) $\left(x + \dfrac{1}{x}\right)^2 = 13$; (ii) $x^2 + \dfrac{1}{x^2} = 11$; and (iii) $x^4 + \dfrac{1}{x^4} = 119$.

50. Find the value of

(i) $x - y$, when $x + y = 9$ and $xy = 14$.
(ii) $x + y$, when $x - y = 9$ and $xy = 36$.
(iii) $x^2 + y^2 + z^2 - xy - xz - yz$, when $x = b + c$, $y = c + a$ and $z = a + b$.

51. Show that (i) $(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$.

(ii) $\left(\dfrac{x+y}{2}\right)^2 - \left(\dfrac{x-y}{2}\right)^2 = xy$.

56. Formula $(a+b)(a-b) = a^2 - b^2$.

$$[\ (a+b)(a-b) = a(a-b) + b(a-b)$$
$$= a^2 - b^2.\ ]$$

That is, *the product of the sum and difference of any two quantities is equal to the difference of their squares.*

Note. *Conversely,* $a^2 - b^2 = (a+b)(a-b)$. *Hence, we can always find the factors of an expression which is of the form* $a^2 - b^2$.

[*When one expression is the product of two or more expressions, each of the latter is called a factor of the former.*]

Example 1. Multiply $3x+5y$ by $3x-5y$.

$(3x+5y)(3x-5y)=(3x)^2-(5y)^2=9x^2-25y^2$.

Example 2. Multiply $a+b-c$ by $a-b+c$.

$$(a+b-c)(a-b+c)=\{a+(b-c)\}\{a-(b-c)\}=a^2-(b-c)^2$$
$$=a^2-(b^2-2bc+c^2)=a^2-b^2+2bc-c^2.$$

Example 3. Multiply x^2+xy+y^2 by x^2-xy+y^2.

$$(x^2+xy+y^2)(x^2-xy+y^2)=\{(x^2+y^2)+xy\}\{(x^2+y^2)-xy\}$$
$$=(x^2+y^2)^2-(xy)^2$$
$$=x^4+2x^2y^2+y^4-x^2y^2=x^4+x^2y^2+y^4.$$

Example 4. Find the product of 512 × 488.

$$512\times488=(500+12)(500-12)$$
$$=500^2-12^2=250000-144=249856.$$

Example 5. Simplify $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$.

The given expression $=\{(a^2+ab+b^2)+(a^2-ab+b^2)\}$
$\times\{(a^2+ab+b^2)-(a^2-ab+b^2)\}$
$=(2a^2+2b^2)\times2ab$
$=2(a^2+b^2)\times2ab=4ab(a^2+b^2)$.

Example 6. Find the value of $(9726854)^2-(9726849)^2$.

The given expression $=(9726854+9726849)(9726854-9726849)$
$=19453703\times5=97268515$.

Example 7. Resolve into factors $9a^2-25$.

The given expression $=(3a)^2-(5)^2$
$=(3a+5)(3a-5)$.

Example 8. Resolve into factors $(a+b)^2-(c-d)^2$.

The given expression $=\{(a+b)+(c-d)\}\{(a+b)-(c-d)\}$
$=(a+b+c-d)(a+b-c+d)$.

Example 9. Resolve into factors $16a^4-81x^4$.

The given expression $=(4a^2)^2-(9x^2)^2=(4a^2+9x^2)(4a^2-9x^2)$.

Again, $4a^2-9x^2=(2a)^2-(3x)^2=(2a+3x)(2a-3x)$.

Hence, the given expression $=(4a^2+9x^2)(2a+3x)(2a-3x)$.

EXERCISE 22

Multiply together :

1. $x+3$ and $x-3$.
2. $5x+13$ and $5x-13$.
3. $x+2a$ and $x-2a$.
4. $ax+by$ and $ax-by$.
5. $\left(3a+\frac{1}{3a}\right)\left(3a-\frac{1}{3a}\right)$.
6. $\left(\frac{3a}{4}+\frac{4b}{5}\right)\left(\frac{3a}{4}-\frac{4b}{5}\right)$.
7. $\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{x}{y}-\frac{y}{x}\right)$.
8. 208×192.
9. 1016×984.
10. $am+n^2$ and $am-n^2$.
11. $xy+yz$ and $xy-yz$.
12. x^2-2yz and x^2+2yz.
13. x^2y+xy^2 and xy^2-x^2y.
14. $x+1$, $x-1$ and x^2+1.
15. a^2+b^2, a^2-b^2 and a^4+b^4.
16. $a+b+c$ and $a+b-c$.
17. $a+b+c$ and $a-b-c$.
18. m^2+mn+n^2 and m^2-mn+n^2.
19. $x^2+2xy+2y^2$ and $x^2-2xy+2y^2$.
20. $ax-by+cz$ and $ax+by-cz$.
21. $-ax+by+cz$ and $ax+by+cz$.
22. $b^2m-c^2n+a^2p$ and $b^2m+c^2n-a^2p$.
23. $a^3-8b^3+27c^3$ and $a^3+8b^3-27c^3$.
24. $a^2x^2-2ax+2$ and $a^2x^2+2ax+2$.
25. $a^4x^4-a^2x^2+1$ and $a^4x^4+a^2x^2+1$.
26. $m^2+\sqrt{2}.mn+n^2$ and $m^2-\sqrt{2}.mn+n^2$.
27. $x^2-\sqrt{2}x+1$, $x^2+\sqrt{2}x+1$ and x^4-1.

Simplify :

28. $(a+b-c)^2-(a-b+c)^2$.
29. $(a-2b+3c)^2-(a+2b-3c)^2$.
30. $(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$.
31. $(x+y-a+b)^2-(x-y+a-b)^2$.
32. $(2a+3b-5c+7d)^2-(2a-3b+5c-7d)^2$.

Find the value of :

33. $2345 \times 2345 - 2343 \times 2343$.
34. $(53497)^2-(53487)^2$.
35. $498567 \times 498567 - 498562 \times 498562$.

Resolve into factors :

36. $25x^2-36$.
37. $9a^2-16c^2$.
38. $16m^2-49n^2$.
39. $4p^2-81q^2$.
40. $a^2x^2-64b^2$.
41. $36x^4-121y^4$.

42. $49-64d^2$. **43.** $144c^2-25d^2$. **44.** $\frac{9x^2}{16y^2}-\frac{25y^2}{36x^2}$.

45. $\frac{x^2}{9}-\frac{49}{y^2}$. **46.** $\frac{a^2}{121}-\frac{144}{25b^2}$. **47.** $(a+b)^2-c^2$.

48. $(a+2b)^2-25c^2$. **49.** $4x^2-(3a-4b)^2$.

50. $4(a^2-b^2)^2-9(b^2-c^2)^2$. **51.** $9\left(x-\frac{1}{3x}\right)^2-16\left(x-\frac{1}{4x}\right)^2$.

52. $\frac{16}{25}\left(5x-\frac{3}{4y}\right)^2-\frac{36}{49}\left(\frac{7}{12}x+\frac{14y}{27}\right)^2$. **53.** $a^2-(2b-3c)^2$.

54. a^4-81b^4. **55.** $(x-y)^2-(a-b)^2$. **56.** $81x^4-625y^4$.

57. $(4a+7b)^2-(3a-8b)^2$. **58.** $(3x+5y)^2-(2x-7y)^2$.

59. $(a+2b-3c)^2-(a+b-c)^2$. **60.** $(2m+3n-5p)^2-(2n+3p)^2$.

61. $(3x-4y+7z)^2-(2x-3y+5z)^2$.

57. Formula $\boldsymbol{(a+b)^3=a^3+3a^2b+3ab^2+b^3}$,

or, $\boldsymbol{=a^3+b^3+3ab(a+b)}$.

$$[\ (a+b)^3=(a+b)(a+b)^2=(a+b)(a^2+2ab+b^2)$$
$$=a(a^2+2ab+b^2)+b(a^2+2ab+b^2)$$
$$=a^3+3a^2b+3ab^2+b^3\ ;$$

and this latter $=a^3+3ab(a+b)+b^3=a^3+b^3+3ab(a+b).\ ]$

Cor. $a^3+b^3=\{a^3+b^3+3ab(a+b)\}-3ab(a+b)$
$$=(a+b)^3-3ab(a+b).$$

Example 1. Find the cube of $3a+5b$.

$$(3a+5b)^3=(3a)^3+3(3a)^2(5b)+3(3a)(5b)^2+(5b)^3$$
$$=27a^3+3(9a^2)(5b)+3(3a)(25b^2)+125b^3$$
$$=27a^3+135a^2b+225ab^2+125b^3.$$

Example 2. Find the cube of 54.

$$54^3=(50+4)^3=50^3+4^3+3.50.4(50+4)$$
$$=125000+64+600\times 54$$
$$=125000+64+32400=157464.$$

Example 3. Simplify

$$(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y).$$

[C. U. Entr. Paper, 1876]

Putting a for $x-y$ and b for $x+y$, we have

the given expression $=a^3+b^3+3a^2b+3b^2a$
$$=a^3+3a^2b+3ab^2+b^3,$$
and $\therefore\ =(a+b)^3=\{(x-y)+(x+y)\}^3=(2x)^3=8x^3.$

Example 4. Simplify

$$64\times 64\times 64+36\times 36\times 36+192\times 64\times 36+108\times 64\times 36.$$

Suppose $x=64$ and $y=36$.

$\therefore\ 3x=192,\ 3y=108.$

$$\begin{aligned}\therefore\ \text{the given expression} &= x^3+y^3+3x.x.y+3y.x.y\\ &= x^3+y^3+3x^2y+3xy^2=(x+y)^3\\ &= (64+36)^3=100^3=1000000.\end{aligned}$$

[Substituting the values of x and y]

Example 5. If $a+b=5$ and $ab=6$, find the value of a^3+b^3.

We have $a^3+b^3=(a+b)^3-3ab(a+b)$,

and $\therefore$ by the given condition

$$=5^3-3\times 6\times 5=125-90=35.$$

Example 6. If $x+\frac{1}{x}=p$, show that $x^3+\left(\frac{1}{x}\right)^3=p^3-3p$.

Since, $a^3+b^3=(a+b)^3-3ab(a+b)$,

$$\begin{aligned}\therefore\quad x^3+\frac{1}{x^3} &= \left(x+\frac{1}{x}\right)^3-3x.\frac{1}{x}\left(x+\frac{1}{x}\right)\\ &= \left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right).\end{aligned}$$

Hence, the reqd. value $=p^3-3p$.

Example 7. Find the cube of $p+q+r$.

$$\begin{aligned}(p+q+r)^3 &= \{(p+q)+r\}^3=(p+q)^3+3(p+q)^2r+3(p+q)r^2+r^3\\ &= (p^3+3p^2q+3pq^2+q^3)+3(p^2+2pq+q^2)r+3(p+q)r^2+r^3\\ &= p^3+q^3+r^3+3p^2q+3pq^2+3p^2r+3pr^2+3q^2r+3qr^2+6pqr.\end{aligned}$$

Example 8. Find the value of $x^3+9x^2y+27xy^2+27y^3$, when $x=5$ and $y=-2$.

$$\begin{aligned}\text{The given expression} &= x^3+3x^2(3y)+3x(3y)^2+(3y)^3=(x+3y)^3\\ &= (5-6)^3=(-1)^3=-1.\end{aligned}$$

Example 9. Show that $x^3+\frac{1}{x^3}=-2$, when $\frac{x^2+1}{x}=1$.

$$\frac{x^2+1}{x}=\frac{x^2}{x}+\frac{1}{x}=x+\frac{1}{x}.$$

Therefore, from the given condition,

$$x+\frac{1}{x}=1.$$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3.x.\frac{1}{x}\left(x+\frac{1}{x}\right)$$

$$=1^3-3.1=1-3=-2.$$

EXERCISE 23

Find the cube of :

1. $x+3$. 2. $2x+1$. 3. $3a+b$. 4. $4x+3y$.
5. x^2+2y. 6. $xy+yz$. 7. a^2b+c^2d. 8. $a+b+2c$.
9. $2x+3y+z$. 10. x^3+y^3. 11. $\frac{1}{2}a+\frac{2}{3}b$. 12. $\frac{x}{y}+\frac{y}{x}$.
13. $\frac{1}{x}+\frac{1}{y}$. 14. $\frac{2}{3a}+\frac{3}{5b}$. 15. 105. 16. 85.

Simplify :

17. $(3m+5n)^3+3(3m+5n)^2(2m-5n)+3(3m+5n)(2m-5n)^2+(2m-5n)^3$.
18. $(3x-8y)^3+(9y-2x)^3+3(x+y)(3x-8y)(9y-2x)$.
19. $(3a-7b)^3+(10b-3a)^3+9b(3a-7b)(10b-3a)$.
20. $(5x-2)^3+(3-4x)^3+3(x+1)(5x-2)(3-4x)$.
21. $(3-7x)^3+(8x-1)^3+3(8x-1)(3-7x)(x+2)$.
22. $(a-b+c)^3+(a+b-c)^3+6a\{a^2-(b-c)^2\}$.
23. $1·75\times1·75\times1·75+·25\times·25\times·25+·75\times1·75\times1·75+·75\times·25\times1·75$.
24. $3·89\times3·89\times3·89+1·11\times1·11\times1·11+15\times3·89\times1·11$.

Find the value of a^3+b^3 :

25. When $a+b=6$ and $ab=7$. 26. When $a+b=7$ and $ab=8$.

27. If $a+\frac{1}{a}=3$, show that $a^3+\left(\frac{1}{a}\right)^3=18$.

28. If $z+\frac{1}{z}=4$, find the value of $z^3+\left(\frac{1}{z}\right)^3$.

29. If $(x+2)+\frac{1}{(x+2)}=5$, find the value of $(x+2)^3+\frac{1}{(x+2)^3}$.

30. If $a+\frac{1}{a}=\sqrt{3}$, prove that $a^3+\frac{1}{a^3}=0$.

31. If $\frac{a^2+1}{a}=4$, show that $\frac{a^6+1}{a^3}=52$.

32. If $\frac{(a+2)^2+1}{a+2}=2$, show that $\frac{(a+2)^6+1}{(a+2)^3}=2$.

Find the value of :

33. $x^3+6x^2+12x+8$, when $x=-2$.

34. $x^3+12x^2+48x+64$, when $x=-5$.

35. $8a^3+36a^2b+54ab^2+27b^3$, when $a=-3$ and $b=2$.

36. $x^3+18x^2+108x+351$, when $x=-11$.

37. If $x+y=5$, show that $x^3+y^3+15xy=125$.

38. If $a^2+b^2=c^2$, show that $a^6+b^6+3a^2b^2c^2=c^6$.

39. If $p+q=2$, show that $p^3+q^3+6pq=8$.

58. Formula $(a-b)^3=a^3-3a^2b+3ab^2-b^3$,

or, $\qquad =a^3-b^3-3ab(a-b)$.

$$\begin{aligned}[\ (a-b)^3&=(a-b)(a-b)^2=(a-b)(a^2-2ab+b^2)\\&=a(a^2-2ab+b^2)-b(a^2-2ab+b^2)\\&=a^3-3a^2b+3ab^2-b^3\ ;\end{aligned}$$

and this latter $=a^3-3ab(a-b)-b^3=a^3-b^3-3ab(a-b)$.]

Cor. $a^3-b^3=\{a^3-b^3-3ab(a-b)\}+3ab(a-b)$

$$=(a-b)^3+3ab(a-b).$$

Example 1. Find the cube of $3x-4y$.

$$\begin{aligned}(3x-4y)^3&=(3x)^3-3(3x)^2(4y)+3(3x)(4y)^2-(4y)^3\\&=27x^3-3(9x^2)(4y)+3(3x)(16y^2)-64y^3\\&=27x^3-108x^2y+144xy^2-64y^3.\end{aligned}$$

Example 2. Find the cube of 297.

$$\begin{aligned}297^3&=(300-3)^3=300^3-3^3-3.300.3(300-3)\\&=27000000-27-810000+8100\\&=(27000000+8100)-(27+810000)\\&=27008100-810027=26198073.\end{aligned}$$

Example 3. If $a-b=5$ and $ab=6$, find the value of a^3-b^3.

$$\begin{aligned}a^3-b^3&=(a-b)^3+3ab(a-b)\\&=5^3+3.6.5 \qquad \text{[Substituting the values]}\\&=125+90=215.\end{aligned}$$

Example 4. Find the cube of $a-b-c$.

$$\begin{aligned}(a-b-c)^3&=\{(a-b)-c\}^3=(a-b)^3-3(a-b)^2c+3(a-b)c^2-c^3\\&=(a^3-3a^2b+3ab^2-b^3)-3(a^2-2ab+b^2)c+3(a-b)c^2-c^3\\&=a^3-b^3-c^3-3a^2b+3ab^2-3a^2c+3ac^2-3b^2c-3bc^2+6abc.\end{aligned}$$

Example 5. Find the value of $27x^3-54x^2+36x-64$, when $x=2\frac{1}{3}$.

The given expression $=(3x)^3-3(9x^2).2+3(3^x).4-8-56$

$=(3x-2)^3-56.$

Hence, the reqd. value $=(7-2)^3-56=125-56=69.$

Example 6. If $\dfrac{x^2-1}{x}=4$, find the value of $\dfrac{x^6-1}{x^3}$.

$$\frac{x^2-1}{x}=\frac{x^2}{x}-\frac{1}{x}=x-\frac{1}{x};$$

$$\therefore\quad x-\frac{1}{x}=4.$$

$$\frac{x^6-1}{x^3}=\frac{x^6}{x^3}-\frac{1}{x^3}=x^3-\frac{1}{x^3}$$

$$=\left(x-\frac{1}{x}\right)^3+3x.\frac{1}{x}\left(x-\frac{1}{x}\right)$$

$$=4^3+3.4=64+12=76.$$

EXERCISE 24

Find the cube of :

1. $x-2.$ 2. $2x-1.$ 3. $2-3a.$ 4. $3-4a.$

5. $2a-3b.$ 6. $5m-4n.$ 7. $2x-5y.$ 8. $\frac{1}{3}a-\frac{2}{3}b.$

9. $3x-\dfrac{1}{3x}.$ 10. $\dfrac{1}{x}-\dfrac{1}{y}.$ 11. 198. 12. 494.

13. $2a-b-c.$ 14. $2x-3y-z.$ 15. $p^2-q^2-r^2.$

Simplify :

16. $(a+2b)^3-3(a+2b)^2(a-2b)+3(a+2b)(a-2b)^2-(a-2b)^3.$

17. $(3x-8y)^3-(2x-7y)^3-3(3x-8y)(2x-7y)(x-y).$

18. $(5x-8)^3-(3x-8)^3-6x(5x-8)(3x-8).$

Find the value of :

19. $m^3-12m^2n+48mn^2-64n^3$, when $m=12$ and $n=3.$

20. $27a^3-135a^2+225a-125$, when $a=4.$

21. $8-9a+27a^2-27a^3$, when $a=3.$

22. $216-144x+108x^2-27x^3$, when $x=3.$

23. $(6a-5b)^3-(3a-4b)^3-3(3a-b)(6a-5b)(3a-4b)$, when $3a-b=0.$

24. If $a-\dfrac{1}{a}=3$, find the value of $a^3-\left(\dfrac{1}{a}\right)^3.$

25. If $c-\dfrac{1}{c}=5$, find the value of $c^3-\left(\dfrac{1}{c}\right)^3$.

26. If $x-\dfrac{1}{x}=p$, show that $x^3-\dfrac{1}{x^3}=p^3+3p$.

27. If $\dfrac{a^2-1}{a}=1$, find the value of $\dfrac{a^6-1}{a^3}$.

28. If $x-y=3$, show that $x^3-y^3-9xy=27$.

29. If $p-2q=4$, show that $p^3-8q^3-24pq=64$.

30. If $2a-3b=5$, show that $8a^3-27b^3-90ab=125$.

59. Formula $(a+b)(a^2-ab+b^2)=a^3+b^3$.

$$\begin{aligned}[\,(a+b)(a^2-ab+b^2)&=a(a^2-ab+b^2)+b(a^2-ab+b^2)\\&=(a^3-a^2b+ab^2)+(a^2b-ab^2+b^3)\\&=a^3+b^3.\,]\end{aligned}$$

Note. *Conversely,* $a^3+b^3=(a+b)(a^2-ab+b^2)$. *Hence, we can always resolve an expression into factors when it is of the form* a^3+b^3.

Example 1. Multiply x^4-x^2+1 by x^2+1.

Putting a for x^2 and b for 1, we have

$$x^4-x^2+1=(x^2)^2-x^2.1+1^2=a^2-ab+b^2.$$

Hence,
$$\begin{aligned}(x^2+1)(x^4-x^2+1)&=(a+b)(a^2-ab+b^2)\\&=a^3+b^3\\&=(x^2)^3+1^3=x^6+1.\end{aligned}$$

Example 2. Multiply $9x^2-12x+16$ by $3x+4$.

Putting a for $3x$ and b for 4, we have

$$\begin{aligned}9x^2-12x+16&=(3x)^2-(3x).4+4^2\\&=a^2-ab+b^2.\end{aligned}$$

Hence,
$$\begin{aligned}(3x+4)(9x^2-12x+16)&=(a+b)(a^2-ab+b^2)\\&=a^3+b^3=(3x)^3+4^3\\&=27x^3+64.\end{aligned}$$

Example 3. Multiply $16a^2-20ab+25b^2$ by $4a+5b$.

Putting x for $4a$ and y for $5b$, we have

$$\begin{aligned}16a^2-20ab+25b^2&=(4a)^2-(4a)(5b)+(5b)^2\\&=x^2-xy+y^2.\end{aligned}$$

Hence,
$$\begin{aligned}(4a+5b)(16a^2-20ab+25b^2)&=(x+y)(x^2-xy+y^2)\\&=x^3+y^3=(4a)^3+(5b)^3\\&=64a^3+125b^3.\end{aligned}$$

Example 4. Resolve $a^3b^3+8c^3$ into factors.

$$\begin{aligned} a^3b^3+8c^3 &= (ab)^3+(2c)^3 \\ &= (ab+2c)\{(ab)^2-(ab)(2c)+(2c)^2\} \\ &= (ab+2c)(a^2b^2-2abc+4c^2). \end{aligned}$$

EXERCISE 25

Multiply :

1. x^2-x+1 by $x+1$. 2. $1-2x+4x^2$ by $1+2x$.

3. $25p^2-5p+1$ by $5p+1$. 4. $49a^2-28ab+16b^2$ by $7a+4b$.

5. $64x^2-24xy+9y^2$ by $8x+3y$. 6. $a^2b^2-4abc+16c^2$ by $ab+4c$.

7. $a^2x^2-5abx+25b^2$ by $ax+5b$. 8. $25a^2-45ab+81b^2$ by $5a+9b$.

Resolve into factors :

9. a^3+1. 10. a^3+8. 11. $8x^3+1$. 12. $27a^3+8$.

13. $8m^3+64$. 14. $64p^3+125$. 15. $8x^3+125y^3$.

16. $8x^3+216y^3$. 17. $27a^3+343y^3$. 18. $27x^6+512y^6$.

19. $216a^3x^3+y^3$. 20. $27a^3b^3+64x^3y^3$.

21. $729a^3b^3c^3+1000x^3y^3z^3$. 22. $1331a^3b^6x^9+729c^3y^6z^9$.

60. Formula $(a-b)(a^2+ab+b^2)=a^3-b^3$.

$$\begin{aligned} [\ (a-b)(a^2+ab+b^2) &= a(a^2+ab+b^2)-b(a^2+ab+b^2) \\ &= (a^3+a^2b+ab^2)-(a^2b+ab^2+b^3) \\ &= a^3-b^3.\ ] \end{aligned}$$

Note. *Conversely,* $a^3-b^3=(a-b)(a^2+ab+b^2)$. *Hence, we can always resolve into factors an expression which is of the form* a^3-b^3.

Example 1. Multiply $4a^2b^4+2ab^2+1$ by $2ab^2-1$.

$$\begin{aligned} (2ab^2-1)(4a^2b^4+2ab^2+1) &= (2ab^2-1)\{(2ab^2)^2+(2ab^2).1+1^2\} \\ &= (2ab^2)^3-1^3=8a^3b^6-1. \end{aligned}$$

Example 2. Resolve $64x^6-a^3y^6$ into factors.

$$\begin{aligned} 64x^6-a^3y^6 &= (4x^2)^3-(ay^2)^3 \\ &= (4x^2-ay^2)\{(4x^2)^2+(4x^2)(ay^2)+(ay^2)^2\} \\ &= (4x^2-ay^2)(16x^4+4ax^2y^2+a^2y^4). \end{aligned}$$

EXERCISE 26

Multiply :

1. $1+2x+4x^2$ by $1-2x$.
2. x^2+3x+9 by $x-3$.
3. $16a^2+4a+1$ by $4a-1$.
4. $x^4+2x^2yz+4y^2z^2$ by x^2-2yz.
5. $9m^2+6mnq+4n^2q^2$ by $3m-2nq$.
6. $16a^2b^2+4ab^2c+b^2c^2$ by $4ab-bc$.

Resolve into factors :

7. $125a^3-1$.
8. $343x^3-8y^6$.
9. $216k^3-125l^3$.
10. $1-512k^3$.
11. $729m^3-64a^3n^6$.
12. $27x^3y^3-1331y^6b^6$.

61. Formula $(x+a)(x+b)=x^2+(a+b)x+ab$.

$$\begin{aligned}[\,(x+a)(x+b)&=x(x+b)+a(x+b)\\ &=x^2+(a+b)x+ab.\,]\end{aligned}$$

Note. *It is easy to see that the above formula includes the following results :*

$$\left.\begin{aligned}&(1)\ (x-a)(x-b)=x^2-(a+b)x+ab\\ &(2)\ (x-a)(x+b)=x^2+(b-a)x-ab\\ &(3)\ (x+a)(x-b)=x^2+(a-b)x-ab\end{aligned}\right\}$$

For instance,
$$\begin{aligned}(x-a)(x-b)&=\{x+(-a)\}\{x+(-b)\}\\ &=x^2+\{(-a)+(-b)\}x+\{(-a)\times(-b)\}\\ &=x^2-(a+b)x+ab.\end{aligned}$$

Similarly, the truth of the other results can be proved, which is left as an exercise for the student.

Hence, we can express the formula more clearly as follows :

$(x+a)(x+b)=x^2+($*algebraic sum of* a *and* $b)x+($*product of* a *and* $b)$.

Example 1. Write down the product of $x+3$ and $x+4$.

Since $3+4=7$ and $3\times4=12$, $\therefore$ the required product $=x^2+7x+12$.

Example 2. Write down the product of $x-7$ and $x+4$.

Since $-7+4=-3$ and $(-7)\times4=-28$, $\therefore$ the required product $=x^2-3x-28$.

Example 3. Write down the product of $x+5$ and $x-9$.

Since $5-9=-4$ and $5\times(-9)=-45$, $\therefore$ the required product $=x^2-4x-45$.

Example 4. Write down the product of $x-2$ and $x+7$.

$$\left.\begin{array}{ll}\text{Since} & -2+7=5 \\ \text{and} & (-2)\times 7=-14\end{array}\right\}, \quad \therefore \text{ the required product } =x^2+5x-14.$$

Example 5. Write down the product of $x-5$ and $x-8$.

$$\left.\begin{array}{ll}\text{Since} & -5-8=-13 \\ \text{and} & (-5)\times(-8)=40\end{array}\right\}, \quad \therefore \text{ the required product } =x^2-13x+40.$$

EXERCISE 27

Write down the product of :

1. $x+1$ and $x+2$.
2. $x+2$ and $x+9$.
3. $x-5$ and $x+6$.
4. $x-3$ and $x-11$.
5. $a-11$ and $a+16$.
6. $m-7$ and $m+19$.
7. $p+13$ and $p-11$.
8. $p+12$ and $p-17$.
9. $x-4$ and $x+9$.
10. $x-5$ and $x-10$.
11. $x-12$ and $x+5$.
12. $k-13$ and $k+2$.
13. $a+5$ and $a+14$.
14. $m-14$ and $m+6$.
15. $x-5$ and $x-13$.
16. $x+7$ and $x+12$.
17. $a-3$ and $a-11$.
18. $x+4$ and $x-13$.
19. $m+5$ and $m-16$.
20. $x-8$ and $x-10$.
21. $a+6$ and $a-12$.
22. $m-7$ and $m+13$.
23. $x-10$ and $x-16$.
24. $x+5$ and $x-18$.
25. $x-16$ and $x+10$.

CHAPTER V

SIMPLE EQUATIONS

62. Definitions. Any two expressions connected by the sign of equality ($=$) constitute an **equation,** and each of the expressions thus connected is called a **side** or **member** of the equation.

The term equation, however, is hardly used in this extended sense. When one expression is put equal to another, the equality may hold *either* for all values of the letters involved, as in $(a+b)(a-b)=a^2-b^2$, *or* for some particular values of the letters only, as in $4x=8$ (which is true only when $x=2$). The latter class of equations alone are called *equations* (more correctly, *Equations of Condition*), whilst any equation of the former class is called an **Identity** (or an Identical Equation).

Thus, $(x+1)+(2x+3)=3x+4$ is an *Identity*,

whereas $(x+1)+(x+3)=3x+2$ is an *Equation* ;

the former being true for *all values* of x, and the latter, *only when* $x=2$.

The letter, to which a particular value or values must be given in order that an equation may be true, is called the ***unknown quantity***. It is usually represented by one of the last letters of the alphabet x, y, z, etc.

Any particular value of the unknown quantity, for which an equation is true, is said to **satisfy** the equation, and is called a **root** or **a solution** of the equation.

To **solve** an equation is to find its root or roots.

An equation containing only one unknown quantity, is said to be an equation of the first degree or a **simple equation**, when the unknown quantity occurs only in the *first* power.

63. Axioms. The process of solving an equation is primarily based upon the following axioms :

(1) If to equals the same quantity, or equal quantities, be added, the sums are equal.

(2) If from equals the same quantity, or equal quantities, be taken, the remainders are equal.

(3) If equals be multiplied by the same quantity, or by equal quantities, the products are equal.

(4) If equals be divided by the same quantity, or by equal quantities, the quotients are equal.

Cor. 1. From axioms (1) and (2), we deduce an important principle which is of great use in solving equations, and which may be enunciated as follows :

Any term may be transposed from one side of an equation to the other by simply changing its sign.

For, let $x-a=b+c$,

then adding a to both sides, we must have

$$x-a+a=b+c+a, \qquad [\text{ Axiom (1) }]$$

or, $$x=b+c+a\,;$$

again, subtracting c from both sides, we have

$$x-a-c=b+c-c \qquad [\text{ Axiom (2) }]$$
$$=b.$$

Thus, $-a$, removed from the left side, appears as $+a$ on the right and $+c$, removed from the right side, appears as $-c$ on the left.

Similarly, if $x-a=b-c+d$, we have $x-a-b+c-d=0$.

Such removal of terms is called **Transposition**.

Cor. 2. *The sign of every term of an equation may be changed without destroying the equality.*

For, let $x-a=b+c$;

then $(x-a)\times(-1)=(b+c)\times(-1)$, [Axiom (3)]

or, $-x+a=-b-c$.

64. Different forms of Simple Equations :

Simple equations are generally of the following types :

I. The unknown quantity with any coefficient is equal to a known quantity (*i.e.*, constant) ; as $3x=12$.

The general form of this type of equation is $ax=b$.

The root of this type of equation is found out by dividing the known quantity by the coefficient of the unknown quantity and is $\frac{b}{a}$.

II. The sum of the unknown quantity with any coefficient, and a known quantity is equal to a known quantity ; as $5x+6=26$.

The general form of this type of equation is $ax+b=c$.

In solving the equation b is to be transposed to the right-hand side and the equation stands as $ax=c-b$; and the root is found out by dividing the algebraical sum of the known quantities by the coefficient of the unknown quantity and is $\frac{c-b}{a}$.

III. In this form of equation there are unknown and known quantities on both sides ; as, $3x+2=5x-4$.

The general form of this type of equation is $ax+b=cx+d$.

To solve a simple equation of this type, unknown quantities are to be grouped in one side and the known in the other ; as $ax-cx=d-b$, or $(a-c)x=d-b$; and the root is found out by dividing the algebraical sum of the known quantities by the algebraical sum of the coefficients of the unknown quantity.

N. B. Thus it is evident that every simple equation is always reducible to type I above.

Simple Examples. We shall now work out some examples illustrating the general method of solving a simple equation by the application of the foregoing principles. The unknown quantity will always be denoted by x.

Example 1. Solve $18x=54$.

N. B. The question may be otherwise put as follows : '*If* $18x=54$, *what is the value of* x ?'

Since, $18x=54$,

dividing both sides by 18, we get

$$\frac{18x}{18}=\frac{54}{18}, \quad \text{or,} \quad x=3.$$

Thus, the required value of x is 3.

Example 2. Solve $12x+9=45$.

$$12x+9=45,$$

or, $12x=45-9$, [by transposition]

or, $12x=36$;

$\therefore$ $x=\frac{36}{12}=3$.

Thus, the required value of $x=3$.

Example 3. Solve $3x+5=x+19$.

N. B. The question may be otherwise put as follows : '*If* $3x+5=x+19$, *what is the value of* x ?'

Since, $3x+5=x+19$,

by transposition we must have

$$3x-x=19-5, \quad \text{or,} \quad 2x=14,$$

and therefore (dividing both sides by 2),

$x=7$. [Axiom (4)]

Thus, the required value of x is 7.

Example 4. Solve the equation $-11x+2(3-x)=32$.

Removing the brackets, we get

$-11x+6-2x=32$,

or, $-13x+6=32$,

or, $-13x=32-6$, [by transposition]

or, $-13x=26$.

Multiplying both sides by -1,

$(-1)\times(-13x)=(-1)\times 26$,

or, $13x=-26$.

$\therefore$ dividing both sides by 13,

$x=-\frac{26}{13}$, *i.e.*, -2.

Thus, the required value of x is -2.

Example 5. Solve $(x+2)(3x+4)-6x=10+(3x+2)(x+1)$.

The left side $=3x^2+10x+8-6x$

$=3x^2+4x+8$;

and the right side $=10+3x^2+5x+2$

$=3x^2+5x+12$.

Hence, $3x^2+4x+8=3x^2+5x+12$.

Removing $3x^2$ from both sides, we have

$4x+8=5x+12$. [Axiom (2)]

Hence, by transposition,

$4x-5x=12-8$,

or, $-x=4$,

and therefore, $x=-4$. [Cor. 2, Art. 63]

Thus, the required value of x is -4.

Note. *The student can easily see for himself that when x has this value, each side of the given equation becomes equal to* 40.

Example 6. Given $\frac{x}{6}+5=\frac{x}{3}+\frac{x}{4}$; find x.

Since, $\frac{x}{6}+5=\frac{x}{3}+\frac{x}{4}$,

multiplying both sides by 12 (which is the L.C.M. of the denominators), we have

$$12\left(\frac{x}{6}+5\right)=12\left(\frac{x}{3}+\frac{x}{4}\right), \quad \text{[Axiom (3)]}$$

or, $2x+60=4x+3x=7x$.

Hence, by transposition,

$2x-7x=-60$, or, $-5x=-60$,

and therefore (dividing both sides by -5), $x=12$.

Thus, the required root is 12.

Note. *When the root is found out, the root may be verified by putting the value of the unknown quantity in the equation. If it is found that equality of both sides is maintained, when the root is substituted in place of the unknown quantity, the root is correct.*

EXERCISE 28

Solve the following equations :

1. $4x=16$. 2. $3x=-15$. 3. $7x=-28$.

4. $-5x=25$. 5. $\frac{x}{5}=-1$. 6. $\frac{-x}{3}=20$.

7. $6x+4=22$. 8. $9x-4=32$.

9. $-8x-4=20$. 10. $-5x+3=-12$.

11. $3x+5(2-x)=-16$. 12. $5(1-x)+3(2-x)=-29$.

13. $4(2-x)+2(3-2x)=30$. 14. $7(3-2x)+5(x-1)=34$.

15. $4x+3=2x+5$. 16. $3x+2=x+6$.

17. $5x-6=2x+3$. 18. $15x-9=11x-25$.

19. $4(x-3)=2(x-6)$. 20. $2(x-15)=5(x-11)+4$.

21. $19-3x=5x+35$. 22. $3(x-2)+7(2x-3)=5(1-2x)-59$.

23. $13x-4(5x-8)+17=0$. 24. $14(x-4)+3(x+5)=6(7-2x)+4$.

25. $8(2x-7)-9(3x-14)=15$. 26. $3x-13(2x-13)=4x-20$.

27. $49+13(5x+27)=8(5+x)-3x$.

28. $16-5(7x-2)=13(x-2)+4(13-x)$.

29. $8x+5(x+7)+9(2x+23)-3(x+6)=0$.

30. $(x-7)(4x-29)=(2x-5)(2x-17)+1$.

31. $(3x+2)(2x-6)=(4-3x)(1-2x)-10$.

32. $(3x+5)(6x-7)=(3x+2)(9x-13)-(3x+1)(3x-1)$.

33. $(x+2)(2x+5)=2(x+1)^2+13$.

34. $(x+1)(4x-7)-(x-1)(x+5)=3(x+2)^2+5$.

35. $\frac{x}{2}+5=\frac{x}{3}+7$. 36. $\frac{x}{6}-\frac{x}{5}=\frac{x}{15}-\frac{x}{3}+7$.

37. $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=2-\frac{x}{6}+\frac{5x}{12}$.

CHAPTER VI

PROBLEMS LEADING TO SIMPLE EQUATIONS

65. Symbolical Expression. The chief difficulty in solving an algebraical problem lies in expressing correctly the condition of the problem by means of symbols. The student should, therefore, be first of all introduced to this art before the solution of any problem is presented to him. The following examples will serve as illustrations.

Example 1. If a man earns x rupees per month, how many twenty-five paise pieces will he earn in half a month ?

Since, 1 rupee $=4$ twenty-five paise pieces,

$\therefore$ x rupees $=4x$ twenty-five paise pieces.

Clearly therefore the man earns $4x$ twenty-five paise pieces per month.

Hence, the number of twenty-five paise pieces earned in half a month $=\frac{1}{2}$ of $4x=2x$.

Example 2. If an insect creeps up a pole x centimetres per minute, how many metres will it rise in y hours ?

Since, 1 centimetre $=\frac{1}{100}$th of a metre,

$\therefore$ x centimetres $=\frac{x}{100}$th of a metre.

Hence, in 1 minute the insect creeps up $\frac{x}{100}$th metres ;

$\therefore$ in 60 minutes " " " " $\frac{x\times 60}{100}$ metres.

Thus, in 1 hour the insect creeps up $\frac{3x}{5}$ metres

Therefore, in y hours it rises $\left(\frac{3x}{5}\times y\right)$ metres

Thus, the required number of metres $=\frac{3xy}{5}$.

Example 3. If a man travels at the rate of x kilometres per hour, in what time will he finish a journey of 10 kilometres ?

Since, x kilometres is travelled in 1 hour,

$\therefore$ 1 kilometre " " " $\frac{1}{x}$ th of an hour ;

$\therefore$ 10 kilometres are " " $\frac{10}{x}$ hours.

Example 4. The digits of a number beginning from the left are x and y. How would you represent the number ?

If the digits be 4 and 5, the number $= 10 \times 4 + 5$;

if the digits be 5 and 7, the number $= 10 \times 5 + 7$;

if the digits be 8 and 4, the number $= 10 \times 8 + 4$;

and so on.

Hence, it is quite clear that when x and y stand for the digits from the left, the number is to be represented by $10x + y$.

EXERCISE 29

1. The sum of two numbers is 15 ; if one of the numbers be x, what is the other ?

2. The difference of two numbers is 20 ; if x be the greater, what is the other ?

3. The difference of two numbers is 25 ; if x be the smaller, what is the greater ?

4. What is the excess of 25 over y ?

5. What is the defect of $2x$ from y ?

6. If x be one factor of 21, what is the other factor ?

7. What number is less than 100 by $3x$?

8. What number taken from $4x$ gives $3y$ as a remainder ?

9. If a man travels x hours at the rate of y kilometres an hour, how many kilometres does he travel ?

10. If a man travels at the rate of y kilometres per hour, in what time will he finish a journey of x kilometres ?

11. A man is x years of age, how old will he be 20 years hence ? How old was he 3 years ago ?

12. In x days a man travels 60 kilometres ; what is his rate per day ?

13. If a train travels 48 kilometres in x hours, how many metres does it travel in one second ?

14. If I spend five paise a week, how many rupees do I save out of a yearly income of $5x$ rupees ?

15. Write down 5 consecutive numbers of which x is the middle one.

16. Write down the sum of 3 consecutive numbers of which the middle one is x.

17. What is the odd number next after $2m+1$?

18. What is the even number next before $2x$?

19. If x men take 10 days to do a work, in what time will y men do it?

20. A room is a metres long and b decimetres wide; what is the measure of the area of the floor in square decimetres?

21. In the last question find the number of square units in the area when the unit of length is 4 decimetres?

22. How many kilometres can a person walk in 20 minutes, if he walks x kilometres in y hours?

23. In what time will a person walk 16 kilometres, if he walks x kilometres in a hours?

24. What is the present age of a man who was $(x-5)$ years old 20 years ago? What will be his age 30 years hence?

25. If the digits of a number beginning from the right are x and y, what is the number?

26. If x, y, z be the digits of a number beginning from the left, what is the number?

27. In the preceding question, if the digits be inverted, how would you represent the new number?

66. **Easy Problems.** We shall now work out some problems which will fairly introduce the beginner to the subject of the present chapter. The unknown quantity will invariably be represented by x.

Example 1. 7 times a number is 28. What is the number?

Suppose, the number is x.

$\therefore \quad 7x=28;$

$\therefore \quad x=\frac{28}{7}=4.$

4 is the required number.

Example 2. 5 times a number increased by 4 is 29. Find the number.

Suppose, x is the number.

$\therefore \quad 5x+4=29,$

or, $\quad 5x=29-4=25;$

$\therefore \quad x=\frac{25}{5}=5.$

5 is the required number.

Example 3. Eight times a number diminished by 10 is equal to the sum of six times the number and 4. Find the number.

Suppose, the number is x.

From the given condition,

$$8x-10=6x+4,$$

or, $8x-6x=4+10,$

or, $2x=14;$

$\therefore\quad x=\frac{14}{2}=7.$

7 is the required number.

Example 4. A and B together start a business with a joint-capital of Rs. 540. If A's share in the capital be double that of B, find the share of each in the joint-fund.

Let x represent B's share.

Then, A's share in the capital is $2x$.

So, the joint-fund $=x+2x$, *i.e.*, $=3x$.

But, the joint-fund is Rs. 540 ;

$\therefore\quad 3x=$ Rs. 540, or, $x=$ Rs. 180,

i.e., B's share is Rs. 180,

and $\therefore$ A's share is Rs. 360.

Example 5. Divide 34 into two parts whose difference is 8.

Let x denote the larger part.

Then, $34-x$ denotes the smaller part.

Hence, by the question,

$$x-(34-x)=8, \qquad \text{or,} \quad 2x-34=8\,;$$

$\therefore\quad 2x=42\,; \qquad \therefore\quad x=21.$

Thus, the larger part is 21 and the smaller part is $(34-21=)$ 13.

Example 6. The sum of three consecutive numbers is 177. Find the numbers.

Let x be the smallest of the consecutive numbers. Since the consecutive numbers differ from each other by 1, the numbers after x are $x+1$ and $x+2$. In this problem, the three consecutive numbers are, therefore, x, $x+1$, $x+2$.

By the given condition,

$$x+(x+1)+(x+2)=177,$$

or, $x+x+1+x+2=177,$

or, $3x+3=177,$

or, $3x=177-3,$

or, $3x=174.$

$\therefore\quad x=\frac{174}{3}=58.$

Hence the consecutive numbers are 58, 59, 60.

Example 7. What number is that of which the ***third*** part exceeds the ***fifth*** part by 4 ?

Let x represent the required number.

Then, by the given condition,

$$\frac{x}{3}-\frac{x}{5}=4, \quad \text{or,} \quad 5x-3x=60,$$

or, $2x=60$; $\therefore$ $x=30$. 30 is the required number.

Example 8. In 10 years A will be twice as old as B was 10 years ago. Find their present ages if A is now 9 years older than B.

Let the present age of B be denoted by x.

Then, the present age of A is $x+9$.

After 10 years A's age $=x+\overline{9+10}=x+19$.

Before 10 „ B's „ $=x-10$.

$\therefore$ by the given condition,

$x+19=2(x-10)$, or, $x+19=2x-20$,

by transposition $2x-x=20+19$, or, $x=39$,

i.e., the present age of $B=39$ years,

„ „ „ „ $A=48$ „ .

EXERCISE 30

1. Ten times a number is 90. What is the number ?

2. A number when increased by 10 becomes 25. Find the number.

3. When 9 is added to nine times a number, the sum is equal to twelve times the number diminished by 18. Find the number.

4. A straight line, whose length is 9 metres, is divided into two portions, one being double of the other. Find the length of each portion.

5. A bag contains as many rupees in it as there are fifty-paise pieces. Find the number of fifty-paise pieces if there be Rs. 30 in all.

6. Find two numbers whose sum is 50, and whose difference is 30.

7. Find a number such that it is equal to five times its defect from 96.

8. Find a number which being multiplied by 8, the product will be greater than half the number by 90.

9. What number is that from which if you subtract 40, the difference will be one-third of the original number ?

10. What number is that of which the excess over 35 is less by 22 than its defect from 67 ?

11. Four times the excess of a number over 16 is equal to the defect of the number from 416 ; find the number.

12. Find 3 consecutive numbers whose sum will be 129.

13. Find a number which when multiplied by 7 is as much above 132 as it was originally below it.

14. Divide 90 into two parts such that three times one of the parts together with four times the other may be equal to 335.

15. The sum of two numbers is 39 and *one-fifth* of one of them is equal to *one-eighth* of the other. Find them.

16. Find a number whose *fourth* part exceeds its *ninth* part by 5.

17. Find a number whose *sixth* part exceeds its *eighth* part by 3.

18. Divide 21 into two parts, so that ten times one of them may exceed nine times the other by 1.

19. A house and a garden cost Rs. 850 and the price of the garden $=\frac{5}{12}$ths of the price of the house ; find the price of each.

20. Divide Rs. 420 among two persons, so that for every ten paise one receives, the other may receive one-fourth rupee.

21. *A* and *B*, two shepherds, owning a flock of sheep, agree to divide its value. *A* takes 72 sheep, while *B* takes 92 sheep and pays *A* Rs. 350. Find the value of a sheep.

22. The ages of two men differ by 10 years, and 15 years ago the elder was just twice as old as the younger ; find the ages of the men.

23. A father's age is three times that of his son, and in 10 years it will be twice as great ; how old are they ?

CHAPTER VII

GRAPHS : PLOTTING OF POINTS

67. Introduction. We have shown in Chapters II and III how certain algebraic ideas and rules may be easily understood by graphical illustrations. In fact, graphical representation of anything, wherever it is possible, greatly helps to realise the nature of the thing represented. In the present chapter we propose to consider how algebraic quantities can be represented by points as a preliminary to geometrical representations of algebraic identities and equations which will be considered later on. Such geometrical representations are called **Graphs.**

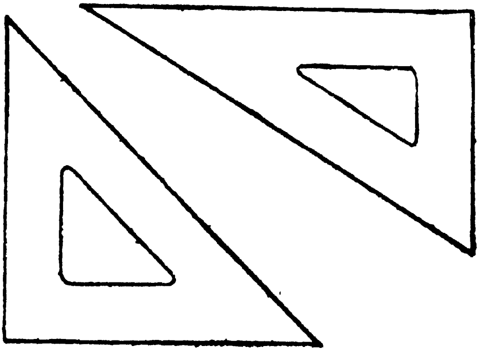
68. Instruments required. The student should first of all provide himself with the following instruments and acquire skill in manipulating them with accuracy and neatness.

(*a*) A hard Pencil.

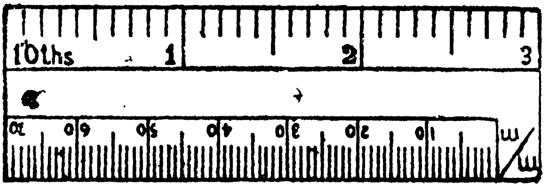
Note. *It must be well-sharpened so that the lines drawn may be very fine.*

(*b*) A pair of Compasses (also called Dividers).

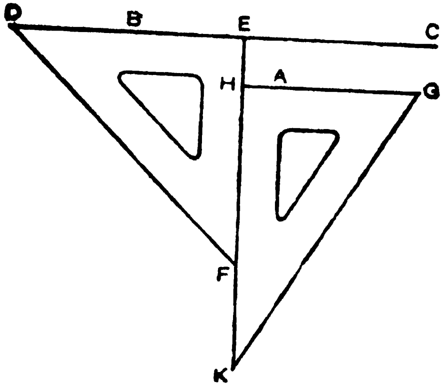
(*c*) Two Set-squares



(*d*) A graduated Flat Ruler (of moderate length) showing millimetres and tenths of an inch.



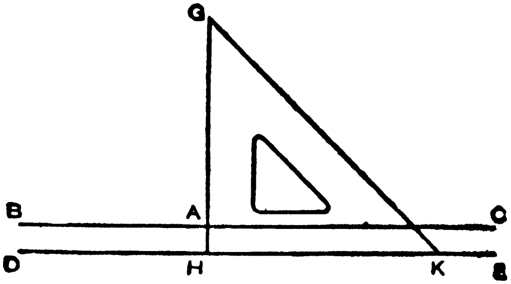
Example 1. Through the point A draw a straight line parallel to BC.



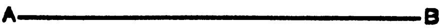
Place the Set-square DEF in such a way that the edge DE may fall along BC. Then slip the other Set-square GHK into the position shown in the diagram, so that HG may pass by A. Now trace a line along HG, which will evidently be parallel to BC.

Example 2. Through the point A in the straight line BC draw a straight line perpendicular to BC.

First trace a line DE, parallel to BC. Then place the Set-square GHK in such a way that HK may fall along DE, and GH may pass by A. Now trace a line along HG, which will evidently be perpendicular to BC.



Example 3. Find the length of the straight line AB.

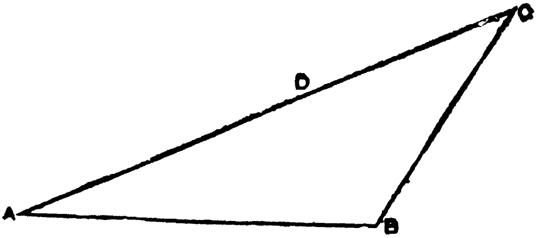


By means of the pair of Compasses and the Millimetre scale we find that the length of AB is equal to 6·4 centimetres.

EXERCISE 31

1. Produce the straight line AB to double its length.

2. On a given straight line AB a point D is taken supposing it to be the middle point. By means of a pair of Compasses, however, it is found that AD is a trifle shorter than BD. How is the mistake to be corrected ?



3. ABC is a triangle and D is a point on AC, as in the above diagram. Through D draw, towards AB, a straight line parallel to CB.

4. In the same diagram, through D draw, away from AB, a straight line parallel to BC.

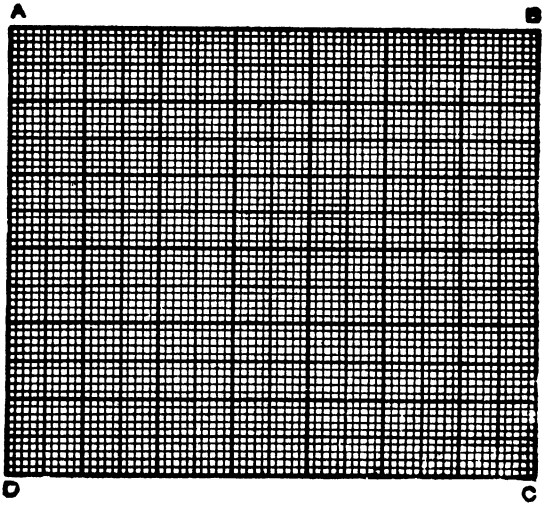
5. In the diagram of example 3, through B, draw a straight line parallel to AC.

6. From the vertices of a given triangle draw perpendiculars to the opposite sides.

7. In example 3, measure the lengths of the sides of the triangle, and also measure the lengths of AD and DC.

69. **Squared Paper.** A specimen of a sheet of squared paper is given below.

We have two sets of parallel straight lines on the paper. One set being parallel to the length, and the other parallel to the breadth of the paper, it is clear that every line of the first set is perpendicular to every line of the second. The distance between every two consecutive parallels is one-tenth of a centimetre, whilst every two consecutive *thick* parallels are a centimetre apart. The whole paper is thus divided into a large number of small squares which are equal to one another, each side of each square being one millimetre (one-tenth of a centimetre) in length. The paper is also divided into a number of thick-bordered squares, each side of each such square being one centimetre in length. It is clear also that one hundred of the small squares are contained in each of the thick-bordered squares.



Note 1. *Lines parallel to AB may be regarded as east-and-west lines, and those parallel to [illegible] north-and-south lines. They may also be considered as horizontal and vertical lines respectively.*

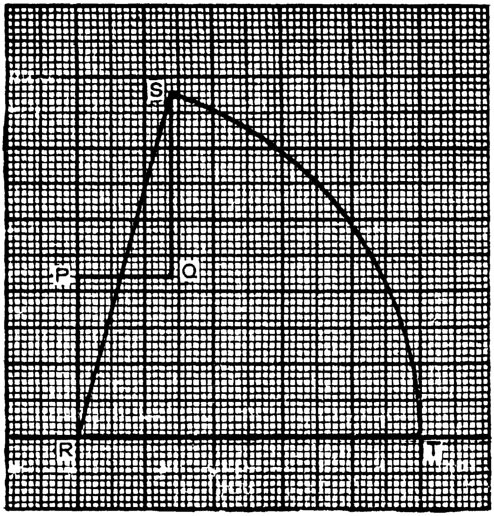
Note 2. *For the sake of convenience the length of a side of a small square may be denoted by the symbol a.*

Note 3. *The paper may also be so ruled that the length of a side of a small square is only one-tenth of an inch instead of one-tenth of a centimetre, i.e., m.m. In that case the distance between every two consecutive thick parallels is evidently an inch. (One inch is approximately equal to 2·54 centimetres).*

Example 1. P, Q, R, S are four stations such that Q is 7 kilometres east of P, R is 11 kilometres south of P, and S is 13 kilometres north of Q. Find the distance between R and S.

Taking the length of a side of a small square (*i.e.*, a) to represent half kilometre we have P, Q, R, S as in the figure on next page where $PQ = 14a$, $PR = 22a$ and $QS = 26a$.

With R as centre and RS as radius describe an arc of a circle cutting the east-and-west line through R at T.



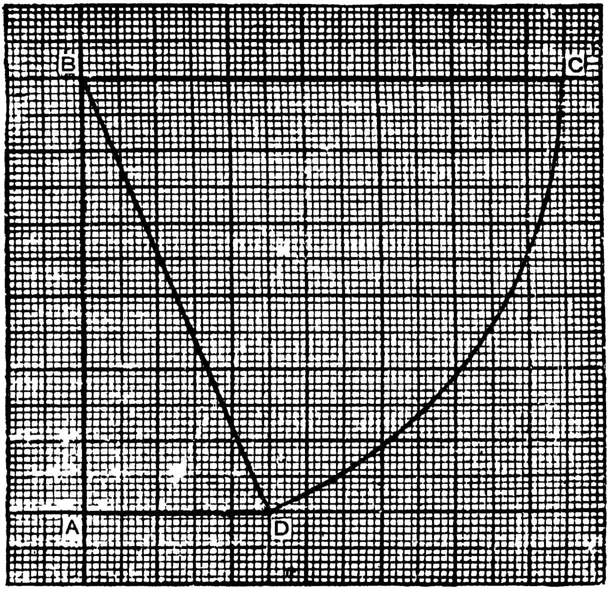
Now as $RT=50a$, we have RS also $=50a$. Hence, the required distance $=$ 25 kilometres.

Example 2. An upright post is 6 metres high. A string of length $6\frac{1}{2}$ metres has one end attached to the top of the post and is held tight with the other end in contact with the ground. How far is this end from the foot of the post?

Let $10a$ (*i.e.*, 10 times the length of a side of a small square) represent one metre. Then 6 metres will be represented by $60a$ and $6\frac{1}{2}$ metres by $65a$.

Let AB represent the post, so that $AB=60a$. Take a point C on the horizontal line through B such that $BC=65a$.

With B as centre and BC as radius describe an arc of a circle cutting the horizontal line through A at D. Join BD; then BD represents the string.



Now, AD is equal to $25a$, which is $20a+5a$. Hence, the required distance $=2\frac{1}{2}$ metres.

EXERCISE 32

1. A is $5\frac{1}{3}$ units of length east of O, and P is 4 units of length north of A. How far is P from O ?

2. B is 3 metres west of O, and Q is $7\frac{1}{5}$ metres south of B. How far is Q from O ?

3. C is 2 metres north of O, and R is $6\frac{2}{7}$ metres west of C. How far is R from O ?

4. D is 2·1 centimetres south of O, and S is 2·8 centimetres east of D. How far is S from O ?

5. A is 2·7 metres east of O. P is north of A and 4·5 metres from O. How far is P from A ?

6. Q is 2·4 metres south of B. O is east of B and 2·5 metres from Q. How far is B from O ?

7. *B* is $4\frac{4}{5}$ metres east of *A*, *C* is $\frac{3}{5}$ metres north of *A*, and *D* is 2 metres north of *B*. How far is *D* from *C* ?

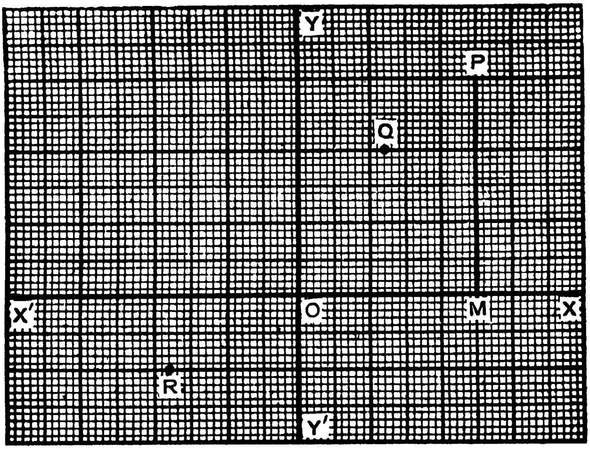
8. *B* is 25 metres north of *A*. *P* is 40 metres west of *A*, and *Q* is 20 metres east of *B*. How far is *Q* from *P* ?

9. Two vertical posts, 14 metres and 3·8 metres high, are 13·6 metres apart. Find the distance between the tops of the posts.

10. A ladder 10 metres long has its foot at a distance of 3 metres from a vertical wall. How far up the wall does it reach ?

70. If in a plane, a point and two straight lines passing through it at right angles to each other be given, the position of any point in the plane can be easily defined.

In the plane of the paper as shown in the diagram given below, let XOX' and YOY' be the two given straight lines at right angles to each other. If *P* be any point in the plane, how to know its position ?



We may regard XOX' as the *east-and-west* line, and YOY' as the *north-and-south* line. Draw *PM* parallel to YOY' meeting XOX' at *M*. Evidently then *M* is due east of *O*, and *P*, due north of *M*. Hence, if *OM* and *MP* be known, we know the position of *P* at once.

Taking the length of a side of a small square as the unit of length, we have $OM = 25$ units of length and $MP = 30$ units of length. Hence the position of *P* may be briefly defined as follows :

25 units east, 30 units north.

Note 1. *If Q be a point whose position is defined to be* 12 *units east*, 20 *units north, to find Q all that we have to do is to take a point* 12 *units due east of O and thence proceed* 20 *units northwards.*

Note 2. *If R be a point whose position is defined to be* 18 *units west*, 10 *units south, to find R all that we have to do is to take a point* 18 *units due west of O and thence proceed* 10 *units southwards.*

EXERCISE 33

[*Squared Paper is to be used in every case.*]

1. Find the points whose positions are defined as follows :

(1) 5 units east, 7 units north. (2) 12 units west, 8 units north.
(3) 15 units west, 18 units south. (4) 22 units east, 9 units south.
(5) 12 units west, 19 units north. (6) 35 units east, 38 units south.

2. It is clear from Chapter II (Positive and Negative Quantities) that '6 units west' is the same as '−6 units east' and '8 units south' is the same as '−8 units north'. Hence, find the points whose positions are defined as follows :

(1) 7 units east, −8 units north.
(2) −15 units east, 9 units north.
(3) −9 units east, −13 units north.

3. In defining the position of a point the words 'east' and 'north' may be omitted if it is accepted as a rule that the distance measured towards the east should invariably be mentioned first. On this convention, find the points whose positions are defined as follows :

(1) 8 units, 9 units. (2) 16 units, −21 units.
(3) −12 units, 15 units. (4) −10 units, −14 units.

4. We may define the position of a point still more briefly if the word 'units' be omitted. Find, then, the points whose positions are defined as follows :

(1) 15, 10. (2) 33, 20. (3) −18, 15.
(4) 20, −15. (5) −25, −33. (6) −23, −38.

71. Definitions. The student is referred to the diagram of the last article. The given lines XOX' and YOY' with reference to which the positions of all points in the plane are defined, are called the **axes of co-ordinates** ; and the point O, where these lines intersect, is called **the origin.**

The straight line XOX' is called the **axis of** x and the straight line YOY', the **axis of** y.

The lengths OM and MP which define the position of the point P are called its **co-ordinates,** OM being called the **abscissa** (or **x-co-ordinate)** and MP, the **ordinate** (or **y-co-ordinate).**

'The point (x, y)' or simply '(x, y)' means 'the point whose abscissa $= x$ units of length, and ordinate $= y$ units of length'.

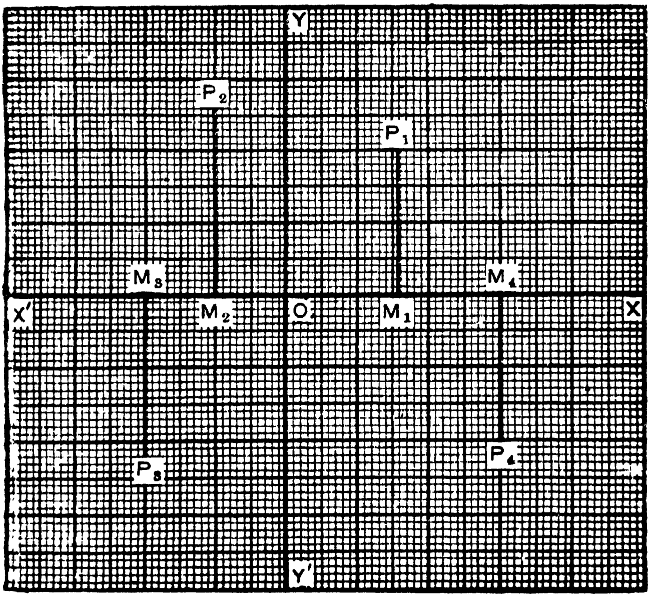
Note 1. *When we speak of the 'x and y' of a point, we mean its abscissa and ordinate.*

Note 2. *The abscissa is positive or negative according as M is on the right or on the left of O. The ordinate is positive or negative according as P is above or below XOX'.*

Note 3. ***'To plot a point'** is to find the position of a point, when its co-ordinates are given.*

Example 1. In the diagram given below write down the co-ordinates of the points P_1, P_2, P_3, P_4.

The figure explains itself. Take two times the length of a side of a small square as the unit of length



(1) $OM_1 = 8$ units and M_1 is on the *right* of O; $M_1P_1 = 10$ units and P_1 is *above* the line XOX'. Hence, the co-ordinates of P_1 are 8 and 10.

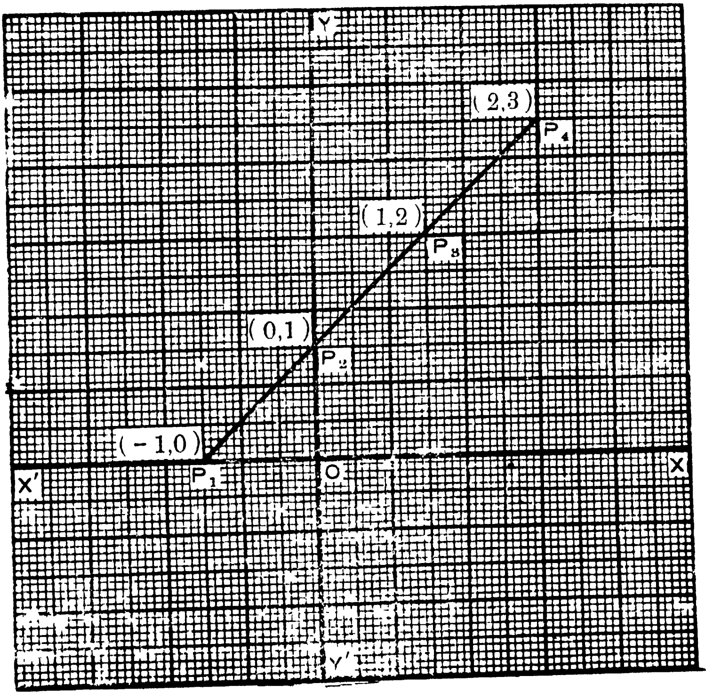
(2) $OM_2 = 5$ units and M_2 is on the *left* of O; $M_2P_2 = 13$ units and P_2 is *above* the line XOX'. Hence, the co-ordinates of P_2 are -5 and 13.

(3) $OM_3 = 10$ units and M_3 is on the *left* of O; $M_3P_3 = 11$ units and P_3 is *below* the line XOX'. Hence, the co-ordinates of P_3 are -10 and -11.

(4) $OM_4 = 15$ units and M_4 is on the *right* of O; $M_4P_4 = 10$ units and P_4 is *below* the line XOX'. Hence, the co-ordinates of P_4 are 15 and -10.

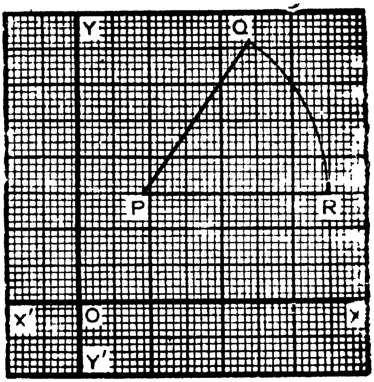
Example 2. Plot the points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and $(2, 3)$, and show that they all lie in a straight line.

Let 15 times the side of a small square represent the unit of length, and let P_1, P_2, P_3, P_4 respectively denote the four given points. Then the positions of the points will be as shown in the figure given below.



Now we find that a Flat Ruler may be so placed that its edge will pass through all the four points. Hence, they all lie in the same straight line.

Example 3. Plot the points (3, 5) and (8, 12), and find the distance between them.

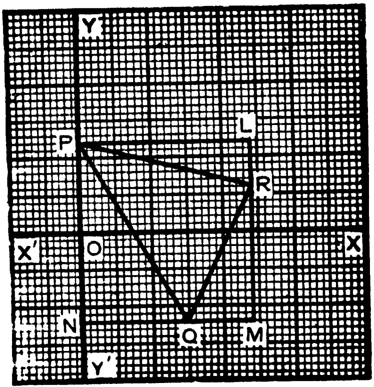


Let 3 times the side of a small square represent the unit of length, and let P and Q respectively denote the two given points. Then the positions of the points will be as shown in the figure.

With centre P and radius PQ draw a circle cutting the east-and-west line through P at R.

The distance required $= PQ = PR = 8\dot{}6$ units. [from the figure]

Example 4. Plot the points $P(0, 4)$, $Q(5, -4)$ and $R(8, 2)$ and find the area of the triangle PQR.



Let 3 times the side of a small square be the unit of length. Then the positions of the points, P, Q, R will be as shown in the diagram. Count the number of small squares falling *wholly* inside the triangle PQR. Of the remaining squares through which the sides pass, find the number of *only* those *half* or *more than half* of which are within the triangle and reject the others. Since nine small squares represent a unit of area, one-ninth of the total number of small squares thus counted will give the area of the triangle pretty accurately.

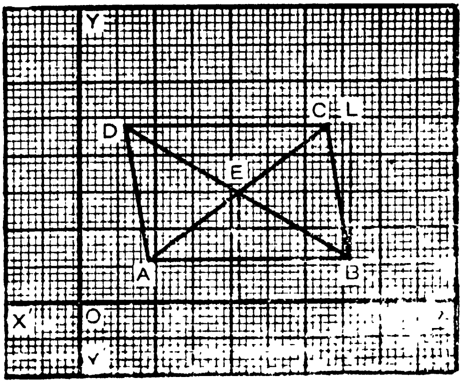
Counting by the above method, the number of small squares in the triangle $PQR = 243$ (nearly).

Hence, the required area = 27 units of area (nearly).

Verification : Through P and Q draw two straight lines parallel to x-axis and through R a straight line parallel to y-axis. Thus the rectangle $PLMN$ is formed.

$$\begin{aligned}\text{Now, } \triangle PQR &= \square PLMN - \triangle PLR - \triangle RQM - \triangle PQN \\ &= PL.PN - \tfrac{1}{2}PL.LR - \tfrac{1}{2}RM.QM - \tfrac{1}{2}PN.NQ \\ &= 8\times 8 - \tfrac{1}{2}.8.2 - \tfrac{1}{2}.6.3 - \tfrac{1}{2}.8.5 \\ &= 64 - 8 - 9 - 20 \\ &= 27.\end{aligned}$$

Example 5. Plot the points $A(3, 2)$, $B(12, 2)$, $C(11, 8)$ and $D(2, 8)$. Find the area of the quadrilateral $ABCD$ and read the co-ordinates of the point of intersection of AC and BD.



Take 3 times the side of a small square as the unit of length. Then the positions of the points, A, B, C and D will be as shown in the diagram.

Counting by the method of Example 4, the number of small squares in the quadrilateral $ABCD = 486$ (nearly).

Hence, the area required = 54 units of area (nearly).

Verification : The quadrilateral is a parallelogram.

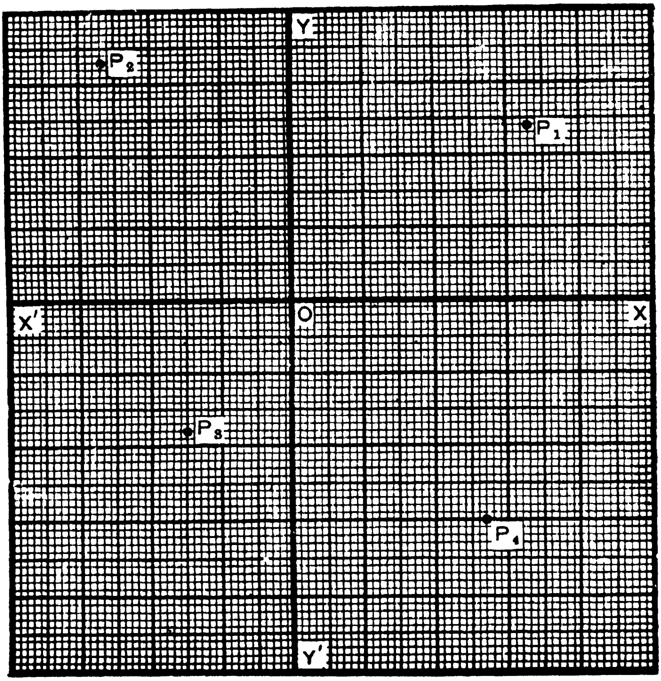
Therefore, its area $= AB.BL$ (base × height)

$= 9 \times 6 = 54$ units of area.

Also, from the diagram, the co-ordinates of E, the point of intersection of AC and BD are 7 and 5.

EXERCISE 34

1. In the diagram given below, what are the co-ordinates of the points P_1, P_2, P_3, P_4, (i) when the unit of length is represented by a side of a small square, (ii) when the unit of length is represented by 5 times the side of a small square ?



2. In the diagram given in page 94 what will be the co-ordinates of the points P_1, P_2, P_3, P_4 if the unit of length be represented by nine times the side of a small square?

3. Plot the points $(-4, -4)$, $(7, 7)$, $(13, 13)$, and satisfy yourself that they lie in a straight line passing through the origin.

4. Plot the points $(-8, 4)$ and $(10, -5)$, and satisfy yourself that the straight line joining them passes through the origin.

5. Plot the points $(8, 5)$ and $(-4, -11)$, and find the distance between them.

6. Plot the points $(-7, 9)$ and $(-12, 21)$, and find the distance between them.

7. Plot the points $(-11, 13)$ and $(3, -35)$, and find the distance between them.

8. Join the points $(0, 0)$ and $(5, 5)$, and produce the straight line both ways. Find the ordinate of the point on this straight line whose abscissa is 11, and the abscissa of the point whose ordinate is -13.

9. Join the points $(0, 7)$ and $(12, 0)$, and produce the straight line both ways. Find the ordinate of the point on the straight line whose abscissa is -18, and the abscissa of the point whose ordinate is -14.

10. Join the points $(-4, 0)$ and $(0, -8)$, and produce the straight line both ways. Find the ordinate of the point on the straight line whose abscissa is -10, and the abscissa of the point whose ordinate is -24.

11. Plot the points $A(3, 2)$, $B(3, 7)$ and $C(8, 5)$, and find the area of the triangle ABC.

12. Plot the points $P(-2, 5)$, $Q(6, 5)$ and $R(8, 9)$, and find the area of the triangle PQR.

13. Plot the points $D(5, 2)$, $E(6, 8)$ and $F(7, 12)$, and find the area of the triangle DEF.

14. Find the area of the quadrilateral whose vertices are $(11, 2)$, $(3, 2)$, $(3, 7)$ and $(11, 7)$. Obtain the co-ordinates of the point of intersection of its diagonals.

15. Find the area of the quadrilateral whose vertices are (i) $(16, 6)$, $(2, 3)$, $(11, 14)$, and $(5, 11)$; (ii) $(3, 6)$, $(5, 4)$, $(17, 16)$ and $(9, 18)$; (iii) $(-12, 5)$, $(-12, -10)$, $(16, -10)$ and $(16, 5)$; (iv) $(0, 1)$, $(10, 8)$, $(3, 13)$ and $(-2, 8)$.

16. Construct a triangle whose base is 12 centimetres and the two other sides are 5 and 13 centimetres respectively. Find the area of the triangle, the altitude and the angle opposite to the longest side.

17. Construct a triangle whose base is 6 centimetres and the two other sides are 3 and 5 centimetres respectively. Measure the altitude as accurately as possible.

18. Plot the following series of points :

(i) $(6, 0)$, $(6, 3)$, $(6, 4)$, $(6, 6)$, $(6, 8)$ and $(6, 10)$;

(ii) $(-2, 7)$, $(3, 7)$, $(5, 7)$, $(7, 7)$, $(8, 7)$ and $(10, 7)$.

Show that they lie on two straight lines respectively parallel to the axis of y and the axis of x. Find the co-ordinates of their point of intersection.

19. Plot the points $(3, 4)$, $(4, 3)$, $(5, 0)$, $(-4, -3)$, $(4, -3)$. Find their distances from the origin and show that they lie on a circle with the origin as centre.

20. Plot the points $A(5, 2)$, $B(9, 2)$, $C(5, 8)$, $D(9, 8)$ and $E(7, 12)$. Find the area of the figure $ABDEC$ and the co-ordinates of the point of intersection of AD and BC.

MISCELLANEOUS EXERCISES II

I

1. From the identity $(a+b)^2=a^2+2ab+b^2$, deduce the square of $x-y-z$ by putting x for a and $-y-z$ for b.

2. Establish the following formulæ :

(i) $a^2+b^2=\frac{1}{2}\{(a+b)^2+(a-b)^2\}$;

(ii) $4ab=(a+b)^2-(a-b)^2$.

3. Prove that

$$(y-z)(y+z-x)+(z-x)(z+x-y)+(x-y)(x+y-z)=0.$$

4. Prove that

$$(a-b)(a+1)(b+1)-a(b+1)^2+b(a+1)^2=(a-b)(a+b+2ab).$$

5. If $a=x+m$, $b=y+m$, $c=z+m$, show that

$$a^2+b^2+c^2-bc-ca-ab=x^2+y^2+z^2-yz-zx-xy.$$

6. If $s=a+b+c$, prove that

$$(as+bc)(bs+ac)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2.$$

7. Divide $(m+n)^3-27p^3$ by $m+n-3p$.

8. Find the quotient when the dividend is $(9x^2-17xy+13y^2)^2$, the remainder is $49y^3(2x+5y)^2$ and the divisor is $3x^2-xy+16y^2$.

9. If $x+\frac{2}{y}=\frac{8}{3}$ and $y+\frac{3}{x}=\frac{9}{2}$, find the value of $x^3y^3+\frac{216}{x^3y^3}$.

10. Show that

$$(x-y+z)^3+(x+y-z)^3+6x(x-y+z)(x+y-z)=8x^3.$$

II

Solve the following equations :

1. $3(x-3)-2(x-2)+x-1=x+3+2(x+2)+3(x+1)$.

2. $(x-3)(x-5)=(x-2)(x-7)$.

3. $2(x+1)(x+3)+8=(2x+1)(x+5)$.

Find the value of x, when

4. $(a+b)(b-x)=b(a-x)$.

5. $\frac{mnx-p}{mn}+\frac{npx-m}{np}+\frac{pmx-n}{pm}=\frac{2p}{mn}+\frac{2m}{np}+\frac{2n}{pm}$.

6. $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}$.

7. $4x-\frac{x-1}{2}=x+\frac{2x-2}{5}+24$.

8. $x-\frac{x-2}{2}=5\frac{3}{4}-\frac{x+10}{5}+\frac{x-2}{4}$.

9. $\frac{2x-1}{2}+\frac{3x-2}{3}+\frac{4x-3}{4}=\frac{1}{12}$.

10. $\frac{2}{3}(x-1)-\frac{5}{6}(2x-3)+\frac{3}{4}(1-2x)=\frac{1}{12}(4x-5)$.

III

1. Find the number to which, if 29 be added, the sum will exceed four times the number by 8.

2. Find a number whose 7th part exceeds the 9th part by 4.

3. A man saves one-tenth of his monthly income and spends one-third of the remainder in buying petty things. At the end of the month, he has Rs. 300 in his pocket after meeting all the current expenses which amounted to two-fifths of the total income. Find his income per month.

4. A merchant invests two-fifths of his capital in sugar business, one-third in jute and half of the remainder in cloth and has Rs. 300 cash. Find his capital and the money invested in each business.

5. *A* is twice as old as *B* and four years older than *C*. The sum of the ages of *A*, *B* and *C* is 96 years. Find the age of each.

6. Two sums of money are together equal to Rs. 52 50 P., and there are as many rupees in the one as there are five paise in the other. Find the sums.

7. Plot the following points on a squared paper and verify that they are the angular points of a rectangle. Show that the length of each of the diagonals is 5 :

$$(1\tfrac{1}{2},\ 2),\ (-1\tfrac{1}{2},\ 2),\ (-1\tfrac{1}{2},\ -2) \text{ and } (1\tfrac{1}{2},\ -2).$$

8. O is a fixed station. A is 20 kilometres north of O. B is 4 kilometres east of A. C is 17 kilometres south of B. Show that the distance between O and C is 5 kilometres.

9. If, in the above example, A be 12 kilometres west of O and P be 5 kilometres north of A, and B be 12 kilometres east of O and Q be 5 kilometres south of B, show that the distance between P and Q is 26 kilometres.

10. Plot the following points on a squared paper and verify that they lie on a straight line through the origin :

$$(-5,\ -10),\ (1,\ 2) \text{ and } (3,\ 6).$$

CHAPTER VIII

HARDER ADDITION AND SUBTRACTION

1. Addition

72. In Chapter III, we have explained the following laws of addition of algebraic quantities and expressions :

(1) If any number of quantities are added together, the result will be the same in whatever order the quantities may be taken. Thus,

$$a+b+c=b+c+a=c+a+b, \text{ etc.} \qquad [\text{ Art. } 31]$$

This is called the **Commutative Law of Addition.**

(2) When any number of quantities are added together, they can be divided into groups and the result expressed as the sum of those groups. Thus,

$$a+b+c=a+(b+c)=(a+b)+c=b+(c+a), \text{ etc.} \qquad [\text{ Art. } 32]$$

This is called the **Associative Law of Addition.**

(3) When any number of *like terms with numerical coefficients* are added, their sum is a *like term* whose coefficient is equal to the sum of the coefficients of the terms added. [Art. 32]

Thus, the sum of $5x$, $-2x$, $7x$, $6x$ is $16x$, since $5+(-2)+7+6=16$.

This process is known as *collecting terms.*

The ordinary rule for adding together compound expressions with like and unlike terms has also been explained in Art. 33.

We have so far applied these rules to simple cases and now propose to consider more difficult problems.

73. Compound expressions with fractional coefficients. If compound expressions with fractional coefficients are to be added, first simplify each expression if necessary and then put the expressions under one another so that like terms stand in the same vertical column, and draw a line below the last expression, then add up each vertical column and put the result below it. Simplify the coefficients in the result by Arithmetical Rules.

The following examples will illustrate the process :

Example 1. Add together :

$$\frac{x}{3}+\frac{y}{5}-\frac{z}{7},\ -\frac{9}{10}y+\frac{12}{7}z+\frac{7}{3}x+12a \text{ and } \frac{3}{7}z-\frac{2}{3}x+\frac{4}{5}y-2b.$$

$$\begin{array}{rrrrrr}
\text{The 1st expression} = & \frac{1}{3}x+ & \frac{1}{5}y- & \frac{1}{7}z & & \\
\text{The 2nd expression} = & \frac{7}{3}x- & \frac{9}{10}y+ & \frac{12}{7}z & +12a & \\
\text{The 3rd expression} = & -\frac{2}{3}x+ & \frac{4}{5}y+ & \frac{3}{7}z & & -2b \\
\hline
\therefore\ \text{the sum} = & 2x+ & \frac{1}{10}y+ & 2z & +12a & -2b
\end{array}$$

[In the sum,

the coefficient of $x=\frac{1}{3}+\frac{7}{3}-\frac{2}{3}=\frac{1+7-2}{3}=\frac{6}{3}=2$,

the coefficient of $y=\frac{1}{5}-\frac{9}{10}+\frac{4}{5}=\frac{2-9+8}{10}=\frac{10-9}{10}=\frac{1}{10}$,

the coefficient of $z=-\frac{1}{7}+\frac{12}{7}+\frac{3}{7}=\frac{-1+12+3}{7}=\frac{14}{7}=2$,

the coefficient of $a=0+12+0=12$,

the coefficient of $b=0+0-2=-2$.]

Note. *Notice that places of like terms in 'a' are vacant in the 1st and 3rd expressions. For convenience, the coefficients of 'a' in these places may be taken to be zero. Similarly, the coefficients of the like terms in 'b' may be taken as zero in the 1st and 2nd expressions.*

Example 2. Find the sum of $\frac{6x-2y}{6}+\frac{4y-3z}{12}+\frac{2z-4x}{8}$,

$$\frac{4x-3y}{12}+\frac{6y-4z}{8}+\frac{3z-6x}{6} \text{ and } \frac{2x-4y}{8}+\frac{3y-2z}{6}+\frac{4z-6x}{12}.$$

Simplifying each of the expressions by collecting terms and proceeding as above, the sum follows. Thus,

$$\text{The 1st exp.} = (\tfrac{6}{6} - \tfrac{4}{8})\,x + (-\tfrac{2}{6} + \tfrac{4}{12})\,y + (-\tfrac{3}{12} + \tfrac{2}{8})\,z$$
$$= (1 - \tfrac{1}{2})\,x + (-\tfrac{1}{3} + \tfrac{1}{3})\,y + (-\tfrac{1}{4} + \tfrac{1}{4})\,z = \tfrac{1}{2}x$$
$$\text{The 2nd exp.} = (\tfrac{4}{12} - \tfrac{6}{6})\,x + (-\tfrac{3}{12} + \tfrac{6}{8})\,y + (-\tfrac{4}{8} + \tfrac{3}{6})\,z$$
$$= (\tfrac{1}{3} - 1)\,x + (-\tfrac{1}{4} + \tfrac{3}{4})\,y + (-\tfrac{1}{2} + \tfrac{1}{2})\,z = -\tfrac{2}{3}x + \tfrac{1}{2}y$$
$$\text{The 3rd exp.} = (\tfrac{2}{8} - \tfrac{6}{12})\,x + (-\tfrac{4}{8} + \tfrac{3}{6})\,y + (-\tfrac{2}{6} + \tfrac{4}{12})\,z$$
$$= (\tfrac{1}{4} - \tfrac{1}{2})\,x + (-\tfrac{1}{2} + \tfrac{1}{2})\,y + (-\tfrac{1}{3} + \tfrac{1}{3})\,z = -\tfrac{1}{4}x$$
$$\therefore \text{ the sum} = -\tfrac{5}{12}x + \tfrac{1}{2}y$$

[In the sum,

the coefficient of $x = \frac{1}{2} - \frac{2}{3} - \frac{1}{4} = \frac{6-8-3}{12} = \frac{-5}{12}$,

the coefficient of $y = 0 + \frac{1}{2} + 0 = \frac{1}{2}$.]

Example 3. Find the numerical value of the sum of

$$\frac{3}{7}x^3 + \frac{5}{11}y^5 - 20a^2 + \frac{49}{2}b^3,\quad 17a^2 - \frac{27}{2}b^3 - \frac{23}{7}x^3,\quad -\frac{y^5}{11} + \frac{3}{2}b^3 - 3a^2$$

and $-\frac{23}{2}b^3 - \frac{4}{11}y^5 + 7a^2 + \frac{20}{7}x^3$, when $x = 98$, $y = 79$, $a = 5$, and $b = 4$.

In this problem, the numerical value can be obtained easily from the sum of the expressions,

$$\begin{array}{lrrrr}
\text{The 1st expression} = & \tfrac{3}{7}x^3 & +\tfrac{5}{11}y^5 & -20a^2 & +\tfrac{49}{2}b^3 \\
\text{The 2nd expression} = & -\tfrac{23}{7}x^3 & & +17a^2 & -\tfrac{27}{2}b^3 \\
\text{The 3rd expression} = & & -\tfrac{1}{11}y^5 & -\ 3a^2 & +\ \tfrac{3}{2}b^3 \\
\text{The 4th expression} = & \tfrac{20}{7}x^3 & -\tfrac{4}{11}y^5 & +\ 7a^2 & -\tfrac{23}{2}b^3 \\
\hline
\therefore\ \text{the sum} = & & & a^2 & +\ \ b^3
\end{array}$$
$$= 5^2 + 4^3 = 5 \times 5 + 4 \times 4 \times 4 = 25 + 64 = 89.$$

[In the result,

the coefficient of $x^3 = \frac{3}{7} - \frac{23}{7} + 0 + \frac{20}{7} = \frac{3-23+0+20}{7} = \frac{0}{7} = 0$,

the coefficient of $y^5 = \frac{5}{11} + 0 - \frac{1}{11} - \frac{4}{11} = \frac{5+0-1-4}{11} = \frac{0}{11} = 0$,

the coefficient of $a^2 = -20 + 17 - 3 + 7 = 24 - 23 = 1$,

the coefficient of $b^3 = \frac{49}{2} - \frac{27}{2} + \frac{3}{2} - \frac{23}{2} = \frac{49-27+3-23}{2}$

$= \frac{52-50}{2} = \frac{2}{2} = 1$.]

74. Compound expressions with literal coefficients. Coefficients which are not wholly numerical are called literal. Thus, the coefficients of x in ax, $6bx$, $(c+d-e)x, \ldots$ being a, $6b$, $(c+d-e), \ldots$ respectively are literal.

The terms ax, $6bx$, $(c+d-e)x$,...... if considered in respect of x differ in their literal coefficients only and are also called *like* when thus considered.

If ax and bx be two like terms in x,

$$\text{their sum}=ax+bx=(a+b)x.$$

Hence, the sum of two like terms is a like term whose coefficient is the sum of the coefficients of the two terms. By Art 47, Cor. 3, this rule for addition will be true even when the number of terms is greater than two.

Thus, the rule for addition of like terms is same for all coefficients numerical as well as literal.

It, therefore, follows that the rule for adding compound expressions is same for both of these coefficients.

The following exmples will illustrate the above rule.

Example 1. Add together :

$$(b+c)x+(c+a)y+(a+b)z,\ ax+by+cz \text{ and } x+y+z.$$

Arranging the expressions so that the like terms may stand in the same vertical column and adding up each such column, the sum follows. Thus,

$$\begin{array}{lccc} \text{The 1st exp.}= & (b+c)x & +(c+a)y & +(a+b)z \\ \text{The 2nd exp.}= & ax & +by & +cz \\ \text{The 3rd exp.}= & x & +\ y & +\ z \\ \hline \end{array}$$

$$\therefore\ \text{the sum}=(a+b+c+1)x+(a+b+c+1)y+(a+b+c+1)z.$$

[In the result,

$$\text{the coefficient of } x=(b+c)+a+1=a+b+c+1,$$
$$\text{the coefficient of } y=(c+a)+b+1=a+b+c+1,$$
$$\text{the coefficient of } z=(a+b)+c+1=a+b+c+1.\ ]$$

Example 2. Add together : $(b-c)x+(c-a)y+(a-b)z$, $(b-c)y+(a-b)x+(c-a)z$ and $(b-c)z+(c-a)x+(a-b)y$.

The expressions contain like terms in respect of x, y and z. Hence, arranging like terms in the same vertical column and proceeding as before, the result follows. Thus,

$$\begin{array}{ll} \text{The 1st expression}= & (b-c)x+(c-a)y+(a-b)z \\ \text{The 2nd expression}= & (a-b)x+(b-c)y+(c-a)z \\ \text{The 3rd expression}= & (c-a)x+(a-b)y+(b-c)z. \\ \hline \end{array}$$

$$\therefore\ \text{the sum}=0.$$

[In the sum,

the coefficient of $x=(b-c)+(c-a)+(a-b)$
$=b-c+c-a+a-b=0.$

Similarly, the coefficients of y and z are zero.]

Example 3. Find the sum of $(ax-by)+(bx-cz)$, $(ay-bx)+(by-cz)$ and $(cz-ax)+(cz-by)$.

Each of these three expressions contains like terms in respect of x, y and z. Arranging each expression in terms of x, y and z and proceeding as in previous examples, the sum is obtained. Thus,

$$\begin{array}{llcrrr}
\text{The 1st exp.} = & ax+bx-by-cz = & & (a+b)x- & by- & cz \\
\text{The 2nd exp.} = & -bx+ay+by-cz = & & -bx+(a+b)y- & & cz \\
\text{The 3rd exp.} = & -ax-by+2cz & = & -ax- & by+ & 2cz. \\
\hline
 & \therefore \text{ the sum} = & & & (a-b)y. &
\end{array}$$

[In the sum,

the coefficient of $x=(a+b)-b-a=a+b-b-a=0,$
the coefficient of $y=-b+(a+b)-b=-b+a+b-b=a-b,$
the coefficient of $z=-c-c+2c=0.$]

Note 1. *When compound expressions with brackets are to be added to like compound expressions, it is more convenient to retain brackets as in Example 2.*

Note 2. *The expressions to be added should be simplified by collecting terms if necessary as in Example 3.*

Example 4. Find the sum of

$$(a^2+b^2)x+(b^2+c^2)y+(c^2+a^2)z,\ (b^2+c^2)m+(c^2+a^2)n,$$
$$(c^2+a^2)p+(a^2+b^2)q \text{ and } (a^2+b^2)j+(b^2+c^2)k.$$

The expressions contain like terms in respect of (b^2+c^2), (c^2+a^2) and (a^2+b^2). Hence, arranging like terms in the same vertical column and proceeding as before,

$$\begin{array}{lrrr}
\text{The 1st expression} = & x(a^2+b^2) & +y(b^2+c^2) & +z(c^2+a^2) \\
\text{The 2nd expression} = & & m(b^2+c^2) & +n(c^2+a^2) \\
\text{The 3rd expression} = & q(a^2+b^2) & & +p(c^2+a^2) \\
\text{The 4th expression} = & j(a^2+b^2) & +k(b^2+c^2) & \\
\hline
\end{array}$$

$\therefore$ the sum $=(x+q+j)(a^2+b^2)+(y+m+k)(b^2+c^2)+(z+n+p)(c^2+a^2)$.

[In the result,

the coefficient of $(a^2+b^2)=x+0+q+j=x+q+j.$

Similarly, the coefficients of (b^2+c^2) and (c^2+a^2) are $(y+m+k)$ and $(z+n+p)$ respectively.]

EXERCISE 35

Add together:

1. $2x^2-5xy+y^2$, $4y^2-7x^2-5x+2y$, $3xy-5+y-6y^2$ and $3-4y+3x$.

2. $abc+a^2b-b^2c^2$, $5a^2b-12b^2c^2-3abc$, $8b^2c^2-4a^2b+2abc$ and $2a^2b+5b^2c^2$.

3. $m^3n^2-3mnp+2m^2n^3+6m^2n^2$, $7mnp-10m^2n^2+5m^3n^2-m^2n^3$, $2m^2n^2-5mnp+3m^2n^3$ and $-7m^3n^2+m^2n^2-4m^2n^3$.

4. $12a^3b^2x-29b^3x^2a+37x^3a^2b+45a^2b^2x^2$, $25b^3x^2a-16a^2b^2x^2-18a^3b^2x-5x^3a^2b$, $32a^2b^2x^2-23x^3a^2b+20a^3b^2x-28b^3x^2a$ and $-9x^3a^2b-14a^3b^2x-60a^2b^2x^2+32b^3x^2a$.

5. $-15a^4b^4c^4+7c^4a^3b^5-24b^4c^3a^5+27a^4b^3c^5$, $19c^4a^3b^5-15a^4b^3c^5+23a^4b^4c^4-8b^4c^3a^5$, $29b^4c^3a^5+11a^4b^4c^4-9a^4b^3c^5-16c^4a^3b^5$ and $-3a^4b^3c^5-10c^4a^3b^5+3b^4c^3a^5-18a^4b^4c^4$.

6. $25a^3b^3-8b^3c^3-23c^3a^3+19a^2b^2c^2$, $16c^3a^3-14a^2b^2c^2-19a^3b^3-12b^3c^3$, $27a^2b^2c^2+13a^3b^3+17c^3a^3-20b^3c^3$, $29b^3c^3-6a^2b^2c^2-21a^3b^3-13c^3a^3$ and $10b^3c^3+3a^3b^3+4c^3a^3-27a^2b^2c^2$.

7. $5a^3-18b^3-53c^3-25abc$, $38c^3-37a^3-7abc+29b^3$, $26abc-17c^3+11b^3+43a^3$, $13b^3-18abc+4a^3+21c^3$ and $-14a^3+12c^3+21abc-34b^3$.

8. $\frac{x}{2}+\frac{y}{3}+\frac{z}{5}$, $\frac{3x}{4}+\frac{2y}{3}+\frac{3z}{5}$ and $\frac{3x}{4}+y+\frac{6z}{5}$.

9. $\frac{3x}{5}+\frac{4y}{7}+\frac{10z}{11}$, $\frac{2y}{7}+\frac{4z}{11}+\frac{x}{5}$ and $\frac{8z}{11}+\frac{6x}{5}+\frac{8y}{7}$.

10. $\frac{4x^2y}{15}+\frac{4y^2z}{13}+\frac{5z^2x}{17}$, $\frac{7y^2z}{13}+\frac{6z^2x}{17}+\frac{7x^2y}{15}$ and $\frac{6z^2x}{17}+\frac{4x^2y}{15}+\frac{2y^2z}{13}$.

11. $\frac{7a^2b}{19}+\frac{9b^2c}{17}+\frac{11ca^2}{21}+\frac{13ab^2}{35}$, $\frac{8b^2c}{17}+\frac{10c^2a}{21}+\frac{12a^2b}{19}+\frac{17bc^2}{35}$ and $\frac{22ab^2}{35}+\frac{18bc^2}{35}+\frac{10ca^2}{21}+\frac{11ac^2}{21}$.

12. $\frac{2}{3}abc^2+\frac{3}{4}bca^2+\frac{4}{7}b^2d$, $\frac{5}{9}cab^2+\frac{1}{3}abc^2+\frac{2}{11}a^2d$, $\frac{1}{4}bca^2+\frac{4}{13}c^2d+\frac{4}{9}cab^2$ and $\frac{9}{11}a^2d+\frac{3}{7}b^2d+\frac{9}{13}c^2d$.

13. $\frac{x-2y}{2}+\frac{2y-3z}{6}+\frac{3z-4x}{12}$, $\frac{2x-3y}{6}+\frac{3y-4z}{12}+\frac{z-2x}{2}$ and $\frac{3z-4y}{12}+\frac{y-2z}{2}+\frac{2z-3x}{6}$.

14. $\frac{2x-3y}{6}+\frac{3y-5z}{15}+\frac{5z-7x}{35}$, $\frac{3x-5y}{15}+\frac{5y-7z}{35}+\frac{2z-3x}{6}$ and $\frac{5x-7y}{35}+\frac{2y-3z}{6}+\frac{3z-5x}{15}$.

15. $\frac{2b-3c}{bc}+\frac{3c-4a}{ca}+\frac{4a-2b}{ab}$, $\frac{2c-3a}{ca}+\frac{3a-4b}{ab}+\frac{4b-2c}{bc}$ and $\frac{2a-3b}{ab}+\frac{3b-4c}{bc}+\frac{4c-2a}{ca}$.

16. $\frac{bx-3ay}{ab}+\frac{2by-4az}{ab}+\frac{3bz-ax}{ab}$, $\frac{cx-4by}{bc}+\frac{3cy-5bz}{bc}+\frac{4cz-bx}{bc}$ and $\frac{ax-2cy}{ca}+\frac{4ay-3cz}{ca}+\frac{5az-cx}{ca}$.

17. $\frac{cy-ax}{caxy}+\frac{az-by}{abyz}+\frac{bx-cz}{bczx}$, $\frac{ay-bx}{abxy}+\frac{bz-cy}{bcyz}+\frac{cx-az}{cazx}$ and $\frac{by-cx}{bcxy}+\frac{cz-ay}{cayz}+\frac{ax-bz}{abzx}$.

If $a=5$, $b=4$, $x=8$, $y=7$, find the numerical value of :

18. $(46a^4+38b^4-87abx^2-105y^4)+(47abx^2+85y^4-56a^4-58b^4)+(57y^4+75b^4+23a^4+63abx^2)+(-33b^4+8y^4-27abx^2-39a^4)+(26a^4-45y^4-22b^4+5abx^2)$.

19. $(35xy^4+207ab^4-98bx^4-62ya^4-83abx^2y)+(68bx^4+102ya^4-65xy^4-87ab^4+53abx^2y)+(26abx^2y-75ab^4-25ya^4+43bx^4+53xy^4)+(28ya^4-29xy^4-65abx^2y+45ab^4+26bx^4)+(-89ab^4-43ya^4+69abx^2y+6xy^4-39bx^4)$.

20. $(57a^4bx+25b^4xy-143x^4ya+37y^4ab-253a^2b^2x^2)+(63x^4ya-92y^4ab-63a^4bx+73a^2b^2x^2-85b^4xy)+(35y^4ab+132b^4xy+82a^2b^2x^2+36x^4ya+96a^4bx)+(-50a^2b^2x^2-78a^4bx+27y^4ab-17x^4ya-52b^4xy)+(61x^4ya-20b^4xy+148a^2b^2x^2-7y^4ab-12a^4bx)$.

Add together :

21. $(a^2+b^2)(m+n)+(a^2-b^2)(p+q)+c^2l$, $(a^2-b^2)(m+n)+(a^2+l^2)\times(p+q)+c^2m$ and $nc^2+l(a^2+b^2)+k(a^2-b^2)$.

22. $(x+y)^2a+(y+z)^2b+(z+x)^2c$, $(x-y)^2a+(y-z)^2b+(z-x)^2c$ and $2(x^2-y^2)a+2(y^2-z^2)b+2(z^2-x^2)c$.

23. $ab(a-b)$, $bc(b-c)$, $ca(c-a)$ and $a^2(c-b)+b^2(a-c)+c^2(b-a)$.

Supply the following omissions :

24. $a^2+b^2+c^2-ab-ac-bc=\{\quad\}-\{(b-c)^2+(c-a)^2+(a-b)^2\}$.

25. $(b+c)x^2+(c+a)y^2+(a+b)z^2=\{\quad\}-(ax^2+by^2+cz^2)$.

26. When $3x-5y+4z$ is subtracted from a number, the remainder is $4y-3x-5z$. Find the number.

27. The middle one of the three consecutive numbers is $3x+4y-1$; find the sum of the numbers.

28. A traveller walked $(2x+3y-4z)$ kilometres on the 1st day, $(3x-4y+5z)$ kilometres on the 2nd day and $(6z+2y-5x)$ kilometres on the 3rd day. Find the distance covered by him in 3 days.

29. The share capitals invested by three shareholders of a Company are $(x+3y+2a-5z)$ rupees, $(7x-6y-2b+2z)$ rupees and $(a+3b-9x+3z)$ rupees. Find the total capital of the Company.

30. Ram earned $(3a+4b-5c)$ rupees, Hari earned $(a-3c)$ rupees more than Ram and Jadu's earnings were equal to those of Ram and Hari. Find their total earnings.

II. Subtraction

75. In Art. 35, we have explained that to subtract a is the same as to add $-a$. Thus, $x-a=x+(-a)$. Similarly, to subtract an expression is to add it with its sign changed. The ordinary rule for subtracting one compound expression from another has already been explained in Art. 38, and has so far been applied to simple cases only. We shall now consider harder examples on subtraction.

Example 1. Subtract $ax+by+cz$ from $(b+c)y+(c+a)z+(a+b)x$.

Arranging like terms in x, y and z and applying the rule explained in Art. 38, the difference required is obtained. Thus,

$$\begin{array}{lcccc}
\text{The minuend} & = & (a+b)x+ & (b+c)y+ & (c+a)z \\
\text{The subtrahend} & = & ax+ & by+ & cz \\
\hline
\therefore\ \text{the difference} & = & bx+ & cy+ & az
\end{array}$$

[In the remainder,

the coefficient of $x=(a+b)-a=a+b-a=b$.

Similarly, the coefficients of y and z are c and a respectively.]

Example 2. Subtract $(b-c)^2yz+(c-a)^2zx+(a-b)^2xy$ from $(b+c)^2yz+(c+a)^2zx+(a+b)^2xy$.

$$\begin{array}{lcccc}
\text{The minuend} & = & (b+c)^2yz+ & (c+a)^2zx+ & (a+b)^2xy \\
\text{The subtrahend} & = & (b-c)^2yz+ & (c-a)^2zx+ & (a-b)^2xy \\
\hline
\therefore\ \text{the remainder} & = & 4bcyz+ & 4cazx+ & 4abxy
\end{array}$$

[In the remainder,

the coefficient of $yz=(b+c)^2-(b-c)^2$

$$=b^2+2bc+c^2-(b^2-2bc+c^2)$$

$$=b^2+2bc+c^2-b^2+2bc-c^2=4bc.$$

Similarly, the coefficients of zx and xy are $4ca$ and $4ab$ respectively.]

Example 3. Supply the omission in the following :

$$(2a+3b)x+(3b+4c)y+(4c+2a)z=(a+b)x+(b+c)y+(c+a)z+\{\quad\}.$$

Evidently, the omission can be obtained by subtracting $(a+b)x+(b+c)y+(c+a)z$ from $(2a+3b)x+(3b+4c)y+(4c+2a)z$. Proceeding as in examples 1 and 2 above, the result of subtraction can be easily found to be $(a+2b)x+(2b+3c)y+(3c+a)z$.

Example 4. Subtract $2\cdot5ax-3\cdot\dot{7}by-8\cdot3\dot{2}z$ from $3\frac{3}{4}ax+2\frac{4}{9}by+6\frac{88}{90}z$.

$$\begin{aligned}\text{The minuend} &= 3\tfrac{3}{4}ax+2\tfrac{4}{9}by+6\tfrac{88}{90}z\\ \text{The subtrahend} &= \underline{2\cdot5ax-3\cdot\dot{7}by-8\cdot3\dot{2}z}\\ \therefore\ \text{the remainder} &= \tfrac{5}{4}ax+\tfrac{56}{9}by+\tfrac{686}{45}z\end{aligned}$$

[In the remainder,

the coefficient of $ax=3\frac{3}{4}-2\cdot5=\frac{15}{4}-\frac{5}{2}=\frac{15-10}{4}=\frac{5}{4}$,

the coefficient of $by=2\frac{4}{9}-(-3\cdot\dot{7})=2\frac{4}{9}+3\cdot\dot{7}=\frac{22}{9}+\frac{34}{9}=\frac{56}{9}$,

the coefficient of $z=6\frac{88}{90}-(-8\cdot3\dot{2})=6\frac{88}{90}+8\cdot3\dot{2}=\frac{628}{90}+\frac{749}{90}$

$$=\frac{628+749}{90}=\frac{1372}{90}=\frac{686}{45}.\ ]$$

Note. *As in addition, fractional coefficients in the remainder must be simplified by Rules of Arithmetic.*

When compound expressions with brackets are to be subtracted, it is more convenient to retain the brackets, as in Examples 1—3.

EXERCISE 36

Subtract :

1. $-7x^5+6x^4y-8x^3y^2-13x^2y^3+9y^4$ from $3x^5-5x^4y+2x^3y^2-7x^2y^3+6y^4$.
2. $3m^3nx-10n^3xm+14x^3mn-20m^2n^2x-27n^2x^2m$ from $5m^3nx-17n^3xm+26x^3mn-13m^2n^2x-19n^2x^2m$.
3. $37x^6-28x^5y+43x^4y^2-54x^3y^3-67x^2y^4+84xy^5-93y^6$ from $48x^6-31x^5y-7x^4y^2-39x^3y^3-41x^2y^4+65xy^5-53y^6$.
4. $-2yzbc^2+4yz^2bc-2ax^4-9y^2zbc+3a^2x^3$ from $3ax^4-5a^2x^3+6yzbc^2-7y^2zbc+8yz^2bc$.

5. $19x^3z^5y - 15x^3y^5z + 27 + 11xyz^4 - 12x^2y^2z^2 - 19xy^3z^5$ from $25 - 16x^3y^5z - 17xy^3z^5 + 21x^3z^5y - 6x^2y^2z^2 + 8xyz^4$

6. $43x^3y^4z^2 - 23x^3y^2z^4 + 25x^4y^3z^2 - 66x^2y^4z^3 + 26x^2y^3z^4 + 35x^4y^2z^3$ from $29x^4y^3z^2 - 37x^3y^4z^2 + 54x^2y^3z^4 - 45x^3y^2z^4 - 67x^4y^2z^3 + 89x^2y^4z^3$.

7. $-29x^4y^3z^5 + 75x^5y^4z^3 + 13x^3y^5z^4 + 53x^3y^4z^5 - 94x^5y^3z^4 - 86x^4y^5z^3$ from $41x^3y^4z^5 - 87x^3y^5z^4 - 28x^4y^5z^3 + 63x^4y^3z^5 - 55x^5y^3z^4 + 37x^5y^4z^3$.

8. What must be added to $3x^2 - 5xy + 6y^2 + 7yz$ in order that the sum may be $-x^2 - y^2 - yz$?

9. What must be added to $-5x^3 + 13x^2y^2 - a^2bx + 5bxy^2 + 7xyab$ in order that the sum may be $x^3 + x^2y^2 + a^2bx - 2bxy^2 - 2xyab$?

10. What must be added to $5x^4 - 6x^3y + 7x^2y^2 - 8xy^3 - 19y^4$ in order that the sum may be $3x^4 + 5x^2y^2 - 12y^4$?

11. What must be added to $-5x^5 - 3x^4y + 6x^3y^2 + 17x^2y^3 + 13xy^4 - 21y^5$ in order that the sum may be $-7x^5 - 4x^3y^2 + 13x^2y^3 + 29y^5$?

12. What must be subtracted from $2a^2 + 5ab - 6b^2$ in order that the remainder may be $a^2 + 2b^2$?

13. What must be subtracted from $5x^2 - 6xy + 4y^2 - 8x - 10y + 15$ in order that the remainder may be $x^2 + 2xy + 3y^2 + 4x + 5y + 6$?

14. What must be subtracted from $3a^3 - 4a^2b + 5ab^2 - 8b^3$ in order that the remainder may be $a^3 - 2ab^2 + 7b^3$?

15. What must be subtracted from $-8x^3y + 4x^2y^2 - 11xy^3 + 12x^2 - 13y + 27$ in order that the remainder may be $4x^3y - 3x^2y^2 - 11xy^3 + 20x^2 - 30y + 56$?

16. From what expression must $3a^2 - 7ab - 8bc + 9b^2$ be subtracted in order that the remainder may be $2a^2 + 3ab + 3bc + 2b^2$?

17. From what expression must $-3x^3 + 5y^2 - 7xy + 8x - 9$ be subtracted in order that the remainder may be $x^3 - 8y^2 + 2xy - 11x + 7$?

18. From what expression must $-7a^3 - 8b^2c - 13ac^2 + 3b^3$ be subtracted in order that the remainder may be $4a^3 - 3b^2c + 7ac^2 - 8b^3$?

19. From what expression must $21x^3 - 37xy^2 + 42y^3 - 18x^2 + 19xy - 39$ be subtracted in order that the remainder may be $-25x^3 + 15xy^2 - 87y^3 + 7x^2 - 43xy + 24$?

Subtract :

20. $\frac{1}{12}x + \frac{27}{35}y + \frac{107}{37}z$ from $\frac{13}{24}x + \frac{213}{168}y + \frac{201}{15}z$.

21. $-\cdot35ax + \frac{13}{28}y + \cdot17mz$ from $-\frac{1}{20}ax + \frac{6}{7}y + \cdot8mz$.

22. $1\cdot17a^2cx + 2\cdot31c^2by - 63\cdot18c^3z$ from $32\cdot39c^2by + 2\cdot37a^2cx - 62\cdot73c^3z$.

23. $\frac{33}{15}a^{\frac{1}{2}}c^{\frac{3}{5}}x+\frac{35}{33}a^{\frac{3}{2}}b^{\frac{5}{4}}y+\frac{31}{26}b^{\frac{3}{8}}c^{\frac{5}{12}}z+2\cdot3lx+3\cdot5my+\frac{32}{7}nz$

from $3\cdot3lx+\frac{35}{44}a^{\frac{3}{2}}b^{\frac{5}{4}}y-\frac{3}{7}nz-\frac{3}{25}b^{\frac{3}{8}}c^{\frac{5}{12}}z-2\cdot5my-\frac{35}{38}a^{\frac{1}{2}}c^{\frac{3}{5}}x.$

24. Supply the omission in the following :

(i) $3\cdot2x+5\cdot3y+5\cdot4z-(\qquad)=2x+3y+6z$;

(ii) $17x+23y+\frac{121}{21}z=52x-1\cdot7y+\frac{40}{7}z-(\qquad)$;

(iii) $1\cdot\dot{2}a+15\cdot52l^2+16m^2+14p$

$=(\qquad)-(2\cdot\dot{2}a+3\cdot52l^2+4m^2+16p).$

Subtract :

25. $bc(b-c)+ca(c-a)+ab(a-b)$ from $bc(b+c)+ca(c+a)+ab(a+b)$.

26. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ from $bc(b-c)+ca(c-a)+ab(a-b)$.

27. $(b-c)^2+(c-a)^2+(a-b)^2$ from $2(a^2+b^2+c^2-ab-bc-ca)$.

28. $(1+a+a^2)x+(1+b+b^2)y+(1+c+c^2)z$

from $(1+a)^2x+(1+b)^2y+(1+c)^2z$.

29. A man earned $(ax+by+cz)$ rupees per month for a year and spent $(10ax+13cz)$ rupees during the same year. How many rupees will he be left with at the end of the year ?

30. If out of $(50x+71y+18z)$ sheep, $(13x+12y)$ and $(15y+8z)$ be sold and $(3z+23x)$ die, find the number of sheep left.

31. Subtract $\{0-(a+b+c)\}$ from $\{0+(a+b+c)\}$.

32. Find the difference between $x+y+z$ and the sum of $\{0+(x+y+z)\}$ and $\{0-(x+y+z)\}$.

33. Ram has $(3x+2y-5z)$ rupees and Shyama has $(x-2y-3z)$ rupees less than what Ram has. Find the amount Shyama has.

CHAPTER IX

HARDER MULTIPLICATION

76. We have explained the following rules of multiplication of Algebraic quantities in Chapter III.

(1) $a\times b=b\times a$, [Art. 42]

$abc=bca=cab$, etc., [Art. 43]

i.e., the value of a product is the same in whatever order the factor may be taken.

This is called the **Commutative Law** of multiplication.

(2) $(ab)\times c = a\times(bc) = b\times(ac) = a\times b\times c$. [Art. 43]

i.e., the factors of a product may be grouped in any way.

This principle is known as **Associative Law** of multiplication.

(3) $a(b+c) = ab+ac$. [Art. 47]

This is known as **Distributive Law** of multiplication.

(4) $a^m \times a^n = a^{m+n}$, where m and n are positive integers.

This is known as **Index Law** of multiplication.

We now proceed to consider products of compound expressions and harder examples on multiplication.

77. To prove that $(a+b)(c+d) = ac+ad+bc+bd$.

Putting x for $c+d$, we have

$$\begin{aligned}(a+b)(c+d) &= (a+b)x = x(a+b)\\ &= xa+xb \qquad \text{[Art. 47]}\\ &= ax+bx = a(c+d)+b(c+d)\\ &= ac+ad+bc+bd.\end{aligned}$$

Cor. Since, $a-b=a+(-b)$ and $c-d=c+(-d)$,

$$\begin{aligned}\therefore\quad (a-b)(c-d) &= \{a+(-b)\}\{c+(-d)\}\\ &= ac+a(-d)+(-b)c+(-b)(-d)\\ &= ac-ad-bc+bd.\end{aligned}$$

78. To prove that $(a+b+c+d+\cdots)(m+n+p+q+\cdots)$
$= a(m+n+p+q+\cdots)+b(m+n+p+q+\cdots)$
$+c(m+n+p+q+\cdots)+d(m+n+p+q+\cdots)+\&c.$

Putting x for $m+n+p+q+\cdots$, we have

$$\begin{aligned}&(a+b+c+d+\cdots)(m+n+p+q+\cdots)\\ &\quad = (a+b+c+d+\cdots)x\\ &\quad = ax+bx+cx+dx+\cdots\\ &\quad = a(m+n+p+q+\cdots)+b(m+n+p+q+\cdots)\\ &\qquad\quad +c(m+n+p+q+\cdots)+d(m+n+p+q+\cdots)+\&c.\end{aligned}$$

Thus, to multiply one multinomial expression by another we have to multiply every term of the one by every term of the other and take the algebraic sum of these partial products.

Example 1. Multiply $2a+3b$ by $4a+5b$.

$$\begin{aligned}(4a+5b)(2a+3b) &= (4a)(2a)+(4a)(3b)+(5b)(2a)+(5b)(3b)\\ &= 8a^2+12ab+10ab+15b^2 = 8a^2+22ab+15b^2.\end{aligned}$$

Example 2. Multiply $3x-7y$ by $2x-5y$.

$$\begin{aligned}(2x-5y)(3x-7y) &= (2x)(3x)+(2x)(-7y)+(-5y)(3x)+(-5y)(-7y)\\ &= 6x^2-14xy-15xy+35y^2\\ &= 6x^2-29xy+35y^2.\end{aligned}$$

EXERCISE 37

Multiply :

1. $2a+3b$ by $a+b$.
2. $2m-3n$ by $m-n$.
3. $a+b+c$ by $a+b+c$.
4. $a-b+c$ by $a-b+c$.
5. $a-b-c$ by $a-b-c$.
6. $a-2b-3c$ by $2a-b-c$.
7. $2x-3y-4z$ by $x-y-z$.
8. $-5x+2a-3b$ by $-x-a+b$.
9. $x^2+y^2+z^2$ by $x-y-z$.
10. $xy+yz+zx$ by $xy-yz-zx$.

79. Arrangement of an expression according to descending or ascending powers of some letter.

When the different terms of an expression contain different powers of any letter, if we arrange the terms in such a way that the term containing the highest power of that letter is put first on the left, the term containing the next highest power is put next ; and so on ; and the term which either contains the lowest power of that letter, or does not contain that letter at all is put last, then we are said to arrange the expression according to **descending** powers of the letter considered. If the order of the terms be reversed, the arrangement is said to be according to **ascending** powers of the letter. Thus, the expression $a^5x^3+3a^4xy-5a^3x^6y^2+4a^2x^4y^3-2ax^2y^4+x^5y^5$ as it stands may be considered as arranged either according to *descending* powers of a, or according to *ascending* powers of y, but if it is arranged as $-5a^3x^6y^2+x^5y^5+4a^2x^4y^3+a^5x^3-2ax^2y^4+3a^4xy$, it is arranged according to descending powers of x.

80. When one expression is to be multiplied by another, arrange both the multiplicand and the multiplier according to descending or ascending powers of some letter common to them, and proceed as exemplified below.

Example 1. Multiply a^2-b^2-ab by $ab-b^2+a^2$.

$$\begin{array}{lllll}\text{Multiplicand} & =a^2-ab-b^2 & & & \\ \text{Multiplier} & =\underline{a^2+ab-b^2} & & & \\ \text{Product by } a^2 & =a^4-a^3b- & a^2b^2 & & \\ \text{Product by } +ab & = \quad +a^3b- & a^2b^2 & -ab^3 & \\ \text{Product by } -b^2 & = & \underline{-\ a^2b^2} & \underline{+ab^3} & \underline{+b^4} \\ \therefore\ \text{complete product} & =a^4 & -3a^2b^2 & & +b^4\end{array}$$

Note. *The process shown above may be described as follows :*

The multiplier has been placed under the multiplicand after having arranged them both according to descending powers of a, and a line has been drawn below the multiplier. The successive products of the multiplicand by the different terms of the multiplier beginning from the left have been placed in different horizontal rows in such a manner that each set of like terms may be in the same vertical column. A line having been now drawn below the lowest of the rows, the complete product has been found by writing down the sum of each vertical column immediately below it.

Example 2. Multiply $2a^2-3x^2-5ax$ by $-3x^2+2a^2+5ax$.

Arranging the multiplicand and the multiplier according to ascending powers of x, we have

$$\begin{array}{lrrrrr}
\text{Multiplicand} = & 2a^2 & -5ax & -3x^2 & & \\
\text{Multiplier} = & 2a^2 & +5ax & -3x^2 & & \\
\hline
 & 4a^4 & -10a^3x & -6a^2x^2 & & \\
 & & +10a^3x & -25a^2x^2 & -15ax^3 & \\
 & & & -6a^2x^2 & +15ax^3 & +9x^4 \\
\hline
\text{Product} = & 4a^4 & & -37a^2x^2 & & +9x^4
\end{array}$$

Example 3. Multiply $2a^3b-5ab^3-a^4+3a^2b^2$
by $2a^4-3a^3b+4ab^3-5a^2b^2$.

Arranging the multiplicand and the multiplier according to descending powers of a, we have

$$\begin{array}{lrrrrrrr}
\text{Multiplicand} = & -a^4 & +2a^3b & +3a^2b^2 & -5ab^3 & & & \\
\text{Multiplier} = & 2a^4 & -3a^3b & -5a^2b^2 & +4ab^3 & & & \\
\hline
 & -2a^8 & +4a^7b & +6a^6b^2 & -10a^5b^3 & & & \\
 & & +3a^7b & -6a^6b^2 & -9a^5b^3 & +15a^4b^4 & & \\
 & & & +5a^6b^2 & -10a^5b^3 & -15a^4b^4 & +25a^3b^5 & \\
 & & & & -4a^5b^3 & +8a^4b^4 & +12a^3b^5 & -20a^2b^6 \\
\hline
\text{Product} = & -2a^8 & +7a^7b & +5a^6b^2 & -33a^5b^3 & +8a^4b^4 & +37a^3b^5 & -20a^2b^6
\end{array}$$

Note. *In this example, the multiplicand and the multiplier are each homogeneous and of the 4th degree, whilst the product also is homogeneous and of the 8th degree. Similarly, it may be seen that whenever the expressions to be multiplied together are homogeneous, the product also is homogeneous, and the degree of the product is equal to the sum of the degrees of the expressions. This law is of great importance in testing the accuracy of a multiplication when the multiplicand and the multiplier are both homogeneous, for in this case if the product obtained does not turn out to be homogeneous, we are sure there has been an error somewhere.*

Example 4. Multiply mx^2-nx-p by x^2+px-1.

$$\begin{array}{rl}
\text{Multiplicand} & = mx^2-nx-p \\
\text{Multiplier} & = x^2+px-1 \\
 & mx^4-nx^3-px^2 \\
 & \qquad +pmx^3-pnx^2-p^2x \\
 & \qquad\qquad -mx^2+nx+p \\
\text{Product} & = mx^4-(n-pm)x^3-(p+pn+m)x^2+(n-p^2)x+p
\end{array}$$

Example 5. Multiply $\frac{11}{8}ax^3+\frac{7}{15}b^2x^2y+3{\cdot}5cxy^2+1{\cdot}05g^2y^3$ by $2lx^2+3{\cdot}5mxy+1{\cdot}5ny^2$.

N. B. To find the product of expressions in which both vulgar fractions and decimal fractions occur as coefficients, it is convenient to reduce all the coefficients to fractions of the same kind (either all vulgar or all decimal) and apply the rule of multiplication.

In this example, as $\frac{7}{15}$, when reduced to a decimal fraction, will involve a very large number of decimal places, we reduce all the coefficients of the multiplicand as also of the multiplier to vulgar fractions.

$$\begin{array}{rl}
\text{Multiplicand} & = \frac{11}{8}ax^3+\frac{7}{15}b^2x^2y+\frac{7}{2}cxy^2+\frac{21}{20}g^2y^3 \\
\text{Multiplier} & = 2lx^2+\frac{7}{2}mxy+\frac{3}{2}ny^2 \\
 & \frac{22}{8}alx^5+\frac{14}{15}b^2lx^4y+7clx^3y^2+\frac{21}{10}g^2lx^2y^3 \\
 & \qquad +\frac{77}{16}amx^4y+\frac{49}{30}b^2mx^3y^2+\frac{49}{4}cmx^2y^3+\frac{147}{40}g^2mxy^4 \\
 & \qquad\qquad +\frac{33}{16}anx^3y^2+\frac{21}{30}b^2nx^2y^3+\frac{21}{4}cnxy^4+\frac{63}{40}g^2ny^4 \\
\text{Product} & = \frac{22}{8}alx^5+(\frac{14}{15}b^2l+\frac{77}{16}am)x^4y+(7cl+\frac{49}{30}b^2m+\frac{33}{16}an)x^3y^2 \\
 & \quad +(\frac{21}{10}g^2l+\frac{49}{4}cm+\frac{21}{30}b^2n)x^2y^3+(\frac{147}{40}g^2m+\frac{21}{4}cn)xy^4+\frac{63}{40}g^2ny^5.
\end{array}$$

Example 6. Multiply together a^2-ab+b^2, a^2+ab+b^2 and $a^4-a^2b^2+b^4$.

$$\begin{array}{rl}
\text{(i)} & a^2-ab+b^2 \\
 & a^2+ab+b^2 \\
 & a^4-a^3b+a^2b^2 \\
 & \quad +a^3b-a^2b^2+ab^3 \\
 & \qquad +a^2b^2-ab^3+b^4 \\
 & a^4 \qquad +a^2b^2 \qquad +b^4\,;
\end{array}$$

$$\begin{array}{rl}
\text{(ii)} & a^4+a^2b^2+b^4 \\
 & a^4-a^2b^2+b^4 \\
 & a^8+a^6b^2+a^4b^4 \\
 & \quad -a^6b^2-a^4b^4-a^2b^6 \\
 & \qquad +a^4b^4+a^2b^6+b^8 \\
 & a^8 \qquad +a^4b^4 \qquad +b^8
\end{array}$$

Thus, the required product $=a^8+a^4b^4+b^8$.

Note. *When the number of factors in a product is more than two, the product is called the* ***continued product*** *of those factors.*

The factors should be arranged in a suitable order so as to lessen the trouble of multiplication in such products.

81. Detached Coefficients. If both the multiplier and the multiplicand contain powers of the same algebraic quantity or be homogeneous expressions of the same quantities, the labour of multiplication may be lessened by detaching the coefficients and placing them in proper relative positions. *If any power be missing, zero must be inserted as its coefficient.*

The following examples will illustrate the process.

Example 1. Multiply x^2-4x+4 by $x-2$.

$$\begin{array}{llll}
x^2 & -4x & +4 & \\
x & -2 & & \\
\hline
1 & -4 & +4 & \\
 & -2 & +8 & -8 \\
\hline
\end{array}$$

$\therefore$ the product $=x^3-6x^2+12x-8$.

Example 2. Multiply $3x^3-2x+4$ by $x+5$.

$$\begin{array}{lllll}
3x^3 & +0x^2 & -2x & +4 & \\
x & +5 & & & \\
\hline
3 & +0 & -2 & +4 & \\
 & +15 & +0 & -10 & +20 \\
\hline
\end{array}$$

$\therefore$ the product $=3x^4+15x^3-2x^2-6x+20$.

81A. To find the coefficient of any particular term in the product without actual multiplication.

The process is being explained below with an illustration.

Example. Find the coefficient of x^5, without actual multiplication, in the product of $(3x^4+4x^3-2x^2+3x+2)$ by $(2x^4-5x^3+2x^2-4x+1)$.

Multiplicand $=3x^4+4x^3-2x^2+3x+2$.

Multiplier $=2x^4-5x^3+2x^2-4x+1$.

To find out the coefficient of x^5 in the product, we are to find out which term of the multiplicand is to be multiplied by a particular term of the multiplier so that the power of x is 5.

So we should proceed as follows :

$$\begin{aligned} 3x^4 \times (-4x) &= -12x^5, \\ 4x^3 \times 2x^2 &= 8x^5, \\ (-2x^2) \times (-5x^3) &= 10x^5, \\ (3x) \times (2x^4) &= 6x^5. \end{aligned}$$

Now the algebraic sum of the coefficients of the products will be the coefficient of x^5 in the product.

$\therefore$ the required coefficient $= -12 + 8 + 10 + 6 = 12$.

EXERCISE 38

Multiply :

1. $25b^2 + 30ab + 9a^2$ by $3a - 5b$. 2. $2a - 3b + 4c$ by $2a + 3b - 4c$.
3. $x^2 - x + 2$ by $x^2 + x + 2$. 4. $a^2 - 2ab + b^2$ by $a^2 + 2ab + b^2$.
5. $x^4 + x^2 + 1$ by $x^4 - x^2 + 1$. 6. $y^3 - x^2y^2 + x^3$ by $x^3 + x^2y^2 + y^3$.
7. $m^4 - m^2n^2 + n^4$ by $m^2 + n^2$. 8. $p^2q^2 + p^4 + q^4$ by $-q^2 + p^2$.
9. $a^3 + 5ab^2 - 6a^2b$ by $5b^2 + a^2 + 6ab$.
10. $x^3 - 3x^2 + 3x - 1$ by $x^2 + 3x + 1$.
11. $2ax^3 + a^4 + 3a^2x^2 + x^4 + 2a^3x$ by $a^2 + x^2 - 2ax$.
12. $a^3 + 3a^2b + b^3 + 3ab^2$ by $3ab^2 - b^3 + a^3 - 3a^2b$.
13. $x^2 - 11 + x^4 - 4x + 2x^3$ by $3 + x^2 - 2x$.
14. $1 + 2x + x^4 + 2x^3 + 3x^2$ by $1 + x^2 - 2x$.
15. $b^4 + a^2b^2 + a^3b + a^4 + ab^3$ by $a^2b^2 - a^3b + b^4 - ab^3 + a^4$.
16. $x^2 - xy - xz + y^2 - yz + z^2$ by $x + y + z$.
17. $a^2 + b^2 + c^2 - bc - ca - ab$ by $a + b + c$.
18. $5a^2b + 4b^3 + 2a^3 - 3ab^2$ by $2ab^2 - 3a^2b + a^3 - 5b^3$.
19. $ax^2 + bx - c$ by $px - q$. 20. $mx^2 - nx - r$ by $nx - r$.
21. $ax^2 - bx + c$ by $x^2 - bx - c$.
22. $ax^3 - bx^2 + cx - d$ by $bx^2 - cx + d$.
23. $px^2 - (q - r)x + s$ by $mx^2 - nx - s$.
24. $ax^2 + 2hxy + by^2$ by $lx + my + n$.
25. $l^2x^2 + m^2xy + n^2y^2 + 2g^2x + 2f^2y + c^2$ by $px^2 + qx + r$.
26. $\frac{7}{6}x^3 + \frac{2}{3}x^2y + \frac{3}{5}xy^2 + \frac{3}{7}y^3$ by $\frac{2}{7}x^2 + \frac{5}{8}xy + \frac{7}{8}y^2$.
27. $\frac{3}{7}x^4 + \frac{5}{7}x^3y + \frac{2}{7}x^2y^2 + \frac{1}{7}xy^3 + \frac{11}{7}y^4$ by $\frac{7}{8}x^2 + \frac{7}{11}y^2$.
28. $1{\cdot}5x^5 + 2{\cdot}3x^3 + 1{\cdot}23x^4 + 3{\cdot}25x^2 + 5$ by ${\cdot}27x^3 + 1{\cdot}39x + 9$.
29. ${\cdot}057a^3 + 1{\cdot}025a^2b + 2{\cdot}021ab^2 + 2{\cdot}8b^3$ by $7a^2 + 2ab + 9b^2$.

30. $2\cdot3x^3+3\cdot15x^2y+1\cdot17xy^2+2\cdot07y^3$ by $lx^2+mxy+ny^2$.

31. $\frac{5}{3}ax^3+\frac{7}{3}bx^2y+\frac{3}{2}cxy^2+2dy^3$ by $\frac{3}{2}ax^2-\frac{8}{7}bxy+\frac{4}{3}cy^2$.

32. $1\cdot5am^3-1\cdot2bm^2n+1\cdot3cmn^2-1\cdot6dn^3$
by $1\cdot5am^3+1\cdot2bm^2n+1\cdot3cmn^2+1\cdot6dn^3$.

Find the continued product of :

33. $2a+3b$, $2a-3b$ and $4a^2+9b^2$.

34. $5ax+6by$, $5ax-6by$ and $25a^2x^2+36b^2y^2$.

35. $x^8+x^4y^4+y^8$, x^2+y^2, $x+y$ and $x-y$.

36. $x^2+3xy+5y^2$, $x^2-3xy+5y^2$ and $x^4-x^2y^2+25y^4$.

37. $a^{12}x^{12}+a^6b^6x^6y^6+b^{12}y^{12}$, $a^4x^4+a^2b^2x^2y^2+b^4y^4$, $ax+by$ and $ax-by$.

Assuming $a^m\times a^n=a^{m+n}$ to be true for *all* values of m and n, prove that :

38. $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$. $\left[a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}}=a^1=a.\right]$

39. $x^{\frac{1}{3}}\times x^{\frac{2}{3}}=x$.

40. $a^{\frac{1}{3}}\times a^{\frac{1}{3}}\times a^{\frac{1}{3}}=a$. $\left[a^{\frac{1}{3}}\times a^{\frac{1}{3}}\times x^{\frac{1}{3}}=a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=a^1=a.\right]$

41. $a^{\frac{1}{4}}=\sqrt[4]{a}$.
$\left[\left(a^{\frac{1}{4}}\right)^4=a^{\frac{1}{4}}\times a^{\frac{1}{4}}\times a^{\frac{1}{4}}\times a^{\frac{1}{4}}=a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}=a^1=a;\ \therefore\ a^{\frac{1}{4}}=\sqrt[4]{a}.\right]$

42. $x^{\frac{2}{3}}=\sqrt[3]{x^2}$.
43. $z^{\frac{3}{4}}=\sqrt[4]{z^3}$.

44. $c^{\frac{2}{3}}\times c^{\frac{4}{3}}\times c^{\frac{3}{3}}=c^3$.
45. $y^2\times y^{\frac{3}{2}}\times y^{\frac{7}{2}}=y^7$.

46. $x^{-2}\times x^5=x^3$. $\left[x^{-2}\times x^5=x^{-2+5}=x^3.\right]$

47. $z^{\frac{3}{2}}\times z^{-\frac{1}{2}}=z$.

48. $a^{-\frac{3}{2}}=\sqrt{a^{-3}}$. $\left[\left(a^{-\frac{3}{2}}\right)^2=a^{-\frac{3}{2}}\times a^{-\frac{3}{2}}=a^{-\frac{3}{2}-\frac{3}{2}}=a^{-3};\ \therefore a^{-\frac{3}{2}}=\sqrt{a^{-3}}.\right]$

49. $b^{-\frac{5}{3}}=\sqrt[3]{b^{-5}}$.
50. $x^{-\frac{5}{3}}\times x^{-\frac{4}{3}}=x^{-3}$

Write down the product of :

51. $-3x^{\frac{1}{3}}$ and $2x^{\frac{2}{3}}$.
52. $5y^{\frac{3}{2}}$ and $-\frac{2}{3}y^{\frac{5}{2}}$.

53. $2x^{\frac{1}{2}}y^{\frac{1}{3}}$ and $3x^{\frac{3}{2}}y^{\frac{1}{2}}$.
54. $-5xy^{\frac{3}{2}}$ and $-3x^{\frac{2}{3}}y^{\frac{1}{2}}$.

55. $4a^{-2}b^3$ and $-\frac{3}{4}a^3b^{-5}$.
56. $\frac{2}{3}a^{\frac{3}{2}}y^3$ and $-\frac{4}{5}a^{\frac{3}{2}}y^{-4}$.

57. $-4a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{3}{4}}$ and $-3a^{\frac{3}{2}}b^{\frac{4}{3}}c^{\frac{5}{4}}$.
58. $-5x^{\frac{2}{3}}y^{\frac{3}{2}}z^{\frac{4}{3}}$ and $-3x^{\frac{1}{3}}y^{\frac{2}{3}}z^{-\frac{1}{3}}$.

59. $-6a^{\frac{5}{6}}b^{-\frac{3}{4}}c^{-\frac{2}{7}}$ and $5a^{\frac{1}{6}}b^{\frac{7}{4}}c^{-\frac{5}{7}}$.

60 $-4a^{\frac{5}{6}}x^{\frac{5}{6}}y^{-\frac{4}{5}}$ and $-19a^{\frac{1}{2}}x^{-\frac{3}{2}}y^{-\frac{6}{5}}$.

Multiply :

61. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

62. $a^{\frac{1}{2}}-b^{\frac{1}{2}}$ and $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

63. $3x^{\frac{2}{3}}-4y^{\frac{1}{3}}$ by $3x^{\frac{2}{3}}+4y^{\frac{1}{3}}$.

64. $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

65. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$.

66. $a^{\frac{3}{2}}-a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$ by $a^{\frac{3}{2}}+a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$.

67. $2x^{\frac{4}{3}}-5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$ by $2x^{\frac{4}{3}}+5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$.

68. $a^{\frac{5}{2}}+a^{2}b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

69. $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$.

70. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

71. $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

72. $a^{2n}-a^{n}x^{n}+x^{2n}$ by $a^{n}+x^{n}$.

73. $a^{-3}-4a^{-2}b+4a^{-1}b^{2}-b^{3}$ by $a^{-2}-2a^{-1}b+b^{2}$.

74. $x^{-3}+3x^{-\frac{3}{2}}y^{\frac{3}{2}}+2y^{3}$ by $x^{-3}-3x^{-\frac{3}{2}}y^{\frac{3}{2}}+2y^{3}$.

75. $2a^{-5}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}-5b^{-3}$ by $2a^{-5}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}+5b^{-3}$.

Apply the method of detached coefficients to find the product of :

76. $2x^{2}+3x+9$ and $3x+5$.

77. $x^{2}-2x-15$ and $2x-3$.

78. $3x^{3}+5x+6$ and $x^{2}+3x+2$.

79. $x^{3}+px+r$ by $px+q$.

80. $\frac{1}{3}x^{4}+\frac{2}{3}x^{2}+5$ by $\frac{2}{3}x^{2}+x+2$.

Without actual multiplication find the coefficient of :

81. x^{3} in $(x^{2}+2x+3)(2x^{2}-3x+1)$.

82. x^{4} in $(x^{3}-3x^{2}+2x+1)(x^{3}+2x^{2}-5x-2)$.

83. x^{6} and x^{3} in $(x^{4}-2x^{3}+x^{2}-3x+2)(2x^{4}+3x^{3}-2x^{2}+2x-3)$.

CHAPTER X

HARDER DIVISION

82. The principal rules for division explained in Chapter III, may be stated as follows :

(i) $a \div b = a \times \frac{1}{b}$;

(ii) $a \div b \div c = a \div bc$;

(iii) $a \div b \times c = a \times c \div b$;

and (iv) $a^m \div a^n = a^{m-n}$, where m and n are positive integers and $m > n$.

The rule (iv) is called the **Index Rule** for division.

The **Law of signs** and the rule for division of a monomial or a multinomial expression by a monomial have been explained in Arts. 50-52. We now propose to consider division of one multinomial expression by another.

83. Division of one multinomial expression by another.

Let us consider a particular example.

$$\begin{aligned} \text{We have } (2a^2+3ab+4b^2)(a+3b) &= 2a^2(a+3b)+3ab(a+3b)+4b^2(a+3b) \\ &= 2a^3+9a^2b+13ab^2+12b^3. \end{aligned}$$

$$\begin{aligned} \text{Hence, } (2a^3+9a^2b+13ab^2+12b^3) \div (a+3b) &= 2a^2+3ab+4b^2. \end{aligned}$$

Now, let us review this result and see in what way, given the dividend and the divisor, we can discover the quotient. The points noticed are :

(i) The dividend and the divisor *both* stand arranged according to descending powers of a common letter, namely, a.

(ii) The *first* term of the quotient, namely, $2a^2 = 2a^3 \div a$, *i.e.*, $=$(the 1st term of the dividend)$\div$(the 1st term of the divisor).

(iii) If we subtract $2a^2(a+3b)$ from the dividend, the remainder is $3a^2b+13ab^2+12b^3$, and the *second* term of the quotient, namely, $3ab = 3a^2b \div a$, *i.e.*, $=$(the 1st term of this remainder)$\div$(the 1st term of the divisor).

(iv) If we subtract $3ab(a+3b)$ from the above remainder, the new remainder is $4ab^2+12b^3$, and the *third* term of the quotient, namely, $4b^2 = 4ab^2 \div a$, *i.e.*, =(the 1st term of this remainder)÷(the 1st term of the divisor).

(v) If we subtract $4b^2(a+3b)$ from the preceding remainder, nothing remains and the division is complete.

The process noted above can be shown as follows :

$$\begin{array}{l|l|l}
a+3b & 2a^3+9a^2b+13ab^2+12b^3 & 2a^2+3ab+4b^2 \\
 & \underline{2a^3+6a^2b} & \\
 & \quad 3a^2b+13ab^2+12b^3 & \\
 & \quad \underline{3a^2b+\ 9ab^2} & \\
 & \qquad\quad 4ab^2+12b^3 & \\
 & \qquad\quad \underline{4ab^2+12b^3} &
\end{array}$$

Hence, we deduce the following rule :

Arrange both the dividend and the divisor according to the descending powers of some common letter and place them in a line as in the process of Division in Arithmetic.

Divide the first term of the dividend by the first term of the divisor and write down the result as the first term of the quotient. Multiply the divisor by the quantity thus found and subtract the product from the dividend.

Regard the remainder as a new dividend and see if it is arranged according to the descending powers of the common letter. Divide its first term by the first term of the divisor and write down the result as the next term of the quotient. Multiply the divisor by this term and subtract the product from the new dividend.

Then go on similarly with the successive remainders until there is no remainder.

Note. *That the rule stated above gives us a correct result is evident. For, the different quantities, that are one by one subtracted from the dividend, being the partial product of the divisor by successive terms of the quotient, their sum is equal to the product of the divisor by the whole quotient ; and as this sum is clearly equal to the dividend, the dividend is equal to the product of the divisor by the quotient, and this is what it should be.*

Example 1. Divide $x^4-4x^2+12x-9$ by x^2-2x+3.

Both the dividend and the divisor, as they are, are arranged according to descending powers of x. Hence, we may proceed at once as follows

$$\begin{array}{r|rrrrr|l}
x^2-2x+3 & x^4 & & -4x^2 & +12x & -9 & x^2+2x-3\\
 & x^4 & -2x^3 & +3x^2 & & & \\ \cline{2-6}
 & & 2x^3 & -7x^2 & +12x & -9 & \\
 & & 2x^3 & -4x^2 & +6x & & \\ \cline{3-6}
 & & & -3x^2 & +6x & -9 & \\
 & & & -3x^2 & +6x & -9 & \\
\end{array}$$

Thus, the required quotient $=x^2+2x-3$.

Note. *In the dividend it must be noticed that the term containing x^3 is wanting and hence the second term which contains x^2, has been put a little apart from the first as if leaving unoccupied the place of the absent term. This point should be attended to, although not strictly required, for the purpose of having like terms placed under one another; for instance, in the above example, if the second term of the dividend stood close to the first, $-2x^3$ would come under $-4x^2$, and $3x^2$ under $12x$, and this might confuse the beginner or otherwise lessen the neatness of the process.*

Example 2. Divide $16x^4+36x^2+81$ by $4x^2+6x+9$.

$$\begin{array}{r|rrrrr|l}
4x^2+6x+9 & 16x^4 & & +36x^2 & & +81 & 4x^2-6x+9\\
 & 16x^4 & +24x^3 & +36x^2 & & & \\ \cline{2-6}
 & & -24x^3 & & & +81 & \\
 & & -24x^3 & -36x^2 & -54x & & \\ \cline{3-6}
 & & & 36x^2 & +54x & +81 & \\
 & & & 36x^2 & +54x & +81 & \\ \cline{4-6}
\end{array}$$

Thus, the required quotient $=4x^2-6x+9$.

Example 3. Divide $x^6-4x^4-2x^3+3x^2+8x-12$ by x^2-4.

N. B. It is not essential to arrange the dividend and the divisor according to descending powers of some letter common to them; the arrangements may as well be according to ascending powers of that letter. The only thing indispensable is that both the expressions should be arranged in the same order, be it descending or ascending. For instance, let us work out the present example by arranging the expressions in the ascending order of the powers of x.

$$\begin{array}{r|rrrrrr|l}
-4+x^2 & -12 & +8x & +3x^2 & -2x^3 & -4x^4 & +x^6 & 3-2x+x^4\\
 & -12 & & +3x^2 & & & & \\ \cline{2-7}
 & & 8x & & -2x^3 & -4x^4 & +x^6 & \\
 & & 8x & & -2x^3 & & & \\ \cline{3-7}
 & & & & & -4x^4 & +x^6 & \\
 & & & & & -4x^4 & +x^6 & \\ \cline{6-7}
\end{array}$$

Thus, the required quotient $=3-2x+x^4$.

Example 4. Divide $a^2b^2+2abc^2-a^2c^2-b^2c^2$ by $ab+ac-bc$.

The dividend, when arranged according to descending powers of a, becomes $(b^2-c^2)a^2+2bc^2.a-b^2c^2$.

The divisor, when so arranged, becomes $(b+c)a-bc$.

Thus, the dividend has become a trinomial and the divisor a binomial.

$$\begin{array}{r|l|l} (b+c)a-bc & (b^2-c^2)a^2+2bc^2.a-b^2c^2 & (b-c)a+bc \\ & \underline{(b^2-c^2)a^2-(b^2c-bc^2)a} & \\ & \quad (b^2c+bc^2)a-b^2c^2 & \\ & \quad \underline{(b^2c+bc^2)a-b^2c^2} & \end{array}$$

Thus, the required quotient $=ab-ac+bc$.

Example 5. Divide $a^3+b^3-c^3+3abc$ by $a+b-c$.

The dividend and the divisor, arranged according to descending powers of a, become respectively $a^3+3bc.a+(b^3-c^3)$ and $a+(b-c)$.

Thus the dividend has become a trinomial and the divisor a binomial.

$$\begin{array}{r|l|l} a+(b-c) & a^3 \qquad\qquad +3bc.a+(b^3-c^3) & a^2-(b-c)a+(b^2+bc+c^2) \\ & \underline{a^3+(b-c)a^2} & \\ & \quad -(b-c)a^2+3bc.a+(b^3-c^3) & \\ & \quad \underline{-(b-c)a^2-(b-c)^2.a} & \\ & \qquad (b^2+bc+c^2)a+(b^3-c^3) & \\ & \qquad \underline{(b^2+bc+c^2)a+(b^3-c^3)} & \end{array}$$

Thus, the required quotient $=a^2+b^2+c^2-ab+ac+bc$.

Example 6. Divide $(b-c)a^3+(c-a)b^3+(a-b)c^3$ by $a^2-ab-ac+bc$.

Let us arrange the dividend and the divisor according to descending powers of a.

The dividend $=(b-c)a^3-b^3a+c^3a+b^3c-bc^3$

$=(b-c)a^3-(b^3-c^3)a+bc(b^2-c^2)$.

The divisor $=a^2-(b+c)a+bc$.

Thus, the dividend has become a trinomial and the divisor also a trinomial.

$$\begin{array}{r|l|l} a^2-(b+c)a+bc & (b-c)a^3-(b^3-c^3)a \quad +bc(b^2-c^2) & (b-c)a+(b^2-c^2) \\ & \underline{(b-c)a^3-(b^2-c^2)a^2+bc(b-c)a} & \\ & (b^2-c^2)a^2-(b^3+b^2c-bc^2-c^3)a+bc(b^2-c^2) & \\ & \underline{(b^2-c^2)a^2-(b^3+b^2c-bc^2-c^3)a+bc(b^2-c^2)} & \end{array}$$

Thus, the required quotient $=ab-ac+b^2-c^2$.

Note. *It must be noted that the expressions which are enclosed within brackets as coefficients of different powers of a are all arranged according to descending powers of b. Such arrangements add to the neatness of the process and lessen the chance of confusion.*

EXERCISE 39

Divide:

1. $x^2-9x+14$ by $x-7$.
2. $3x^2-17x+10$ by $3x-2$.
3. $12x^2-8x-32$ by $4x-8$.
4. $55x^2-67x-14$ by $11x+2$.
5. $2a^2-7ab+6b^2$ by $a-2b$.
6. $x^4+x^2y^2+y^4$ by x^2+xy+y^2.
7. $4x^2-9a^2$ by $2x+3a$.
8. x^3+a^3 by $x+a$.
9. $a^3-a^2b-7ab^2+3b^3$ by $a-3b$.
10. $\frac{1}{2}x^3+\frac{23}{10}x^2+\frac{42}{5}x+18$ by $\frac{1}{2}x^2+\frac{4}{5}x+6$.
11. $\frac{3}{2}x^3-\frac{15}{8}x^2+\frac{67}{48}x-\frac{5}{12}$ by $\frac{3}{4}x^2-\frac{9}{16}x+\frac{5}{12}$.
12. $\frac{1}{6}a^3y^3-\frac{5}{24}a^2y^2b+\frac{67}{432}ayb^2-\frac{5}{108}b^3$ by $\frac{a^2}{12}y^2-\frac{ab}{16}y+\frac{5}{108}b^2$.
13. $\frac{7}{2}a^3m^3+\frac{161}{10}a^2m^2n+\frac{294}{5}amn^2+126n^3$ by $\frac{7}{2}a^2m^2+\frac{28}{5}amn+42n^2$.
14. $\frac{4}{9}x^4-x^2y^2+\frac{8}{3}xy^3-\frac{16}{9}y^4$ by $\frac{2}{9}x^2-\frac{xy}{3}+\frac{4}{9}y^2$.
15. $\frac{1}{7}y^5-\frac{3}{7}xy^4+\frac{2}{21}x^2y^3+\frac{5}{21}x^3y^2-\frac{1}{3}x^4y+\frac{2}{21}x^5$ by $\frac{1}{21}y^2-\frac{1}{7}xy+\frac{1}{21}x^2$.
16. $\frac{13}{12}mn^3+\frac{1}{6}m^2n^2+\frac{m^4}{2}-\frac{1}{12}m^3n+\frac{1}{3}n^4$ by $\frac{1}{6}mn+\frac{1}{8}m^2+\frac{1}{24}n^2$.
17. $\frac{2}{3}a^2y^3+\frac{1}{4}y^5+\frac{a^5}{12}-\frac{3}{4}a^3y^2-\frac{1}{6}ay^4-\frac{1}{12}a^4y$ by $\frac{1}{6}ay-\frac{1}{4}y^2+\frac{1}{12}a^2$.
18. If $x+y+z=-3a$, find the quotient when
$(2x-y-z)(2y-z-x)(2z-x-y)$ is divided by $a^2+a(x+y)+xy$.

Divide:

19. $\frac{1}{3}[(x-y)^3+(y-z)^3+(z-x)^3]$ by $(x-y)(y-z)$.
20. $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.
21. $2x^3y^3+y^6+x^6$ by $2xy+x^2+y^2$.
22. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$ by $x+c$.
23. $x^3+(b-c-a)x^2+(ca-ab-bc)x+abc$ by $x^2+(b-a)x-ab$.
24. $a^3+a^2b+a^2c-abc-b^2c-bc^2$ by a^2-bc.
25. $a^2(b+c)-b^2(c+a)+c^2(a+b)+abc$ by $a-b+c$.
26. $a^2(b+c)+b^2(a-c)+c^2(a-b)+abc$ by $a+b+c$.
27. $x^3-2ax^2+(a^2-ab-b^2)x+a^2b+ab^2$ by $x-a-b$.
28. $a^3+b^3+c^3-3abc$ by $a+b+c$.
29. $x^3+y^3-1+3xy$ by $x+y-1$.
30. $x^3-8y^3-27z^3-18xyz$ by $x-2y-3z$.
31. $x^3-y^3+z^3+3xyz$ by $x-y+z$.
32. $8x^3-27y^3-z^3-18xyz$ by $4x^2+9y^2+z^2+6xy+2xz-3yz$.
33. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $a-b$.

34. $(x^2-bx+ca)(x-b)(x+a)+(x-b+c)x^2$ by $(x+a)(x-b)$.
35. $c(ab-x^2)+(a-c)(x-c)x+x(x^2-ab)$ by $(x-b)(x-c)$.
36. $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $ab+bc-ac-b^2$.
37. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$ by $a^2b-bc^2-ac^2+a^2c$.
38. $xy^3+2y^3z-xy^2z+xyz^2-x^3y-2yz^3+x^3z-xz^3$ by $y+z-x$.
39. $b(x^3+a^3)+x(x^3-a^3)+a^3(x+a)$ by $(a+b)(x+a)$.
40. $(a-b)^2c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$ by $(a-b)c^2-(c-a)b^2$.

[Arrange the given expressions according to descending powers of c.]

41. $(ax+by)^3+(ax-by)^3-(ay-bx)^3+(ay+bx)^3$
by $(a+b)^2x^2-3ab(x^2-y^2)$.
[C. U. Entr., 1888]

[Simplify the dividend and the divisor and then arrange the two expressions according to descending powers of x.]

42. $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+4xyz$
by $1+xy+yz+zx$.
[C. U. Entr., 1878]

[Arrange the expressions according to descending powers of x.]

43. $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$.
[B. U. Entr., 1884]

Assuming the formula $a^m \div a^n = a^{m-n}$ to be true for *all* values of m and n, show that :

44. $a^0=1$. [$a^0=a^{m-m}=a^m \div a^m=1$.]

45. $a^{-n}=\frac{1}{a^n}$. [$a^{-n}=a^{0-n}=a^0 \div a^n=1 \div a^n$.]

46. $x^{\frac{5}{3}} \div x^{\frac{2}{3}}=x$.
47. $x^{-\frac{3}{4}} \div x^{-\frac{7}{4}}=x$.

Divide :

48. $a^2b^{\frac{2}{3}}$ by $a^{-1}b^{-\frac{1}{3}}$.
49. $a^{-2}b^{\frac{1}{2}}c^{\frac{5}{3}}$ by $a^{-3}b^{\frac{2}{3}}c^2$.
50. $15xyz$ by $-5x^{\frac{2}{3}}y^{\frac{3}{2}}z^{\frac{4}{3}}$.
51. $9x^{\frac{4}{3}}-16y^{\frac{2}{3}}$ by $3x^{\frac{2}{3}}+4y^{\frac{1}{3}}$.
52. $a+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.
53. $a^3+a^{\frac{3}{2}}b^{\frac{3}{2}}+b^3$ by $a^{\frac{3}{2}}+a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$.
54. $4x^{\frac{8}{3}}-37x^{\frac{4}{3}}y^{\frac{4}{3}}+9y^{\frac{8}{3}}$ by $2x^{\frac{4}{3}}+5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$.
55. $a-b^2$ by $a^{\frac{1}{2}}-b^{\frac{1}{1}}$.
56. $4a^{-10}+12a^{-\frac{15}{2}}b^{-\frac{3}{2}}+9a^{-5}b^{-3}-25b^{-6}$ by $2a^{-5}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}-5b^{-3}$.
57. $9x^{-\frac{5}{2}}-25x^{-\frac{5}{4}}y^{-\frac{3}{4}}+70x^{-\frac{5}{8}}y^{-\frac{9}{8}}-49y^{-\frac{3}{2}}$ by $3x^{-\frac{5}{4}}+5x^{-\frac{5}{8}}y^{-\frac{3}{8}}-7y^{-\frac{3}{4}}$.
58. a^3-b^3 by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.
59. $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

84. Inexact Division. It may so happen that the dividend is not exactly divisible by the divisor. For instance, if in example 2, Art. 83, the dividend were $16x^4+36x^2+6x+86$, the second remainder would be $36x^2+60x+86$, and hence the final remainder $6x+5$. As $6x+5$ cannot be divided by $4x^2+6x+9$, the division in this case would be incomplete and the result might be expressed as in Arithmetic, thus :

$$\frac{16x^4+36x^2+6x+86}{4x^2+6x+9}=4x^2-6x+9+\frac{6x+5}{4x^2+6x+9}.$$

The right-hand side is called the *complete quotient.* The portion of the dividend which is thus left as a residue not divisible by the divisor is spoken of as the *remainder* in division. Hence, if D denote the dividend, d the divisor, Q the quotient, and R the *remainder*, we have the following invariable relation between these symbols $D=d\times Q+R$.

85. Detached Coefficients. If both the dividend and the divisor contain powers of the same algebraic quantity or be *homogeneous* expressions of the same algebraic quantities, the labour of long division can be much saved by detaching the coefficients and placing them in proper relative positions.

The process is illustrated by the following examples :

Example 1. Divide $6x^4+13x^3+39x^2+37x+45$ by $3x^2+2x+9$.

$$\begin{array}{rrrrrrl}
3+2+9\;\big) & 6 & +13 & +39 & +37 & +45 & \big(\;2+3+5\\
 & \underline{6} & \underline{+\ 4} & \underline{+18} & & & \\
 & & +\ 9 & +21 & +37 & & \\
 & & \underline{+\ 9} & \underline{+\ 6} & \underline{+27} & & \\
 & & & 15 & +10 & +45 & \\
 & & & \underline{15} & \underline{+10} & \underline{+45} &
\end{array}$$

$\therefore$ the required quotient is $2x^2+3x+5$.

By the ordinary Method :

$$\begin{array}{rrrrrrl}
3x^2+2x+9\;\big) & 6x^4 & +13x^3 & +39x^2 & +37x & +45 & \big(\;2x^2+3x+5\\
 & \underline{6x^4} & \underline{+\ 4x^3} & \underline{+18x^2} & & & \\
 & & 9x^3 & +21x^2 & +37x & & \\
 & & \underline{9x^3} & \underline{+\ 6x^2} & \underline{+27x} & & \\
 & & & 15x^2 & +10x & +45 & \\
 & & & \underline{15x^2} & \underline{+10x} & \underline{+45} &
\end{array}$$

$\therefore$ the required quotient is $2x^2+3x+5$.

Example 2. Divide x^3-27 by x^2+3x+9.

N. B. If any power of x either in the dividend or in the divisor be absent, the term, involving that power is to be supplied with a zero coefficient.

$$\begin{array}{rrrrrl}
1+3+9\;\big) & 1 & +0 & +0 & -27 & \big(\;1-3\\
 & \underline{1} & \underline{+3} & \underline{+9} & & \\
 & & -3 & -9 & -27 & \\
 & & \underline{-3} & \underline{-9} & \underline{-27} &
\end{array}$$

$\therefore$ the required quotient is $x-3$.

EXERCISE 40

Apply the method of detached coefficients to find the quotient of the following :

1. $2m^3-9m^2n+13mn^2-6n^3$ by $2m-3n$.
2. $a^4-3a^3b+3ab^3-b^4$ by a^2-b^2.
3. $2x^4-3x^3y-3xy^3-2y^4$ by x^2+y^2.
4. $2a^4-36a^2x^2-16ax^3$ by $2a^2+8ax$.
5. $3+2x+4x^2+5x^3-4x^4+2x^5$ by $1+2x^2$.
6. $x^4-4x^2+12x-9$ by x^2+2x-3.
7. $4a^4-9a^2b^2+24ab^3-16b^4$ by $2a^2-3ab+4b^2$.
8. $a^4+4a^2x^2+16x^4$ by $a^2+2ax+4x^2$.
9. a^4+4b^4 by $a^2+2ab+2b^2$.
10. $2x^5-7x^4-2x^3+18x^2-3x-8$ by x^3-2x^2+1.
11. x^4-81 by $x-3$. 12. a^5-32 by $a-2$.
13. $3-9x+2x^2+5x^3-7x^4+2x^5$ by $1-3x+x^2$.
14. $82x^2+40-45x^3+18x^4-67x$ by $6x^2+8-7x$.
15. $64-x^6$ by $2-x$. 16. $1+x^6-2x^3$ by x^2+1-2x.
17. $13ab^3+2a^2b^2+6a^4-a^3b+4b^4$ by $4ab+b^2+3a^2$.
18. $a^3b-15b^4-8a^2b^2+a^4+19ab^3$ by a^2+3b^2-2ab.
19. x^6-a^6 by $x^3-2x^2a+2xa^2-a^3$.
20. $8a^2b^3+3b^5+a^5-9a^3b^2-2ab^4-a^4b$ by $2ab-3b^2+a^2$.
21. $y^6+x^6-2x^3y^3$ by x^2+y^2-2xy.

Find the complete quotient of :

22. $\dfrac{x^2+11x+35}{x+5}$. 23. $\dfrac{x^3+\frac{1}{27}y^3}{x-\frac{1}{3}y}$.

24. Find the remainder when x^3+px^2+qx+r is divided by x^2+px+q.

25. Divide $1+2x+4x^2$ by $3-x$, retaining four terms in the quotient.

86. A few important results.

The student already knows that

$$x^2-a^2=(x-a)(x+a),$$

and $$x^3-a^3=(x-a)(x^2+xa+a^2).$$

Hence, x^4-a^4 [which $=x^3(x-a)+a(x^3-a^3)$]

$$=(x-a)\{x^3+a(x^2+xa+a^2)\}$$
$$=(x-a)(x^3+x^2a+xa^2+a^3).$$

$$\begin{aligned} \text{Hence, } x^5-a^5\ [\text{ which} &= x^4(x-a)+a(x^4-a^4)\ ] \\ &= (x-a)\{x^4+a(x^3+x^2a+xa^2+a^3)\} \\ &= (x-a)(x^4+x^3a+x^2a^2+xa^3+a^4). \end{aligned}$$

Similarly, it may be shown that $x-a$ is a factor of x^6-a^6, of x^7-a^7, of x^8-a^8; and so on; hence, generally, $x-a$ is a factor of x^n-a^n where n is any whole number.

We conclude, therefore, that for *all* positive integral values of n, x^n-a^n is *divisible* by $x-a$.

Again, since, $x^n+a^n=(x^n-a^n)+2a^n$, of which x^n-a^n is divisible by $x-a$ and $2a^n$ is not, $\therefore$ x^n+a^n is *not* divisible by $x-a$.

Thus, when n is a positive integer,

$$\left.\begin{array}{rr} & x-a \textit{ always} \text{ divides } x^n-a^n, \\ \text{but} & \textit{never} \text{ divides } x^n+a^n. \end{array}\right\} \quad \cdots \quad \text{(A)}$$

Cor. 1. $x+a$ divides x^n-a^n *only* when n is an *even* integer.

$$\left.\begin{array}{l} \text{For, when } n \text{ is even, } (-a)^n=\ \ a^n,\ \dagger \text{ and } \therefore x^n-a^n=x^n-(-a)^n, \\ \qquad \text{when } n \text{ is odd, } (-a)^n=-a^n,\ \dagger \text{ and } \therefore x^n-a^n=x^n+(-a)^n; \end{array}\right\}$$

$$\text{also, } x+a=x-(-a).$$

Now, from (A), we know that $x-(-a)$ divides $x^n-(-a)^n$, but not $x^n+(-a)^n$. Hence, $x+a$ divides x^n-a^n when n is even, but not when n is odd, *i.e.*, $x+a$ divides x^n-a^n *only* when n is an *even* integer.

Cor. 2. $x+a$ divides x^n+a^n *only* when n is an *odd* integer.

$$\left.\begin{array}{l} \text{For, when } n \text{ is odd, } (-a)^n=-a^n, \text{ and } \therefore x^n+a^n=x^n-(-a)^n, \\ \qquad \text{when } n \text{ is even, } (-a)^n=\ \ a^n, \text{ and } \therefore x^n+a^n=x^n+(-a)^n; \end{array}\right\}$$

$$\text{also, } x+a=x-(-a).$$

Now, from (A), we know that $x-(-a)$ divides $x^n-(-a)^n$, but not $x^n+(-a)^n$. Hence, $x+a$ divides x^n+a^n when n is odd, but not when n is even, *i.e.*, $x+a$ divides x^n+a^n *only* when n is an *odd* integer.

Thus, we have obtained the following results ‡ :

$$\left.\begin{array}{r} x-a \text{ divides } x^n-a^n \textit{ always}, \\ x^n+a^n \textit{ never}. \end{array}\right\}$$

$$\left.\begin{array}{r} x+a \text{ divides } x^n-a^n \textit{ only} \text{ when } n \text{ is } \textit{even}, \\ x^n+a^n \textit{ only} \text{ when } n \text{ is } \textit{odd}. \end{array}\right\}$$

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† This follows from repeated applications of the laws of signs in multiplication; thus, $(-a)^2=a^2$; hence, $(-a)^3=(-a)\times(-a)^2=(-a)\times a^2=-a^3$; hence, $(-a)$ $=(-a)\times(-a)^3=(-a)(-a^3)=a^4$; hence, $(-a)^5=(-a)(-a)^4=(-a)\times a^4=-a^5$; and so on. That is, any power of $-a$ is positive or negative according as the index of that power is an even or an odd integer.

‡ These results have been formally proved in Chapter XXIII.

EXERCISE 41

Verify by actual division that the following expressions are divisible by $x+a$:

1. x^3+a^3. 2. x^4-a^4. 3. x^5+a^5.

4. x^6-a^6. 5. x^7+a^7. 6. x^8-a^8.

Verify by actual division that the following expressions are *not* divisible by $x+a$:

7. x^3-a^3. 8. x^4+a^4. 9. x^5-a^5.

10. x^6+a^6. 11. x^7-a^7. 12. x^8+a^8.

Write down the quotient of:

13. x^4-1 by $x-1$. 14. x^4-y^4 by $x+y$. 15. x^5-1 by $x-1$.

16. x^5+y^5 by $x+y$. 17. x^6-1 by $x-1$. 18. x^6-y^6 by $x+y$.

19. x^7-1 by $x-1$. 20. x^7+y^7 by $x+y$.

CHAPTER XI

FORMULÆ AND THEIR GRAPHICAL REPRESENTATION

87. The different formulæ established in Chapter IV are stated below to facilitate any reference to them. A complete knowledge of these special products is essential for performing many algebraical operations with neatness and accuracy. It is, therefore, desired that the student should commit them to memory so that the necessity even for occasional references may be altogether done away with.

(i) $(a+b)^2=a^2+2ab+b^2$.

(ii) $(a-b)^2=a^2-2ab+b^2$.

(iii) $(a+b)(a-b)=a^2-b^2$.

(iv) $(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b)$.

(v) $(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab(a-b)$.

(vi) $a^3+b^3=(a+b)(a^2-ab+b^2)=(a+b)^3-3ab(a+b)$.

(vii) $a^3-b^3=(a-b)(a^2+ab+b^2)=(a-b)^3+3ab(a-b)$.

(viii) $(x+a)(x+b)=x^2+(a+b)x+ab$.

(ix) $(x-a)(x+b)=x^2+(b-a)x-ab$.

(x) $(x-a)(x-b)=x^2-(a+b)x+ab$.

88. Application of Formulæ.

Example 1. Find the product of (i) 999×999 and (ii) 9988×10012.

(i) We have
$$\begin{aligned} 999 \times 999 &= 999^2 \\ &= (1000-1)^2 \\ &= 1000^2 - 2 \times 1000 \times 1 + 1^2 && [\text{ Formula (ii) }] \\ &= 1000000 - 2000 + 1 \\ &= 998001. \end{aligned}$$

(ii)
$$\begin{aligned} 9988 \times 10012 &= 10012 \times 9988 \\ &= (10000+12)(10000-12) \\ &= 10000^2 - 12^2 && [\text{ Formula (iii) }] \\ &= 100000000 - 144 \\ &= 99999856. \end{aligned}$$

Example 2. Find the value of $2931^3 + 1069^3 + 12000 \times 2931 \times 1069$.

Putting a for 2931 and b for 1069,

$$\begin{aligned} \text{the given expression} &= a^3 + b^3 + 12000ab \\ &= a^3 + b^3 + 3ab(a+b) \\ & \qquad [\text{ since, } a+b = 2931 + 1069 = 4000 \text{ }] \\ &= (a+b)^3 && [\text{ Formula (iv) }] \\ &= (4000)^3 \\ &= 4000 \times 4000 \times 4000 \\ &= 64000000000. \end{aligned}$$

Note. *The student is referred to the examples worked out in Chapter IV for further illustrations.*

89. Algebraic quantities expressed as the difference of two squares.

We have $a^2 + 2ab + b^2 = (a+b)^2$,

and $a^2 - 2ab + b^2 = (a-b)^2$.

Subtracting, $4ab = (a+b)^2 - (a-b)^2$,

or, $$ab = \tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2.$$

Hence, the product of any two factors

= square of (½ × the sum of the factors)

− square of (½ × the difference of the factors).

Example 1. Express $(x+y+2z)(x+y)$ as the difference of two squares.

$$(x+y+2z)(x+y)=\left\{\frac{(x+y+2z)+(x+y)}{2}\right\}^2-\left\{\frac{(x+y+2z)-(x+y)}{2}\right\}^2$$
$$=\left(\frac{2x+2y+2z}{2}\right)^2-\left(\frac{x+y+2z-x-y}{2}\right)^2$$
$$=(x+y+z)^2-z^2.$$

Example 2. Express $(x+1)(2x+3)(x+5)$ as the difference of two squares.

The given expression

$$=\{(x+1)(2x+3)\}(x+5)=(2x^2+5x+3)(x+5)$$
$$=\left\{\frac{(2x^2+5x+3)+(x+5)}{2}\right\}^2-\left\{\frac{(2x^2+5x+3)-(x+5)}{2}\right\}^2$$
$$=(x^2+3x+4)^2-(x^2+2x-1)^2.$$

Example 3. Express $(x+a)(x+2a)(x+3a)(x+4a)$ as the difference of two squares.

The given expression

$$=\{(x+a)(x+4a)\}\{(x+2a)(x+3a)\}$$
$$=(x^2+5ax+4a^2)(x^2+5ax+6a^2)$$
$$=\left\{\frac{(x^2+5ax+4a^2)+(x^2+5ax+6a^2)}{2}\right\}^2-\left\{\frac{(x^2+5ax+6a^2)-(x^2+5ax+4a^2)}{2}\right\}^2$$
$$=(x^2+5ax+5a^2)^2-(a^2)^2.$$

Example 4. Express $(x+2a)(x+4a)(x+6a)(x+8a)+7a^4$ as the difference of two squares.

The given expression

$$=\{(x+2a)(x+8a)\}\{(x+4a)(x+6a)\}+7a^4$$
$$=(x^2+10ax+16a^2)(x^2+10ax+24a^2)+7a^4$$
$$=\left\{\frac{(x^2+10ax+16a^2)+(x^2+10ax+24a^2)}{2}\right\}^2-\left\{\frac{(x^2+10ax+24a^2)-(x^2+10ax+16a^2)}{2}\right\}^2+7a^4$$
$$=(x^2+10ax+20a^2)^2-(4a^2)^2+7a^4$$
$$=(x^2+10ax+20a^2)^2-16a^4+7a^4$$
$$=(x^2+10ax+20a^2)^2-(3a^2)^2.$$

EXERCISE 42

[The following examples are to be worked out with the help of the formulæ of Art. 87]

Find the squares of the following :

1. $5x+9y$. 2. $16a-13b$. 3. $x+100$. 4. $y+500$.
5. $a+999$. 6. $y+10001$. 7. 988. 8. 1012.
9. 100·5. 10. 99·6.

Find the cubes of the following :

11. $2x+5$. 12. 105. 13. 99·5. 14. 800·6.

15. Show that $(a+b)^2+(a-b)^2=2(a^2+b^2)$.

Hence, find the value of a^2+b^2, when

(i) $a=5004$, $b=4996$; (ii) $a=1012$, $b=988$.

16. Show that $(a+b)^2-(a-b)^2=4ab$.

Hence, express the following as the difference of two squares :

(i) $4(x+2y)(2x+y)$; (ii) $(6x+10y)(4x+6y)$; (iii) $(x+98)(x+102)$;
(iv) 505×495 ; (v) $(2x+100{\cdot}4)(2x+99{\cdot}6)$.

Find the following products :

17. $(a+x)(a-x)(a^2+x^2)$. 18. $(2a+3)(2a-3)(4a^2+9)$.
19. $(a^2-ab+b^2)(a^2+ab+b^2)(a^4-a^2b^2+b^4)$.
20. $98\times 102\times 10004$. 21. $96\times 104\times 10016$.
22. $(2a+x)(4a^2+4ax+x^2)$.
23. $(a-2)(a+2)(a^2+4a+4)(a^2-4a+4)$.
24. $(x+4)(x^2-4x+16)$. 25. $(2y-3)(4y^2+6y+9)$.
26. $(x+2)(x^2+2x+4)(x-2)(x^2-2x+4)$.
27. $(2x+105)(2x+15)$. 28. $(6x-25)(6x+43)$. 29. $(6x-25)(6x-43)$

Simplify the following :

30. $(2a+x+y)^2+2(2a+x+y)(8a-x-y)+(8a-x-y)^2$.
31. $(17a+20x+19y)^2-2(19x+18y+17a)(20x+19y+17a)$
$+(19x+18y+17a)^2$
32. $(16a+x+y)^3+(4a-x-y)^3+3(16a+x+y)^2(4a-x-y)$
$+3(16a+x+y)(4a-x-y)^2$.
33. $(121a+x+y)^3-(116a+x+y)^3-15a(121a+x+y)(116a+x+y)$.
34. $(5a-8x)^3+(6a+8x)^3+33a(5a-8x)(6a+8x)$.
35. $(2x+3y-16z)^3+3(3x-3y+16z)^2(2x+3y-16z)$
$+(3x-3y+16z)^3+3(3x-3y+16z)(2x+3y-16z)^2-120x^3$.

Resolve into factors :

36. $(5a+8b+2)^2-(4a+6)^2$. **37.** $8x^3+125y^3$.

38. $(8a+13x)^3-64$. **39.** $(15a+3b)^2-4$.

Find the value of :

40. $(16a+2b)^2-2(13a+2b)(16a+2b)+(13a+2b)^2$, when $a=5$ and $b=7891$.

41. $(91x+5y)^3-3(91x+5y)^2(87x+5y)$ $+3(91x+5y)(87x+5y)^2-(87x+5y)^3$, when $x=2$ and $y=83$

42. $(589963)^2-2\times589963\times589863+(589863)^2$.

43. $90{\cdot}002\times89{\cdot}998$. **44.** 9238^2-9233^2.

45. $49856\times49856\times49856-3\times49856\times49855-49855\times49855\times49855$.

46. Factorize $(x+2)(2x+1)(5x+2)-3x^4$ by expressing it as the difference of two squares.

47. Show that $(ax+b)(bx+a)\{abx^2-(a^2+b^2)x+ab\}$ can be expressed as the difference of two squares.

48. Express $(5x+1)(2x+5)(3x+5)(4x+3)$ as the difference of two squares.

49. Express $(7x+3a)(7x+5a)(7x+9a)(7x+11a)+61a^4$ as the sum of two squares.

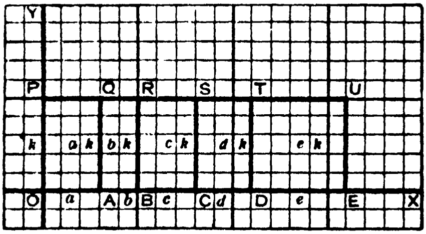
50. Show that $(3x+2y)^3-(2x+3y)^3-3(3x+2y)(2x+3y)(x-y)$ is a perfect cube. [G. U. 1950]

90. Graphical Representation of Algebraic Formulæ. Some of the formulæ are illustrated below by their geometrical representations on squared paper.

(1) To demonstrate graphically, the identity

$$(a+b+c+d+e)k=ak+bk+ck+dk+ek.$$

Let OX and OY be the co-ordinate axes, O being the origin.



Let A, B, C, D, E be the points on OX, such that $OA=a$, $AB=b$, $BC=c$, $CD=d$ and $DE=e$. Also, let P be a point on OY, such that

$OP=k$. Complete the rectangle $OPUE$. Through A, B, C, D draw AQ, BR, CS, DT parallels to OP so as to meet PU in Q, R, S, T respectively, so that the figures $OPQA$, $AQRB$, $BRSC$, $CSTD$, $DTUE$ are each a rectangle.

Now, rect. $PE=$rect. $PA+$rect. $QB+$rect. $RC+$rect. SD $+$rect. TE. --- (1)

But rect. $PE=OE.OP=(OA+AB+BC+CD+DE).OP$
$=(a+b+c+d+e).k$;

and rect. $PA=OA.OP$
$=ak$;

rect. $QB=AB.AQ=AB.OP$
$=bk$;

rect. $RC=BC.BR=BC.OP$
$=ck$;

rect. $SD=CD.CS=CD.OP$
$=dk$;

rect. $TE=DE.DT=DE.OP$
$=ek$;

$\therefore$ from (1), $(a+b+c+d+e)k=ak+bk+ck+dk+ek$.

(2) To demonstrate graphically, the identity

$$(a+b)^2=a^2+2ab+b^2.$$

Let OX and OY be the co-ordinate axes, and O be the origin.

Let A and B be two points taken on OX, such that $OA=a$ and $AB=b$; also let L and P be two points on OY, such that $OL=a$ and $LP=b$. Then, $OB=OP=a+b$. Complete the square $OPRB$. Let AQ be drawn through A parallel to OY to meet PR in Q; also let LMN be drawn through L parallel to OX to meet AQ in M and BR in N.

Then, fig. $OR=$fig. $OM+$fig. $AN+$fig. $LQ+$fig. MR. --- (1)

Now, fig. $OR=OB.OP$
$=OB.OB$ [$\because OP=OB$]
$=OB^2=(a+b)^2$;

fig. $OM=OA.OL=OA.OA$
$=a^2$;

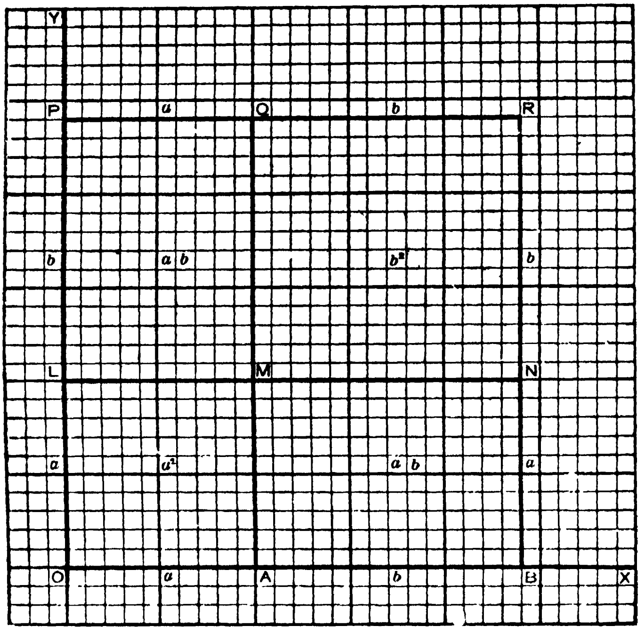
fig. $AN=AM.AB=OL.AB$
$=ab$;

fig. $LQ = LM.LP$

$= PQ.LP = ab$;

fig $MR = MN.MQ = QR.LP$

$= b.b = b^2$.



from (1), $(a+b)^2 = a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$.

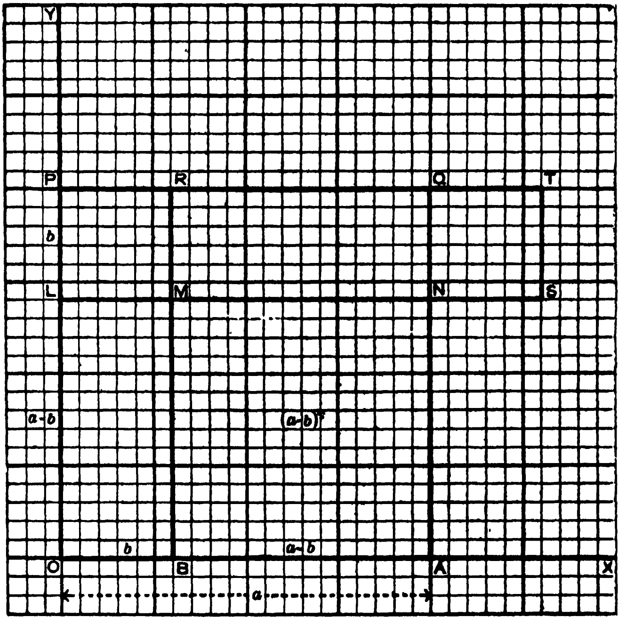
(3) To demonstrate graphically, the identity

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Let OX and OY be the co-ordinate axes, and O, the origin.

Take two points A and B on OX, such that $OA = a$ and $OB = b$. Complete the square $OPQA$, on OA. Through B, draw BR parallel to OY to meet PQ in R; cut off a length PL from PO, equal to b. Through L, draw LMN parallel to OX to meet BR and AQ in M and N respectively. Produce PQ to T, making $QT = PR$ ($= b$). Complete the square $QTSN$, on QT.

Since, $OA=a$ and $OB=b$,
$\therefore$ $BA=a-b$.
Also, since, $OP=OA=a$,
and $PL=b$,
$\therefore$ $OL=a-b$;
$\therefore$ $AB=OL$.



Now, fig. $BN=$ fig. $OQ+$ fig. $NT-$ fig. $OR-$ fig. RS. ... (1)

But, fig. $BN=BA.BM=BA.OL$
$=BA.BA=BA^2$
$=(a-b)^2$;
fig. $OQ=OA.OP=OA.OA$
$=OA^2=a^2$;
fig. $NT=$ sq. on QT
$=$ sq. on PR
$=b^2$;

fig. $OR = OP.OB = OA.OB$

$= ab$;

fig. $RS =$ fig. $RN +$ fig. QS

$=$ fig. $RN +$ fig. PM

$=$ fig. PN [$\because$ fig. $QS =$ fig. PM, each being equal to b^2.]

$= PQ.PL$

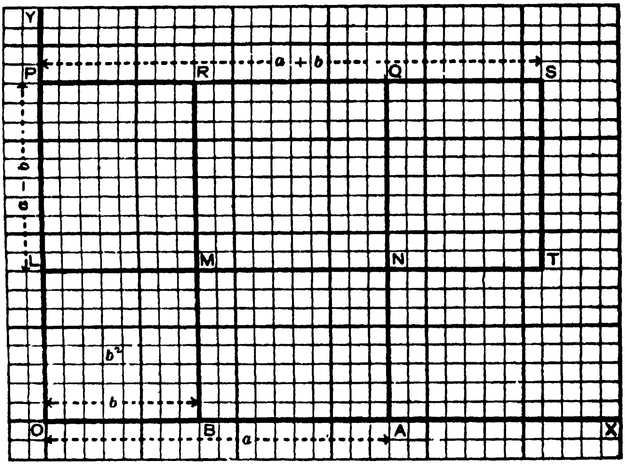
$= ab$.

$\therefore$ from (1), $(a-b)^2 = a^2 + b^2 - ab - ab$, *i.e.*, $= a^2 + b^2 - 2ab$

(4) To demonstrate graphically, the identity

$$a^2 - b^2 = (a-b)(a+b).$$

Let OX and OY be the co-ordinate axes, and O, the origin.



Take two points, A and B on OX, such that $OA = a$ and $OB = b$; also, take two points, P and L on OY, such that $OP = a$ and $OL = b$.

Complete the squares $OPQA$ and $OLMB$. Produce BM to meet PQ in R and LM to meet AQ in N; also produce MN to T, making $NT = NA$ $(= b)$; and complete the rectangle $NTSQ$.

Thus, rect. $BN =$ rect. QT,

also $PL = OP - OL = a - b$,

and $AB = OA - OB = a - b$.

$\therefore$ $PL = AB$.

Now, fig. PA − fig. BL = fig. PN + fig. BN
= fig. PN + fig. QT
= fig. PT. ... ... (1)

But, fig. PA = sq. on OA
$= a^2$;

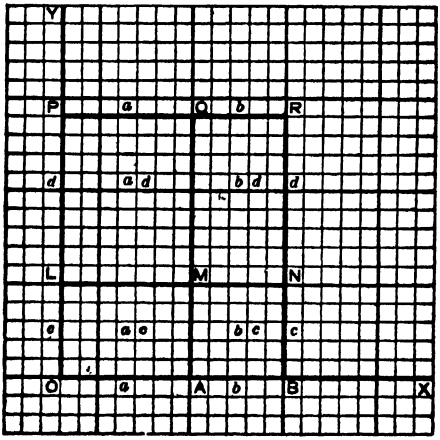
fig. BL = sq. on OB
$= b^2$;

fig. $PT = PS.PL$
$= (PQ + QS) PL$
$= (PQ + NT).PL$
$= (a + b)(a - b)$.

∴ from (1), $a^2 - b^2 = (a - b)(a + b)$.

(5) To demonstrate graphically, the identity
$$(a+b)(c+d)=ac+bc+ad+bd.$$

Let OX and OY be the co-ordinate axes, and O, the origin.



On OX, take two points, A and B, making $OA = a$ and $AB = b$; also, on OY take two points, P and L, making $OL = c$ and $LP = d$.

Complete the rectangles $OPRB$ and $OLNB$.

Through A, draw AMQ parallel to OY to meet LN in M and PR in Q.

Now, fig. OR = fig. OM + fig. AN + fig. LQ + fig. MR. ... (1)

But, fig. $OR = OB.OP$

$$= (OA + AB)(OL + LP)$$

$$= (a + b)(c + d);$$

fig. $OM = OA.OL = ac$;

fig. $AN = AB.AM$

$$= AB.OL = bc;$$

fig. $LQ = PQ.PL$

$$= OA.PL = ad$$

fig. $MR = QR.QM$

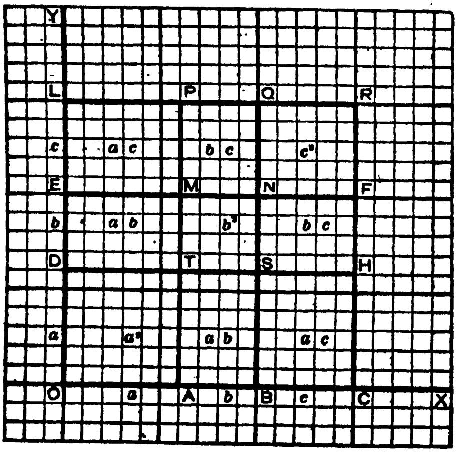
$$= AB.PL = bd;$$

$\therefore$ from (1), $(a + b)(c + d) = ac + bc + ad + bd$.

(6) To demonstrate graphically, the identity

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac.$$

Let OX and OY be two perpendicular straight lines, through O.



Take three points, A, B and C on OX, such that $OA = a$, $AB = b$, $BC = c$.

Complete the square $OCRL$ on OC, so that

$$OL = OC = OA + AB + BC = a + b + c.$$

Let D and E be points on OL, such that

$$OD = a \text{ and } DE = b, \text{ whence } EL = c.$$

Through A and B, draw AP and BQ parallel to OY to meet LR in P and Q respectively; also, through D and E draw $DTSH$ and $EMNF$ parallel to OX to meet AP, BQ, CR in points, T, S, H and M, N, F respectively.

Then, $$\text{fig. } OR = \text{fig. } OT + \text{fig. } TN + \text{fig. } NR + \text{fig. } DM + \text{fig. } AS + \text{fig. } PN + \text{fig. } NH + \text{fig. } EP + \text{fig. } BH \quad \cdots \quad (1)$$

Now, $$\text{fig. } DM = DT.DE = OA.AB = ab,$$
and $$\text{fig. } AS = AT.AB = OD.AB = ab.$$

Similarly, $$\text{fig. } NP = \text{fig. } NH = bc,$$
and $$\text{fig. } EP = \text{fig. } BH = ac.$$

Also, $$\text{fig. } OR = \text{sq. on } OC = OC^2 = (OA + AB + BC)^2 = (a + b + c)^2,$$
$$\text{fig. } OT = OA.OD = OA.OA = OA^2 = a^2,$$
$$\text{fig. } TN = TM.TS = AB.DE = AB^2 = b^2,$$
$$\text{fig. } NR = NQ.NF = EL.BC = BC^2 = c^2;$$

∴ from (1),

$$(a + b + c)^2 = a^2 + b^2 + c^2 + ab + ab + bc + bc + ac + ac,$$
$$i.e., \quad = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac.$$

EXERCISE 43

Find, graphically, the value of:

1. (i) $(5+6) \times 11$; (ii) 7^2; (iii) $(\frac{5}{2} - \frac{1}{2})^2$.

2. Verify graphically:
   (i) $9^2 - 7^2 = 32$; (ii) $(7+3)^2 = 100$;
   (iii) $(3+5) \times 2 = 3 \times 2 + 5 \times 2$;
   (iv) $(x+a)(x+b) = x^2 + (a+b)x + ab$;
   (v) $(x-a)(x-b) = x^2 - (a+b)x + ab$;
   (vi) $(x-a)(x+b) = x^2 - ax + bx - ab$.

3. Calculate, graphically, the area of a square described on a straight line whose length is equal to twelve centimetres.

4. Find, graphically, the area of a room, 5 metres long and 3 metres broad.

5. A rectangular garden of length 9 metres and breadth 3 metres has got a path of uniform breadth surrounding it. If the breadth of the path be one metre, find, graphically, the total area of the garden and the path together.

6. In a square plot of land of side 10 metres, a square pond of length four metres is dug. Find, graphically, the area of the remaining portion of the land.

7. Find, graphically, the area of a rectangular plot of land whose length is 50 metres, and is five times its breadth.

8. A rectangular court-yard of length 10 metres and breadth 5 metres is to be paved with square stones. If the side of the stone be one metre, find, graphically, the number of stones necessary for the purpose.

9. A square garden of side 20 metres has within it a walk of uniform breadth equal to one metre running round it. Find, graphically, the area of the path.

10. A rectangular court of length 20 metres and breadth 10 metres has two paths, each of breadth one metre joining the middle points of the opposite sides, and *symmetrically* situated about the lines joining those middle points ; find, graphically, the area of that portion of the court which is not covered by the path.

CHAPTER XII

SIMPLE FACTORS

91. Definitions. When an expression is the product of two or more others, each of these latter is called a **factor** of the former.

An expression is said to be *resolved into factors* when those expressions of which it is the product are found.

[A few simple cases of resolution into factors have already been incidentally treated in the chapter on *Simple Formulæ and their Application.* These cases, however, will not be altogether passed over in the following articles as the present chapter is intended for a more systematic treatment of the subject.]

Note. *In this chapter we shall confine our attention to rational expressions only (i.e., expressions free from radical signs), and by the factors of an expression will be meant the rational expressions of which it is the product.*

92. Simple Cases. Any expression, *all the terms of which have got a common factor* may, on inspection, be at once resolved into two factors, one of which is a simple expression and the other compound; thus,

(1) $a^2x+ax^2=ax(a+x)$. (2) $2a^3b^2-3a^2b^3=a^2b^2(2a-3b)$

(3) $24x^4a^3-40x^3a^4+56x^2a^5=8x^2a^3(3x^2-5xa+7a^2)$.

EXERCISE 44

Resolve into factors:

1. $ab+ac$. 2. $a^2b^3+a^3b^2$. 3. $x^3y^4-2x^4y^3$.
4. $2x^2yz+4xy^2z-6xyz^2$. 5. $4a^5b-6a^4b^2-8a^3b^3$.
6. $ax^2y-5a^2x^3y^2+3ax^3$. 7. $3x^4y^3z^2-12x^2y^4z^3+21x^3y^2z^4$.
8. $28a^3b^5-42a^5b^6$. 9. $72x^{10}y^8+108x^8y^{10}$.
10. $39a^5b^7c^7-65b^5c^7a^7-91c^5a^7b^7$.

93. Expressions of the form a^2-b^2.

The method of resolving into factors an expression of this form has already been treated in Art. 56, Note. A few more examples are added here for the exercise of the student.

EXERCISE 45

Resolve into factors:

1. $9a^2-16b^2$. 2. $4a^3-25ax^2$. 3. $36x^4-1$.
4. $16x^4-1$. 5. $16x^5-9x$. 6. $16x^5-81x$.
7. $1-16a^4$. 8. x^2-81x^6. 9. $36-\dfrac{x^4}{a^2}$.
10. $64a^4-49x^6$. 11. $121-m^6$. 12. $49x^6a^{10}-81$.
13. $a^2b^2-25c^2d^2$. 14. $81x^{12}-64a^{10}$. 15. $p^2q^4-100p^2$.
16. $144x^7-25x^3a^4$. 17. $192a^9-243a^5x^4$. 18. $98a^3x^5-128ax$.
19. $324x^{17}a^9-484x^5a^3$. 20. $245m^{23}n^{13}-605m^{15}n^7$.
21. $(a+3b)^2-25c^2$. 22. $a^2-(3b-5c)^2$. 23. $(x+y)^2-(x-y)^2$.
24. $(3a+2x)^2-(2a+x)^2$. 25. $4(a-b)^2-9(c-d)^2$.
26. $49x^2-(5y-3z)^2$. 27. $(8x+5)^2-(2x-7)^2$.
28. $(a+b-c)^2-(a-b+c)^2$. 29. $(2a-3b+4c)^2-(a+4b-5c)^2$.
30. $64(a+3x-4y)^2-9(2a-x+3y)^2$.
31. $(4x^2-5a^2)^2-(5x^2-4a^2)^2$. 32. $(5a^2-3a+7)^2-(5a^2-3a-7)^2$.
33. $(x+y)^3(a+b)-(x+y)(a+b)^3$.

94. Expressions which by mere inspection can be put into the form a^2-b^2. The following examples are intended for illustration.

Example 1. Resolve into factors $a^4+a^2b^2+b^4$.

$$\begin{aligned} a^4+a^2b^2+b^4 &= (a^4+2a^2b^2+b^4)-a^2b^2 \\ &= (a^2+b^2)^2-(ab)^2=\{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\} \\ &= (a^2+ab+b^2)(a^2-ab+b^2). \end{aligned}$$

Example 2. Resolve into factors x^4+4.

$$\begin{aligned} x^4+4 &= (x^4+4x^2+4)-4x^2 \\ &= (x^2+2)^2-(2x)^2 \\ &= \{(x^2+2)+2x\}\{(x^2+2)-2x\} \\ &= (x^2+2x+2)(x^2-2x+2). \end{aligned}$$

Example 3. Resolve into factors x^4-6x^2+1.

$$\begin{aligned} x^4-6x^2+1 &= (x^4-2x^2+1)-4x^2 \\ &= (x^2-1)^2-(2x)^2 \\ &= \{(x^2-1)+2x\}\{(x^2-1)-2x\} \\ &= (x^2+2x-1)(x^2-2x-1). \end{aligned}$$

Example 4. Resolve into factors $a^2-b^2+2bc-c^2$.

$$\begin{aligned} a^2-b^2+2bc-c^2 &= a^2-(b^2-2bc+c^2) \\ &= a^2-(b-c)^2=\{a+(b-c)\}\{a-(b-c)\} \\ &= (a+b-c)(a-b+c). \end{aligned}$$

Example 5. Resolve into factors $2(ab+cd)-a^2-b^2+c^2+d^2$.

$$\begin{aligned} \text{The given expression} &= (c^2+2cd+d^2)-(a^2-2ab+b^2) \\ &= (c+d)^2-(a-b)^2 \\ &= \{(c+d)+(a-b)\}\{(c+d)-(a-b)\} \\ &= (c+d+a-b)(c+d-a+b). \end{aligned}$$

EXERCISE 46

Resolve into factors :

1. x^4+x^2+1. 2. x^8+x^4+1. 3. $a^4+a^2x^2+x^4$.
4. $a^8+a^4x^4+x^8$. [C. U. Entrance, 1887]
5. x^4+64. 6. $4x^4+81$. 7. $9x^4+36$.
8. a^4+2a^2+9. 9. x^4-7x^2+9. 10. $4x^4+8x^2+9$.
11. $4x^4-16x^2+9$. 12. $4x^4+3x^2+9$
13. $4a^4-37a^2+9$ 14. $4a^4+625$.

15. $9x^4+23x^2+16.$

16. $9a^4-25a^2+16.$

17. $9x^4-33x^2+16.$

18. $9a^4-a^2+16.$

19. $16x^4+4x^2a^2+25a^4.$

20. $9a^4-19a^2x^2+25x^4.$

21. $x^4+8x^2+144.$

22. $a^4-35a^2b^2+25b^4.$

23. $36a^4-16a^2b^2+b^4.$

24. $49m^4+16n^4-60m^2n^2.$

25. $64a^4+81x^4.$

26. $4x^4+(7a)^4.$

27. $x^2-y^2+2yz-z^2.$

28. $4a^2-b^2-9c^2+6bc.$

29. $9x^2-4y^2+12yz-9z^2.$

30. $a^2-4b^2-25c^2+20bc.$

31. $30xz+16y^2-9x^2-25z^2.$

32. $a^2+4b^2-9c^2-4d^2-4ab+12cd.$

33. $(x^2-2xy)-(z^2-2yz).$

34. $4x^2-1+9a^2-25b^2+12xa-10b.$

35. $9x^2-4y^2-49z^2-30x+28yz+25.$

36. $16a^2-16c^2-9b^2-24a+24bc+9.$

37. $49y^2+20z+x^2-14xy-25z^2-4.$

38. $16x^2+42by-9y^2+40xa-49b^2+25a^2.$

39. $49x^2-1+16y^2-64z^2+16z-56xy.$

40. $a^2-b^2-c^2+d^2-2(ad-bc).$

95. Expressions of the form a^3+b^3 or a^3-b^3.

The resolution of such expressions into factors has already been considered in Articles 59 and 60, Notes. A few cases, however, of a little more complicated character may, with advantage, be added here.

Example 1. Resolve into factors a^9+x^9.

Since, $a^3+b^3=(a+b)(a^2-ab+b^2)$,

we have
$$\begin{aligned} a^9+x^9 &= (a^3)^3+(x^3)^3 \\ &= (a^3+x^3)\{(a^3)^2-(a^3)(x^3)+(x^3)^2\} \\ &= (a^3+x^3)(a^6-a^3x^3+x^6) \\ &= (a+x)(a^2-ax+x^2)(a^6-a^3x^3+x^6). \end{aligned}$$

Example 2. Resolve into factors a^9-x^9.

Since, $a^3-b^3=(a-b)(a^2+ab+b^2)$,

we have
$$\begin{aligned} a^9-x^9 &= (a^3)^3-(x^3)^3 \\ &= (a^3-x^3)\{(a^3)^2+(a^3)(x^3)+(x^3)^2\} \\ &= (a^3-x^3)(a^6+a^3x^3+x^6) \\ &= (a-x)(a^2+ax+x^2)(a^6+a^3x^3+x^6). \end{aligned}$$

Example 3. Resolve into factors $64x^7 - xa^6$.

$$\begin{aligned}64x^7 - xa^6 &= x(64x^6 - a^6)\\ &= x\{(8x^3)^2 - (a^3)^2\}\\ &= x(8x^3 + a^3)(8x^3 - a^3)\\ &= x\{(2x)^3 + a^3\}\{(2x)^3 - a^3\}\\ &= x\{(2x+a)(4x^2 - 2xa + a^2)\}\{(2x-a)(4x^2 + 2xa + a^2)\}\\ &= x(2x+a)(2x-a)(4x^2 - 2xa + a^2)(4x^2 + 2xa + a^2).\end{aligned}$$

Otherwise :

$$\begin{aligned}64x^7 - xa^6 &= x(64x^6 - a^6)\\ &= x\{(4x^2)^3 - (a^2)^3\}\\ &= x(4x^2 - a^2)(16x^4 + 4x^2a^2 + a^4)\\ &= x(2x+a)(2x-a)\{(16x^4 + 8x^2a^2 + a^4) - 4x^2a^2\}\\ &= x(2x+a)(2x-a)\{(4x^2 + a^2)^2 - (2xa)^2\}\\ &= x(2x+a)(2x-a)(\overline{4x^2 + a^2} + 2xa)(\overline{4x^2 + a^2} - 2xa)\\ &= x(2x+a)(2x-a)(4x^2 + 2xa + a^2)(4x^2 - 2xa + a^2).\end{aligned}$$

Note. *Although the resolution can be effected in either of the two ways shown above, it is generally found convenient to adopt the first method.*

Example 4. Resolve into factors $x^4 - \dfrac{64x}{y^3}$.

$$\begin{aligned}x^4 - \frac{64x}{y^3} &= x\left(x^3 - \frac{64}{y^3}\right)\\ &= x\left\{(x)^3 - \left(\frac{4}{y}\right)^3\right\}\\ &= x\left(x - \frac{4}{y}\right)\left(x^2 + \frac{4x}{y} + \frac{16}{y^2}\right).\end{aligned}$$

EXERCISE 47

Resolve into factors :

1. $a^3 - 8b^3$.
2. $a^4 - 27ax^3$.
3. $512x^9 + 1$.
4. $a^9 - 512b^9$.
5. $27a^6 + 125x^6$.
6. $m^6 - n^6$.
7. $343x^3 + 512y^3$. [C. U. Entrance, 1882]
8. $64x^{12} - 1$.
9. $a^6 - 64x^{12}$.
10. $125x^9 - 216a^9$.
11. $54a^{13}b + 343ab^{13}$.
12. $729x^{20}y^2 - 64x^2y^{20}$.
13. $(a^2 + b^2)^3 + 8a^3b^3$.
14. $(2x^2 - 3y^2)^3 + y^6$.
15. $(2a^3 - b^3)^3 - b^9$.
16. $a^6 + \dfrac{b^6}{27}$.
17. $a^3 - \dfrac{8}{b^3}$.
18. $\dfrac{1}{8x^3} + \dfrac{8}{y^3}$.

96. Expressions of the form x^2+px+q resolved into factors by inspection.

From the relation $x^2+(a+b)x+ab=(x+a)(x+b)$, it is clear that to resolve an expression of the form x^2+px+q into two factors we have to find two quantities a and b such that $a+b=p$ and $ab=q$. This can be done by inspection whenever a and b are rational and integral. The student can very well refer himself to the examples worked out after Art. 61, for a clearer comprehension of such cases.

Example 1. Resolve into factors $x^2+17x+30$.

We have to find two numbers whose sum $=17$, and product $=30$.

The pairs of numbers whose product is 30 are : (i) 1 and 30, (ii) 2 and 15, (iii) 3 and 10, (iv) 5 and 6. Out of these 4 pairs we must pick out that of which the sum is 17 ; the second pair, therefore, is the one sought.

Thus, 2 and 15 are the numbers required.

Hence, $x^2+17x+30=x^2+(2+15)x+30=x^2+2x+15x+30$

$$=x(x+2)+15(x+2)=(x+2)(x+15).$$

Example 2. Resolve into factors $x^2-11x+24$.

We must find two numbers whose product $=+24$, and sum $=-11$. Clearly then the two numbers must be *both* negative.

The pairs of negative numbers whose product is 24 are : (i) -1 and -24, (ii) -2 and -12, (iii) -3 and -8, (iv) -4 and -6. Out of these 4 pairs we must pick out that of which the sum is -11 ; the third pair, therefore, is the one sought.

Thus, the required numbers are -3 and -8.

Hence, $x^2-11x+24=x^2-(3+8)x+24=x^2-3x-8x+24$

$$=x(x-3)-8(x-3)=(x-3)(x-8).$$

Example 3. Resolve into factors $x^2+6x-40$.

We must find two numbers whose product $=-40$, and sum $=+6$.

The pairs of numbers whose product is -40 are : (i) 1 and -40, (ii) -1 and 40, (iii) 2 and -20, (iv) -2 and 20, (v) 4 and -10, (vi) -4 and 10, (vii) 5 and -8, (viii) -5 and 8. Out of these 8 pairs we must pick out that of which the sum is $+6$; the sixth pair, therefore, is the one sought.

Thus, the required numbers are -4 and 10.

Hence, $x^2+6x-40=x^2+(10-4)x-40=x^2+10x-4x-40$

$$=x(x+10)-4(x+10)=(x+10)(x-4).$$

Note. *From the fact that the sum of the two numbers is positive it is clear that the positive number must be numerically greater than the negative. Hence, we might at once reject the first, third, fifth and seventh of the above pairs.*

Example 4. Resolve into factors $x^2-5x-36$.

We have to find two numbers whose product $=-36$, and sum $=-5$. Clearly, then the numbers must have different signs and the negative number must be numerically greater than the positive one.

Hence, the only admissible pairs of numbers whose product is -36 are : (i) 1 and -36, (ii) 2 and -18, (iii) 3 and -12, (iv) 4 and -9. Out of these 4 pairs we must pick out that of which the sum is -5 ; the last pair, therefore, is the one sought.

Thus, the required numbers are 4 and -9.

Hence, $$x^2-5x-36=x^2-(9-4)x-36=x^2-9x+4x-36$$ $$=x(x-9)+4(x-9)=(x-9)(x+4).$$

Example 5. Resolve into factors $a^2+7ab+12b^2$.

The factors will evidently be $a+pb$ and $a+qb$, where p and q are such that $p+q=7b$, and $pq=12b^2$.

Arguing as before it is easy to see that $3b$ and $4b$ are the numbers whose sum is $7b$, and product $12b^2$.

Hence, $$a^2+7ab+12b^2=a^2+(3b+4b)a+12b^2=a^2+3ab+4ab+12b^2$$ $$=a(a+3b)+4b(a+3b)=(a+3b)(a+4b).$$

Example 6. Resolve into factors $3m^2-36mn+60n^2$.

$$3m^2-36mn+60n^2=3(m^2-12mn+20n^2).$$

We have to find two numbers whose sum $=-12n$, and product $=20n^2$

Arguing in the usual way we find that $-10n$ and $-2n$ are the required numbers.

Hence, $$3m^2-36mn+60n^2=3(m^2-12mn+20n^2)$$ $$=3\{m^2-(10n+2n)m+20n^2\}=3\{m^2-10nm-2nm+20n^2\}$$ $$=3\{m(m-10n)-2n(m-10n)\}=3(m-10n)(m-2n).$$

Example 7. Resolve into factors a^4-a^2-12.

Putting x for a^2, the given expression becomes x^2-x-12, and it is easy to see that $x^2-x-12=(x-4)(x+3)$.

Hence, $a^4-a^2-12=(a^2-4)(a^2+3)=(a+2)(a-2)(a^2+3)$.

Example 8. Resolve into factors $(x^2+2x)^2-3(x^2+2x)-18$.

Putting a for x^2+2x, the given expression becomes $a^2-3a-18$. and it is easy to see that

$$a^2-3a-18=(a-6)(a+3).$$

Hence, the given expression $=\{(x^2+2x)-6\}\{(x^2+2x)+3\}$ $$=(x^2+2x-6)(x^2+2x+3).$$

Example 9. Resolve into factors

$$(5a+b)^2+(5a+b)(a+2b)-20(a+2b)^2.$$

Putting x for $5a+b$ and y for $a+2b$, the given expression becomes $x^2+xy-20y^2$.

Now it can be easily seen that

$$x^2+xy-20y^2=(x+5y)(x-4y).$$

Hence, the given expression

$$=\{(5a+b)+5(a+2b)\}\{(5a+b)-4(a+2b)\}$$
$$=(10a+11b)(a-7b).$$

96A. Expressions of the form px^2+qx+r resolved into factors by inspection.

Suppose, the two factors of the expression px^2+qx+r are $(ax+b)$ and $(cx+d)$.

From multiplication we know

$$(ax+b)(cx+d)=acx^2+bcx+adx+bd=acx^2+(bc+ad)x+bd\ ;$$
$$\therefore\quad px^2+qx+r=acx^2+(bc+ad)x+bd\ ;$$
$$\therefore\quad p=ac,\ q=bc+ad \text{ and } r=bd\ ;$$
$$\therefore\quad pr=(ac)\times(bd)=(bc)\times(ad).$$

It is evident from the above relation between a, b, c and d that the algebraic sum of the two factors of pr is equal to q.

Therefore the product of the coefficient of x^2 and the term independent of x is to be resolved into two factors, the algebraic sum of which is equal to the coefficient of x.

The process is being explained in the first method of the following illustrations :

Example 10. Resolve into factors $8x^2+2x-3$.

***First Method*:** Find the product of the *coefficient of x^2* and *the term independent of x*.

In the present case, the *product* $=8\times(-3)=-24$.

Now, resolve -24 into two factors whose algebraic *sum* = the coefficient of x, *i.e.*, 2.

By trial, the factors are 6 and -4.

Thus, the given expression $=8x^2+6x-4x-3$

$$=2x(4x+3)-(4x+3)=(4x+3)(2x-1).$$

***Second Method*:** The given expression $= 8x^2 + 2x - 3$
$$= \tfrac{1}{8}(8 \times 8x^2 + 2 \times 8x - 3 \times 8)$$
$$= \tfrac{1}{8}(a^2 + 2a - 24). \quad [\text{ Putting } a \text{ for } 8x\ ]$$

Now it can be easily seen that $a^2 + 2a - 24 = (a+6)(a-4)$.

Hence, the given expression $= \tfrac{1}{8}(a+6)(a-4) = \tfrac{1}{8}(8x+6)(8x-4)$
$$= \tfrac{1}{8}\{2(4x+3) \times 4(2x-1)\} = (4x+3)(2x-1).$$

Example 11. Resolve into factors $12x^2 + 7x - 10$.

***First Method*:** Find the product of *the coefficient of* x^2 and *the term independent of* x; resolve the *product* into two factors whose *algebraic sum* is equal to the coefficient of x.

In the present case, the *product* $= 12 \times (-10) = -120$.

By trial, the factors of (-120), whose algebraic sum = the coefficient of x, *i.e.*, $+7$, are $+15$ and -8.

Thus, the given expression $= 12x^2 + 15x - 8x - 10$
$$= 3x(4x+5) - 2(4x+5) = (4x+5)(3x-2).$$

***Second Method*:** The given expression $= 12x^2 + 7x - 10$
$$= \tfrac{1}{12}(12 \times 12x^2 + 7 \times 12x - 10 \times 12)$$
$$= \tfrac{1}{12}(a^2 + 7a - 120). \quad [\text{ Putting } a \text{ for } 12x\ ]$$

Now it can be easily seen that $a^2 + 7a - 120 = (a+15)(a-8)$.

Hence, the given expression $= \tfrac{1}{12}(12x+15)(12x-8)$
$$= \tfrac{1}{12}\{3(4x+5) \times 4(3x-2)\} = (4x+5)(3x-2).$$

Example 12. Resolve into factors $26x^3 - 40ax^2 + 14a^2x$.
$$26x^3 - 40ax^2 + 14a^2x = 2x(13x^2 - 20ax + 7a^2).$$

***First Method*:** Find the product of *the coefficient of* x^2 and *the term independent of* x. In the present case, the product $= 13 \times 7a^2 = 91a^2$. Now, resolve $91a^2$ into two factors whose *algebraic sum* = the coefficient of x, *i.e.*, $-20a$.

By trial, the factors are $-7a$ and $-13a$.

Thus, the given expression
$$= 2x(13x^2 - 13ax - 7ax + 7a^2)$$
$$= 2x\{13x(x-a) - 7a(x-a)\} = 2x(x-a)(13x-7a).$$

Second Method : The given expression

$$=2x\{13x^2-20ax+7a^2\}$$

$$=2x\{\tfrac{1}{13}(13\times 13x^2-20a\times 13x+13\times 7a^2)\}$$

$$=2x\{\tfrac{1}{13}(y^2-20ay+91a^2)\} \quad [\text{ Putting } y \text{ for } 13x]$$

$$=2x\{\tfrac{1}{13}(y^2-13ay-7ay+91a^2)\}$$

$$=2x[\tfrac{1}{13}\{y(y-13a)-7a(y-13a)\}]$$

$$=2x[\tfrac{1}{13}(y-13a)(y-7a)]\ ;$$

$\therefore$ the expression $=2x[\tfrac{1}{13}(13x-13a)(13x-7a)]$

$$=2x[\tfrac{1}{13}\times 13(x-a)(13x-7a)]$$

$$=2x(x-a)(13x-7a).$$

EXERCISE 48

Resolve into factors :

1. x^2+3x+2.
2. x^2+5x+6.
3. a^2+4a+3.
4. x^2-5x+4.
5. $x^2+7x+10$.
6. $x^2-7x+12$.
7. $x^2+8x+15$.
8. $x^2-2x-15$.
9. $x^2-13x+36$.
10. $x^2-5x-36$.
11. $x^2-14x+24$.
12. $x^2-22x+40$.
13. $x^2+7x-30$.
14. $x^2+2x-48$.
15. $x^2+16x-36$.
16. $x^2+9x-36$.
17. $x^2+11x-42$.
18. $x^2+14x-72$.
19. $x^2-3x-40$.
20. $x^2-11x-80$.
21. $x^2-29x-96$.
22. $x^2-10x-56$.
23. x^2-x-42.
24. x^2-x-72.
25. $x^2+22x+120$.
26. $x^2+16x-80$.
27. $x^2-21x-72$.
28. $x^2+5x-84$.
29. $x^2-20x+96$.
30. $x^2+23x-78$.
31. $x^2-6x-72$.
32. $x^2-25x+84$.
33. $x^2-26x+88$.
34. $x^2+7x-120$.
35. $x^2-2x-80$.
36. $x^2+8x-84$.
37. a^2-a-56.
38. $m^2-9m-90$.
39. $a^2+17a-60$.
40. $a^2-15a+54$.
41. $p^2-22p-48$.
42. m^2+m-72.
43. $m^2+27m-90$.
44. $a^2-29a+120$.
45. $x^2+7x-78$.
46. $a^2-49a-102$.
47. $a^2-19a+60$.
48. $x^2+12x-64$.
49. $a^2-26a-120$.
50. $x^2+8x-105$.
51. $x^2-xy-42y^2$.
52. $a^2-12ab+32b^2$.
53. $m^2+mn-30n^2$.
54. $a^2+ab-12b^2$.
55. $a^2-2ab-15b^2$.
56. $x^2-7xy-8y^2$.
57. $x^2+3xy-40y^2$.
58. $p^2-14pq+48q^2$.
59. $p^2+2pq-80q^2$.
60. $3x^2+60xy-288y^2$.
61. a^4+4a^2-5.
62. x^4+2x^2-15.
63. $2x^4+6x^2-56$.
64. x^6+2x^3-3.
65. a^6-10a^3+16.
66. x^6+26x^3-27.
67. a^6+7a^3-8.
68. x^8-20x^4+64.
69. a^8-11a^4-80.

70. $x^{12}-7x^6-8$.
71. $(a^2+2a)^2-(a^2+2a)-2$.
72. $(x^2+3x)^2+3(x^2+3x)+2$.
73. $(x^2-2x)^2-2(x^2-2x)-3$.
74. $(a^2-3a)^2-3(a^2-3a)-4$.
75. $(x^2-4x)^2-4(x^2-4x)-5$.
76. $(x^2-x)^2-8(x^2-x)+12$.
77. $(x^2-5x)^2+10(x^2-5x)+24$.
78. $(a^2+7a)^2-8(a^2+7a)-180$.
79. $(a^2+6a)^2-32(a^2+6a)-320$.
80. $(x^2-8x)^2-29(x^2-8x)+180$.
81. $2x^2+x-15$.
82. $6a^2-a-15$.
83. $8m^2-6m-9$.
84. $18x^3+21x^2y-72xy^2$.
85. $10a^2-41ab+21b^2$.
86. $12m^2-mn-20n^2$.
87. $12x^2+28xy-5y^2$.
88. $20a^2+ab-30b^2$.
89. $18x^2-51xy+35y^2$.
90. $60x^3y+115x^2y^2-120xy^3$.

97. Quantities of the form x^2+px+q resolved into factors by expressing them as the difference of two squares.

The method will be best illustrated by the solution of a few typical cases.

Example 1. Resolve into factors $x^2-7x+12$.

$$x^2-7x+12=x^2-7x+(\tfrac{7}{2})^2-(\tfrac{7}{2})^2+12$$

[adding and subtracting $(\tfrac{7}{2})^2$]

$$=\{x^2-7x+(\tfrac{7}{2})^2\}-(\tfrac{49}{4}-12)=(x-\tfrac{7}{2})^2-\tfrac{1}{4}$$
$$=\{(x-\tfrac{7}{2})+\tfrac{1}{2}\}\{(x-\tfrac{7}{2})-\tfrac{1}{2}\}=(x-3)(x-4).$$

Note. *It must be noticed that we have added to x^2-7x the square of half of 7 (i.e., the square of the half the coefficient of x) to get a perfect square. Generally speaking, x^2+2ax (or, x^2-2ax) becomes a complete square when a^2 is added to it.*

97A. Expressions of the form px^2+qx+r can be resolved into factors by expressing them as the difference of two squares in the following method :

$$px^2+qx+r=p\left(x^2+\frac{q}{p}x+\frac{r}{p}\right)$$
$$=p\left\{x^2+2\cdot\frac{q}{2p}\cdot x+\left(\frac{q}{2p}\right)^2-\left(\frac{q}{2p}\right)^2+\frac{r}{p}\right\}$$
$$=p\left\{\left(x+\frac{q}{2p}\right)^2-\left(\frac{q^2}{4p^2}-\frac{r}{p}\right)\right\}$$
$$=p\left\{\left(x+\frac{q}{2p}\right)^2-\frac{q^2-4pr}{4p^2}\right\}$$
$$=p\left\{\left(x+\frac{q}{2p}\right)^2-\left(\frac{\sqrt{q^2-4pr}}{2p}\right)^2\right\}$$
$$=p\left\{x+\frac{q}{2p}+\frac{\sqrt{q^2-4pr}}{2p}\right\}\left\{x+\frac{q}{2p}-\frac{\sqrt{q^2-4pr}}{2p}\right\}.$$

Example 2. Resolve into factors $3x^2+11x-4$.

$$\begin{aligned}3x^2+11x-4&=3(x^2+\tfrac{11}{3}x-\tfrac{4}{3})=3\{x^2+\tfrac{11}{3}x+(\tfrac{11}{6})^2-(\tfrac{11}{6})^2-\tfrac{4}{3}\}\\&=3\{(x+\tfrac{11}{6})^2-(\tfrac{121}{36}+\tfrac{4}{3})\}=3\{(x+\tfrac{11}{6})^2-\tfrac{169}{36}\}\\&=3\{(x+\tfrac{11}{6})+\tfrac{13}{6}\}\{(x+\tfrac{11}{6})-\tfrac{13}{6}\},\qquad[\because\ \tfrac{169}{36}=(\tfrac{13}{6})^2]\\&=3(x+4)(x-\tfrac{1}{3})=(x+4)(3x-1).\end{aligned}$$

Example 3. Resolve into factors $8x^2-10x+3$.

$$\begin{aligned}8x^2-10x+3&=8\{x^2-\tfrac{10}{8}x+\tfrac{3}{8}\}=8\{x^2-\tfrac{5}{4}x+(\tfrac{5}{8})^2-(\tfrac{25}{64}-\tfrac{3}{8})\}\\&=8\{(x-\tfrac{5}{8})^2-\tfrac{1}{64}\}=8\{(x-\tfrac{5}{8})+\tfrac{1}{8}\}\{(x-\tfrac{5}{8})-\tfrac{1}{8}\}\\&=8(x-\tfrac{1}{2})(x-\tfrac{3}{4})=\{2(x-\tfrac{1}{2})\}\{4(x-\tfrac{3}{4})\}\\&=(2x-1)(4x-3).\end{aligned}$$

Example 4. Resolve into factors $2a^2+5ab-12b^2$.

$$\begin{aligned}2a^2+5ab-12b^2&=2(a^2+\tfrac{5}{2}ab-6b^2)\\&=2\left\{a^2+\frac{5}{2}ab+\left(\frac{5b}{4}\right)^2-\left(\frac{25b^2}{16}+6b^2\right)\right\}\\&=2\{(a+\tfrac{5}{4}b)^2-\tfrac{121}{16}b^2\}\\&=2\{(a+\tfrac{5}{4}b)+\tfrac{11}{4}b\}\{(a+\tfrac{5}{4}b)-\tfrac{11}{4}b\}\\&=2(a+4b)(a-\tfrac{3}{2}b)=(a+4b)(2a-3b).\end{aligned}$$

Example 5. Resolve into factors $ax^2+(a^2+1)x+a$.

$$\begin{aligned}ax^2+(a^2+1)x+a&=a\left\{x^2+\frac{a^2+1}{a}\cdot x+1\right\}\\&=a\left\{x^2+\frac{a^2+1}{a}\cdot x+\left(\frac{a^2+1}{2a}\right)^2-\left(\frac{a^4+2a^2+1}{4a^2}-1\right)\right\}\\&=a\left\{\left(x+\frac{a^2+1}{2a}\right)^2-\frac{a^4-2a^2+1}{4a^2}\right\}\\&=a\left\{\left(x+\frac{a^2+1}{2a}\right)+\frac{a^2-1}{2a}\right\}\left\{\left(x+\frac{a^2+1}{2a}\right)-\frac{a^2-1}{2a}\right\}\\&=a\left(x+a\right)\left(x+\frac{1}{a}\right)\\&=(x+a)(ax+1).\end{aligned}$$

Similarly, it may be shown that

$$\begin{aligned}ax^2-(a^2+1)x+a&=(x-a)(ax-1),\\ax^2+(a^2-1)x-a&=(x+a)(ax-1),\\ax^2-(a^2-1)x-a&=(x-a)(ax+1).\end{aligned}$$

Note. *It is useful to remember these results as we are thus enabled to write down at once the factors of any expression which agrees in form with any of those considered above. For instance, we can at once say that :*

$$3x^2-10x+3=(x-3)(3x-1),$$
$$4x^2-15x-4=(x-4)(4x+1),$$
$$5x^2+24x-5=(x+5)(5x-1)\text{; } \textit{and so on.}$$

Example 6. Resolve into factors

$$4(x^2+2x+5)^2+17(x^2+2x+5)(x^2+6x)+4(x^2+6x)^2.$$

Putting a for x^2+2x+5 and b for x^2+6x, the given expression becomes $4a^2+17ab+4b^2$, and it is easy to see that

$$4a^2+17ab+4b^2=(a+4b)(4a+b).$$

Hence, the given expression

$$=\{(x^2+2x+5)+4(x^2+6x)\}\{4(x^2+2x+5)+(x^2+6x)\}$$
$$=(5x^2+26x+5)(5x^2+14x+20)$$
$$=(x+5)(5x+1)(5x^2+14x+20).$$

EXERCISE 49

Resolve the following expressions into factors applying the method of this article :

1. $x^2+4x+3.$
2. $x^2+6x+5.$
3. $x^2+8x+15.$
4. $x^2-10x+21.$
5. $x^2-2x-48.$
6. $x^2-4x-45.$
7. $x^2-12x+32.$
8. $x^2-6x-55.$
9. $3x^3-6x^2-105x.$
10. $2x^2-5x-3.$
11. $3x^2-5x-2.$
12. $3x^2+14x+8$
13. $4x^2+7x-2.$
14. $6x^2+x-2.$
15. $6x^2-5x-4.$
16. $6x^2+7x-3.$
17. $8x^2+2x-15.$
18. $4x^2+4x-35.$
19. $6x^2-x-12.$
20. $3x^2-16x-12.$
21. $2x^2-9x-35.$
22. $2x^2+5x-42.$
23. $3x^2+13x-30.$
24. $12x^2+x-6.$
25. $2a^2+7ab-15b^2.$
26. $6x^2-13xy+6y^2.$
27. $6m^2-11mn-10n^2.$
28. $3p^2+5pq-12q^2.$
29. $8a^2-14ab-15b^2.$
30. $10m^2+11mn-6n^2.$
31. $12x^2+13xy-4y^2.$
32. $15a^2-11ab-12b^2.$
33. $2a^2-5ab+2b^2.$
34. $3a^2-8ab-3b^2.$
35. $3x^2+8xy-3y^2.$
36. $4a^2+15a-4.$
37. $4a^2-17ab+4b^2.$
38. $5x^2-24x-5.$
39. $5x^2-26xy+5y^2.$
40. $6x^2+37x+6.$
41. $6a^2+35ab-6b^2$
42. $6a^2-35ab-6b^2.$

43. $7a^2-50ab+7b^2$.
44. $7a^2+48ab-7b^2$.
45. $7a^2-48ab-7b^2$.
46. $8x^2+63xy-8y^2$.
47. $9x^2-82xy+9y^2$.
48. $10x^2+99xy-10y^2$.
49. $2(a+b)^2+3(a+b)-2$.
50. $2(x^2+y^2)^2-3xy(x^2+y^2)-2x^2y^2$.
51. $2(a^2+b^2)^2+5ab(a^2+b^2)+2a^2b^2$.
52. $4(x^2-4xy+y^2)^2+15xy(x^2-4xy+y^2)-4x^2y^2$.
53. $2x^4-5x^2-12$.
54. $8a^4-14a^2b^2-9b^4$.
55. $9a^4+2a^2b^2-32b^4$.
56. $8x^6-65x^3+8$.
57. $4a^8-17a^4b^4+4b^8$.

CHAPTER XIII

EASY IDENTITIES

98. We have explained the significance of 'Identity' in Art. 62. In fact, an Identity is a statement that two expressions are equal for *all* values of the letters involved. Each of the two expressions constituting an identity is called a *side* or a *member* of the identity.

Thus, $5x=2x+3x$ is an identity, since the expressions $5x$ and $2x+3x$ are equal for all values of x. The sides of this identity are $5x$ and $2x+3x$, $5x$ being the *left-hand* side and $2x+3x$, the *right-hand* side.

Similarly $(a+b)^2=a^2+2ab+b^2$ is an identity, since the equality of both sides holds for all values of a and b. As a matter of fact, every formula established in Chapter IV is an identity.

99. An identity is proved when its two sides are shown to be equal.

To establish the equality of the two sides of an identity, reduce each side to its simplest form. Identity is proved if these forms are found to be equal. A better method, however, is to reduce one of the sides of the identity to the form of the other by simplification and transformation with the aid of the formulæ enumerated in Chapter XI.

Sometimes the sides of an identity may be conveniently expressed in simpler forms by substituting letters for groups of terms in the identity. Such substitutions must be effected wherever necessary.

The following examples will illustrate the process :

Example 1. Prove that $(a+3b)^2+(a-3b)^2=2a^2+18b^2$.

The left-hand side $=(a^2+6ab+9b^2)+(a^2-6ab+9b^2)$ [Arts. 54, 55]

$=2a^2+18b^2$.

Example 2. Prove that

$$a^2+b^2+c^2-ab-bc-ca=\tfrac{1}{2}[(b-c)^2+(c-a)^2+(a-b)^2].$$

The left-hand side $=\tfrac{1}{2}[2a^2+2b^2+2c^2-2ab-2bc-2ca]$

$=\tfrac{1}{2}[(a^2+b^2)+(b^2+c^2)+(c^2+a^2)-2ab-2bc-2ca]$

$=\tfrac{1}{2}[(b^2-2bc+c^2)+(c^2-2ca+a^2)+(a^2-2ab+b^2)]$

$=\tfrac{1}{2}[(b-c)^2+(c-a)^2+(a-b)^2]$. [Art. 55]

Example 3. Prove that

$$(x+5y-3z)^3+(x-5y+3z)^3+6x(x+5y-3z)(x-5y+3z)=8x^3.$$

Substituting a or $x+5y-3z$ and b for $x-5y+3z$, we have

the left-hand side $=a^3+b^3+6x.ab$

$=a^3+b^3+3ab(a+b)$ [since, $a+b=(x+5y-3z)+(x-5y+3z)=2x$]

$=(a+b)^3$ [Art. 57]

$=(2x)^3=8x^3$.

Example 4. Prove that

$$(b+c)(b-c)+(c+a)(c-a)+(a+b)(a-b)=0.$$

The left-hand side $=(b^2-c^2)+(c^2-a^2)+(a^2-b^2)$ [Art. 56]

$=b^2-c^2+c^2-a^2+a^2-b^2=0$.

Example 5. If $s=a+b+c$, prove that

$$(as+bc)(bs+ca)(cs+ab)=(a+b)^2(b+c)^2(c+a)^2.$$ [C. U. 1902]

$as+bc=a(a+b+c)+bc$

$=a^2+a(b+c)+bc=a^2+ab+ac+bc$

$=a(a+b)+c(a+b)=(a+b)(a+c)$. [Art. 61]

Similarly, $bs+ca=b(a+b+c)+ca=b^2+b(a+c)+ac$

$=b^2+ab+bc+ac=(b+c)(b+a)$;

and $cs+ab=c(a+b+c)+ab=c^2+c(a+b)+ab$

$=c^2+ca+cb+ab=(c+a)(c+b)$.

$\therefore$ the left-hand side

$=(a+b)(a+c)(b+c)(b+a)(c+a)(c+b)$

$=(a+b)^2(b+c)^2(c+a)^2$.

Example 6. Prove that $4a^2b^2-(a^2+b^2-c^2)^2$

$$=s(s-2a)(s-2b)(s-2c), \text{ where } s=a+b+c.$$

The left-hand side $=(2ab)^2-(a^2+b^2-c^2)^2$

$$\begin{aligned}
&=\{2ab+(a^2+b^2-c^2)\}\{2ab-(a^2+b^2-c^2)\}\\
&=\{(a^2+2ab+b^2)-c^2\}\{c^2-(a^2+b^2-2ab)\}\\
&=\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}\\
&=(\overline{a+b}+c)(\overline{a+b}-c)(c+\overline{a-b})(c-\overline{a-b})\\
&=(a+b+c)(a+b-c)(c+a-b)(c-a+b)\\
&=(a+b+c)(\overline{a+b+c}-2c)(\overline{c+a+b}-2b)(\overline{b+c+a}-2a)\\
&=s(s-2c)(s-2b)(s-2a)\\
&=s(s-2a)(s-2b)(s-2c).
\end{aligned}$$

Example 7. If $2s=a+b+c$, prove that

$$(s-a)^3+(s-b)^3+3(s-a)(s-b)c=c^3.$$

We have, $c=2s-(a+b)=(s-a)+(s-b)$

Hence, $(s-a)^3+(s-b)^3+3(s-a)(s-b)c$

$$\begin{aligned}
&=(s-a)^3+(s-b)^3+3(s-a)(s-b)\{(s-a)+(s-b)\}\\
&=\{(s-a)+(s-b)\}^3=c^3.
\end{aligned}$$

Example 8. Prove that $(x-y)^2+(y-z)^2+(z-x)^2$

$$=2(x-y)(x-z)+2(y-z)(y-x)+2(z-x)(z-y).$$

Putting $\left.\begin{array}{l} a \text{ for } x-y\\ b \text{ for } y-z\\ c \text{ for } z-x\end{array}\right\}$ we have $a+b+c=0$.

Hence,

$$\begin{aligned}
&\{(x-y)^2+(y-z)^2+(z-x)^2\}-\{2(x-y)(x-z)+2(y-z)(y-x)+2(z-x)(z-y)\}\\
&=(a^2+b^2+c^2)-\{2a(-c)+2b(-a)+2c(-b)\}\\
&=a^2+b^2+c^2+2ac+2ab+2bc=(a+b+c)^2=0\,;
\end{aligned}$$

$\therefore\ (x-y)^2+(y-z)^2+(z-x)^2$

$$=2(x-y)(x-z)+2(y-z)(y-x)+2(z-x)(z-y).$$

Example 9. If $2s=a+b+c$, show that

$$2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)=2s^2-a^2-b^2-c^2.$$

Since, $2x+2y+2z=(x+y)+(y+z)+(z+x)$,

we must have, $2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)$

$$\begin{aligned}
&=\{(s-a)(s-b)+(s-b)(s-c)\}+\{(s-b)(s-c)\\
&\qquad+(s-c)(s-a)\}+\{(s-c)(s-a)+(s-a)(s-b)\}.
\end{aligned}$$

Now, $(s-a)(s-b)+(s-b)(s-c)=(s-b)\{(s-a)+(s-c)\}$
$=(s-b)\{2s-a-c\}=(s-b)b.$

Similarly, $(s-b)(s-c)+(s-c)(s-a)=(s-c)c,$
and $(s-c)(s-a)+(s-a)(s-b)=(s-a)a.$

Hence, the given expression $=(s-b)b+(s-c)c+(s-a)a$
$=s(b+c+a)-b^2-c^2-a^2$
$=2s^2-a^2-b^2-c^2.$

Example 10. If $a+b+c=0$, prove that $a^3+b^3+c^3=3abc.$

Since, $a+b+c=0,\ c=-(a+b),$

$\therefore$ the left-hand side $=a^3+b^3+\{-(a+b)\}^3$
$=a^3+b^3-\{a^3+b^3+3ab(a+b)\}$ [Art. 57]
$=-3ab(a+b)=3ab\{-(a+b)\}=3abc.$

Note. *Evidently the identity $a^3+b^3+c^3=3abc$ is true only if $a+b+c=0$. Such identities which are true only for some particular value of the symbols involved is called* ***Conditional Identities.***

Example 11. If $a+b+c=0$, prove that
$a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2.$ [Allahabad, 1923]

Since, $a+b+c=0$, we have by transposition,
$a=-(b+c),\ b=-(c+a),\ c=-(a+b)$;

$\therefore$ $a^2+ab+b^2=\{-(b+c)\}^2+\{-(b+c)\}b+b^2$
[since, $a=-(b+c)$]
$=(b+c)^2-(b+c)b+b^2$
$=b^2+2bc+c^2-b^2-bc+b^2$
$=b^2+bc+c^2.$

Also, $a^2+ab+b^2=a^2+a\{-(c+a)\}+\{-(c+a)\}^2$
[since, $b=-(c+a)$]
$=a^2-a(c+a)+(c+a)^2$
$=a^2-ca-a^2+c^2+2ca+a^2=c^2+ca+a^2.$

Hence, $a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2.$

Alternative Method :

$a^2+ab+b^2=a(a+b)+b^2$
$=\{-(b+c)\}(-c)+b^2=(b+c)c+b^2$
$=bc+c^2+b^2=b^2+bc+c^2.$

Also, $b^2+bc+c^2=b(b+c)+c^2$

$$=\{-(c+a)\}(-a)+c^2$$

$$=(c+a)a+c^2=ca+a^2+c^2=c^2+ca+a^2.$$

Hence, $a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2$.

Example 12. If $x=b-c+a$, $y=c-a+b$, $z=a-b+c$, prove that

$$(b-a)x+(c-b)y+(a-c)z=0.$$

We have

$$(b-a)x=(b-a)(b-c+a)=(b-a)\{(b+a)-c\}$$

$$=(b-a)(b+a)-(b-a)c=b^2-a^2-bc+ac\,; \qquad \text{[Art. 56]}$$

$$(c-b)y=(c-b)(c-a+b)=(c-b)\{(c+b)-a\}$$

$$=(c-b)(c+b)-(c-b)a=c^2-b^2-ca+ab\,;$$

$$(a-c)z=(a-c)(a-b+c)=(a-c)\{(a+c)-b\}$$

$$=(a-c)(a+c)-(a-c)b=a^2-c^2-ab+bc\,;$$

$$\therefore \quad (b-a)x+(c-b)y+(a-c)z$$

$$=b^2-a^2+c^2-b^2+a^2-c^2-bc+ac-ca+ab-ab+bc=0.$$

Example 13. If $x=b+c$, $y=c+a$, $z=a+b$, prove that

$$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

The left-hand side $=\frac{1}{2}[2x^2+2y^2+2z^2-2yz-2zx-2xy]$

$$=\tfrac{1}{2}[(x^2-2xy+y^2)+(y^2-2yz+z^2)$$

$$+(z^2-2zx+x^2)] \qquad \text{[re-arranging terms]}$$

$$=\tfrac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2] \qquad \text{[Art. 55]}$$

$$=\tfrac{1}{2}[\{(b+c)-(c+a)\}^2+\{(c+a)-(a+b)\}^2$$

$$+\{(a+b)-(b+c)\}^2] \qquad \text{[substituting for } x, y, z \text{]}$$

$$=\tfrac{1}{2}[(b-a)^2+(c-b)^2+(a-c)^2]$$

$$=\tfrac{1}{2}[(b^2-2ba+a^2)+(c^2-2cb+b^2)+(a^2-2ac+c^2)] \quad \text{[Art. 55]}$$

$$=\tfrac{1}{2}[(2a^2+2b^2+2c^2-2bc-2ca-2ab) \qquad \text{[collecting terms]}$$

$$=a^2+b^2+c^2-bc-ca-ab.$$

Example 14. If $2s=a+b+c$, prove that

$$(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2. \qquad \text{[Allahabad, 1926]}$$

The left-hand side

$$=(s^2-2as+a^2)+(s^2-2bs+b^2)+(s^2-2cs+c^2)+s^2$$

$$=4s^2-2s(a+b+c)+a^2+b^2+c^2$$

$$=4s^2-2s\times 2s+a^2+b^2+c^2$$

$$=4s^2-4s^2+a^2+b^2+c^2=a^2+b^2+c^2.$$

EXERCISE 50

Show that :

1. $(a^2+ax-x^2)(a^2-ax+x^2)=a^4-a^2x^2+2ax^3-x^4$.
2. $(a^2-ax+x^2)(ax-a^2+x^2)=x^4-a^2x^2+2a^3x-a^4$.
3. $(a+b+c)(a-b-c)+(b+c-a)(a-b+c)=2b(a-b-c)$.
4. $2(x^3-x)+3x(x+1)=x(x+1)(2x+1)$.
5. $x^4+x+x(x+1)(2x+1)-2x(x+1)=x^2(x+1)^2$.
6. $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$.
7. $(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2=2(a+b+c+d)(a-d)$.
8. $(a+b+c-d)(d-a-b+c)=c^2-(a+b-d)^2$.
9. The product of $(b+c)^2-a^2$ and $a^2-b^2-c^2+2bc$ is $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.
10. $(a+b+c)^2-(a+b-c)^2+(a+c-b)^2-(b+c-a)^2=8ac$.

Prove that :

11. $(a^2+b^2+c^2)^2-(b^2+c^2-a^2)^2-(a^2-b^2+c^2)^2$
$$+(a^2+b^2-c^2)^2=8a^2b^2.$$
12. $(b-c+d+a)(d+a-b+c)+(c-d+a+b)(b+c+d-a)=4(ad+bc)$.
13. $(b+c+a-d)(b+c-a+d)=2(ad+bc)-(a^2-b^2-c^2+d^2)$.
14. $4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$
$$=(a+d+b-c)(a+d-b+c)(b+c+a-d)(b+c-a+d).$$
15. $(x-y+z)^2+(y-z+x)^2+(z-x+y)^2+2(x-y+z)(y-z+x)$
$$+2(y-z+x)(z-x+y)+2(z-x+y)(x-y+z)=(x+y+z)^2.$$
16. $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$
$$=(ay-bx)^2+(cx-az)^2+(bz-cy)^2.$$
17. $(a+c)^3-(b+c)^3-3(a+c)(b+c)(a-b)=(a-b)^3$.
18. $(x-ay+bz)^3+(x+ay-bz)^3+6x(x-ay+bz)(x+ay-bz)=8x^3$.
19. $4(a+b+c)^2=(a+b)^2+(b+c)^2+(c+a)^2$
$$+2(a+b)(b+c)+2(b+c)(c+a)+2(c+a)(a+b).$$
20. $8(a+b+c)^3=(a+b)^3+(b+2c+a)^3+6(a+b)(b+2c+a)(a+b+c)$.
21. $27(a+b+c)^3=(a+3b+2c)^3+(2a+c)^3$
$$+9(a+3b+2c)(2a+c)(a+b+c).$$
22. If $s=a+b+c$, show that
$$(s-3a)^2+(s-3b)^2+(s-3c)^2=3\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$
23. If $ab+bc+ca=0$, prove that
    (i) $a^2+b^2+c^2=(a+b+c)^2$;
    (ii) $a^2b^2+b^2c^2+c^2a^2=-2abc(a+b+c)$.

24. If $2s=x+y+z$, prove that
$$4y^2z^2-(y^2+z^2-x^2)^2=16s(s-x)(s-y)(s-z).$$

25. Prove that
$$(x+2y+19z)^3+(x-2y-19z)^3+6x(x+2y+19z)(x-2y-19z)$$
$$=(5x+6y-z)^3+(z-6y-3x)^3+6x(5x+6y-z)(z-6y-3x).$$

26. Prove that $(a+2b+3c)^2+(a-b-3c)^2+2(a+2b+3c)(a-b-3c)$
$$=(3a+y+z)^2+(a+y+z-b)^2-2(3a+y+z)(a+y+z-b).$$

27. Show that $(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$.

28. Prove that $(x-y)^2-(y-z)(z-x)$
$$=(y-z)^2-(z-x)(x-y)=(z-x)^2-(x-y)(y-z)$$
$$=-\{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)\}.$$

29. Prove that $(a-b)^2-(b-c)^2-(c-a)^2=2(b-c)(c-a)$,
$$(b-c)^2-(c-a)^2-(a-b)^2=2(c-a)(a-b),$$
$$(c-a)^2-(a-b)^2-(b-c)^2=2(a-b)(b-c).$$

30. Prove that $(a-b)^2+(a-b)(b-c)+(b-c)^2$
$$=(b-c)^2+(b-c)(c-a)+(c-a)^2$$
$$=(c-a)^2+(c-a)(a-b)+(a-b)^2.$$

MISCELLANEOUS EXERCISES III

I

1. Arrange the following expression : (i) according to descending powers of y, and (ii) according to ascending powers of z :
$$x^3z+xy^3-x^3y-xy^2z-xz^3+xyz^2-2yz^3-2y^3z.$$

2. Find the value of :
$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}\left(7x-4y\right)\right\}\right],$$
when $x=-\frac{1}{2}$ and $y=2$.

3. If $x-\dfrac{1}{x}=p$, prove that $x^3-\dfrac{1}{x^3}=p^3+3p$.

4. Write down the quotient of x^5-y^5 by $x-y$.

5. Simplify $(a+b+c)^2-(a-b+c)^2+(a+b-c)^2-(b+c-a)^2$, and find its numerical value when $a=b=c=-4$.

6. Find the sum of $x^2-(x-y+z)(x+y-z)$,
$y^2-(y-x+z)(y+x-z)$ and $z^2-(z-x+y)(z+x-y)$.

7. Reduce $(a-b+c+d)(a+b+c-d)$ to the form A^2-B^2.

8. Resolve into factors $4x^2+12xy+9y^2-8x-12y$.

II

1. Find an expression which exceeds $ax^3+bx^2y+3cxy^2+dy^3$ by as much as it falls short of four times

$$2ax^3+\frac{1}{2}(3a-b)x^2y+\frac{1}{2}(3a-c)xy^2+5dy^3.$$

2. Resolve the sum of the following expressions into simple factors:

$$(b-1)m^4+am^3+(c-b)m^2-bm-2,\ am^3-(c-a)m^2+(a+b)m+1$$
$$\text{and } (a-b+1)m^4-(2a-b)m^3+(a+b)m^2-(a-2b)m+1.$$

3. Multiply $x^{\frac{4}{3}}+2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+2x^{\frac{2}{3}}+1$ by $x^{\frac{1}{3}}-2x^{\frac{1}{3}}+1$.

4. Prove that $\{(ac+bd)x+(ad-bc)y\}^2+\{(ac+bd)y-(ad-bc)x\}^2$
$$=(a^2+b^2)(c^2+d^2)(x^2+y^2).$$

5. Find the continued product of $x-a$, $x-b$ and $x-c$. Hence show that $(x-3)^3=x^3-9x^2+27x-27$.

6. Divide $x^5-px^4+qx^3-qx^2+px-1$ by $x-1$.

7. Find the quotient when the product of $a^6+a^5b-a^3b^3+ab^5+b^6$ and a^2-ab+b^2 is divided by $a^4-a^2b^2+b^4$ and show that its defect from $(a^2+b^2)^2$ is a^2b^2.

8. Resolve into factors:

(i) $ab-ac-b^2+bc$; (ii) $b^2-12ac-4a^2-9c^2$.

III

1. Find the sum of

$(\sqrt{b}-\sqrt{c}+\sqrt{a})x^3+(\sqrt{bc}-\sqrt{ca}+\sqrt{ab})x^2+(\sqrt{abc}-2m+n)x+3u$, $(\sqrt{c}-\sqrt{a}+\sqrt{b})x^3+(\sqrt{ca}-\sqrt{ab}+\sqrt{bc})x^2+(\sqrt{abc}-2n+m)x+2(v-u)$ and $(p-2\sqrt{b})x^3+(q-2\sqrt{bc})x^2+(m+n+r-2\sqrt{abc})x+(s-u-2v)$.

2. Subtract the sum of $3a^3-5a^2b+2b^3$, $8a^2b-3b^3+2ab^2$, $5ab^2-4a^3-3a^2b$ and $2a^3-6ab^2+4b^3$ from $a(a^2+b^2)$.

3. If $a+b=8$ and $ab=5$, find the value of a^3+b^3.

4. Find the value of $49c^2+9(a+b)^2-42(a+b)c$,
when $a=89$, $b=-69$, $c=8$.

5. Divide $x^3(y-z)+y^3(z-x)+z^3(x-y)$ by y^2-xz-z^2+xy.

6. Resolve into factors $a-3+16a^2+64a^3$ after reducing it to the form of $(A-B)+(A^2-B^2)+(A^3-B^3)$.

7. Show that $(1+x+x^2)^2-(1-x+x^2)^2=4x(1+x^2)$.

8. If $a_1+a_2+a_3+\cdots\cdots+a_n=\frac{n}{2}s$, show that

$$(s-a_1)^2+(s-a_2)^2+(s-a_3)^2+\cdots\cdots+(s-a_n)^2$$
$$=a_1^2+a_2^2+a_3^2+\cdots\cdots+a_n^2.$$

IV

1. Simplify the expression

$$(l^2r-3lmn+2m^3)p^3+3(lmr+m^2n-2ln^2)p^2q+3(2m^2r-lnr-mn^2)pq^2$$
$$+(3mnr-lr^2-2n^3)q^3,$$

where $p=-m$ and $q=l$.

2. What must be subtracted from $\frac{1}{3}a^3x^4+3.7a^2bx^3-3.257ab^2x^2+\frac{5}{3}b^3x+9$ so as to make the difference equal to the sum of $4\frac{1}{3}a^2bx^3-.007ab^2x^2+2\frac{2}{3}b^3x-5\frac{2}{3}a^3x^4+6$, $5\frac{1}{6}x-3\frac{1}{4}a^2bx^3+a^3x^4-.05ab^2x^2+11$ and $2a^3x^4-1\frac{3}{8}a^2bx^3-6.2ab^2x^2-10\frac{1}{3}b^3x-20$?

3. Multiply $a^{\frac{5}{2}}-2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}-8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}-32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+2b^{\frac{1}{3}}$.

4. Arrange the following expressions according to descending powers of a :

(i) $a^3+b^3+c^3-3abc$; (ii) $a^2(b-c)+b^2(c-a)+c^2(a-b)$;
(iii) $a^4(b-c)+b^4(c-a)+c^4(a-b)$.

5. Find the product of $x+a$, $x+b$ and $x+c$.

Hence, deduce the coefficients of x^2 and x in

$$(x-7)(x+8)(x-12).$$

6. Prove that $(ab+cd+ac+bd)(ab+cd-ac-bd)$
$$=a^2b^2+c^2d^2-a^2c^2-b^2d^2.$$

7. If $a=q+r+s$, $b=r+s-p$, $c=p+q+r$, prove that
$$a^2+b^2+c^2-2ab-2ac+2bc=r^2.$$

8. Divide $a^3+8b^3+27c^3-18abc$ by $a^2+4b^2+9c^2-6bc-3ca-2ab$.

V

1. Find the value of $49a^2+126ab+81b^2$, when $a=46$, $b=-37$.

2. Find the expression which falls short of $bx^4y-dx^2y^3-fy^5$ by as much as it exceeds $ax^5-cx^3y^2+exy^4$.

3. If $2s = a + b + c$, show that

$$(s-a)^2 + (s-b)^2 + (s-c)^2 + s^2 = a^2 + b^2 + c^2.$$

4. Simplify $(5a-7c)^3 + (8c-3a)^3 + 3(2a+c)(5a-7c)(8c-3a)$.

5. Reduce the following to its simplest form :

$$(2x^3 - x^2 + 3x - 4)(2x^3 + x^2 + 3x + 4)$$
$$+ (2x^3 + x^2 - 3x + 4)(2x^3 + x^2 + 3x - 4).$$

6. Show that $\dfrac{x^8 + x^4 + 1}{x + \sqrt{x} + 1} = (x - \sqrt{x} + 1)(x^2 - x + 1)(x^4 - x^2 + 1)$.

7. Divide $a - b$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.

8. Resolve into factors :

(i) $6a^4x^2 + a^3x - 6a^3x^3 - a^2x^2$; (ii) $xy(1+z^2) + z(x^2+y^2)$.

VI

1. Find the value of $8765943 \times 8765943 - 8765938 \times 8765938$.

2. Find the value of

$$27a^3 + 108a^2b + 144ab^2 + 64b^3, \text{ when } a = 29,\ b = -23.$$

3. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$; and hence show that $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$.

4. Find the quotient when $(ax+b)^2 + (cx+d)^2 + (bx-a)^2 + (dx-c)^2$ is divided by $a^2 + b^2 + c^2 + d^2$.

5. Express $(x-1)(x-3)(x-4)(x-6) + 34$ as the sum of two squares ; hence show that it is always a positive quantity and that its value is equal to 25 when $x^2 - 7x + 9 = 0$.

6. Resolve $(a^2 - b^2 - c^2 + d^2)^2 - 4(ad - bc)^2$ into four factors.

7. Resolve into factors :

(i) $a^2 - 2ab + b^2 + 2a - 2b$; (ii) $6a^2 - ab - b^2 + 6a - 3b$;

(iii) $15x^2 - 4xy - 4y^2 + 10x + 4y$.

8. Divide $(2x-y)^2a^4 - (x+y)^2a^2x^2 + 2(x+y)ax^4 - x^6$

by $(2x-y)a^2 - (x+y)ax + x^3$.

VII

1. If $x + y + z = 8$ and $x^2 + y^2 + z^2 = 50$, find the value of $xy + yz + zx$.

2. Prove that $(2a - 3b)^2 + (3b - 5c)^2 + (5c - 2a)^2$

$$= 2(2a-3b)(2a-5c) + 2(3b-5c)(3b-2a) + 2(5c-2a)(5c-3b).$$

3. Find the product of

$$x+y+z-x^{\frac{1}{2}}y^{\frac{1}{2}}-y^{\frac{1}{2}}z^{\frac{1}{2}}-z^{\frac{1}{2}}x^{\frac{1}{2}} \text{ and } x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}.$$

4. Divide $a^2(x^2-a^2)-ab(x+a)^2+b(x^3+a^3)$

by $a^2(x-a)+bx(x-2a)$.

5. Show that

$$(16x^5-20x^3+5x)^2+(1-x^2)\{16(1-x^2)^2-20(1-x^2)+5\}^2=1.$$

6. Find the continued product of

$x+y+z$, $x-y+z$, $x+y-z$ and $z-x+y$.

7. Resolve into factors :

(i) $6x^2+x-15$; (ii) $35(x-y)^2-41(x-y)+12$;

(iii) $11x^2-54xy^2+63y^4$.

8. If $x+y+z=0$, show that

$(x+y)(y+z)(z+x)=-xyz$ and $x^3+y^3+z^2=3xyz$.

VIII

1. Multiply together the expressions $1+ax+\frac{a(a-1)}{2}x^2$ and $1+bx+\frac{b(b-1)}{2}x^2$ as far as the term involving x^2.

2. If $x+y+z=15$ and $xy+yz+zx=85$, find the value of $x^2+y^2+z^2$.

3. If $a^2+b^2=1=c^2+d^2$, show that $(ad-bc)(ad+bc)=(a-c)(a+c)$.

4. Divide $(ax+by)^3+(ax-by)^3+(bx-ay)^3+(bx+ay)^3$

by $(a+b)^2x^2-3ab(x^2-y^2)$.

5. Evaluate $x^2+\frac{1}{x^2}$, $x^3+\frac{1}{x^3}$ and $x^4+\frac{1}{x^4}$, when $x+\frac{1}{x}=a$.

6. If $bx=ay$, prove that $(x^2+y^2)(a^2+b^2)=(ax+by)^2$. [B. U. 1910]

7. Show that $(x^2+y^2)(x^2+z^2)+2x(x^2+yz)(y+z)+4x^2yz$

$=(x^2+xy+xz+yz)^2$.

8. Resolve into factors : $x^4-11x^2y^2+y^4$. [B. U. 1897]

IX

1. (i) Multiply a^2+ax+x^2 by a^2-ax+x^2.

(ii) Find, without actual multiplication, the coefficient of x^4 in the product of

$$(2x^3-3x^2+4x+4)(5x^4+4x^2-2x+1).$$

2. Show that $(a^2+2ab+b^2-c^2)(a^2-2ab+b^2+c^2)$
$$=(a^2-b^2)^2+(4ab-c^2)c^2$$

3. If $a^2+b^2=1=c^2+d^2$, show that $(ac-bd)^2+(ad+bc)^2=1$.

4. Write down the expansion of $\left(x+\frac{2}{x}\right)^5$.

5. Show that $(a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2)=a^4+b^4$.

6. Divide $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$ by $ab+bc+ca$.

7. Show that $(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)$
$$=a^2(b-c)+b^2(c-a)+c^2(a-b).$$

8. Solve the equation $(7+x)(8-x)-\frac{7x}{3}=17x+1-x^2$.

X

1. If $x+\frac{1}{x}=2(a+m)$, $x-\frac{1}{x}=2b$, $y+\frac{1}{y}=2(c+n)$ and $y-\frac{1}{y}=2d$ find the value of $xy+\frac{1}{xy}$.

2. Simplify $\left(\frac{a}{b}+\frac{b}{a}\right)^4-2\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)^2+\left(\frac{a}{b}-\frac{b}{a}\right)^4$.

3. Show that $(1+a)^2(1+c^2)-(1+c)^2(1+a^2)=2(a-c)(1-ac)$.

4. Show that $(b^2-c^2)(b+c-2a)^2+(c^2-a^2)(c+a-2b)^2$
$$+(a^2-b^2)(a+b-2c)^2=0, \text{ if } a+b+c=0.$$

5. Multiply $a+b^{\frac{2}{3}}+c^{\frac{1}{2}}-b^{\frac{1}{3}}c^{\frac{1}{4}}-c^{\frac{1}{4}}a^{\frac{1}{2}}-a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{3}}+c^{\frac{1}{4}}$.

6. Resolve $15x^2-41x+14$ into simple factors.

7. Find the value of x for which
$$\frac{x-1}{4}-\frac{2(x+1)}{9}+\frac{5(x-5)}{12}-4=\frac{x+1}{18}.$$

8. A and B have the same income. A lays by a fifth of his income; but B, by spending annually Rs. 80 more than A, at the end of 4 years finds himself Rs. 220 in debt. What was their income?

CHAPTER XIV

HIGHEST COMMON FACTORS

(*By factorisation*)

100. Definitions. A *common factor* of two or more algebraical expressions is an expression which divides each of them without a remainder.

N. B. By expressions we shall mean rational and integral expressions only (i.e., expressions free from radical signs and in which no letter occurs in the denominator of any term).

An *elementary* common factor is one which cannot itself be resolved into factors.

Thus, 2, 3, a, a, b, b, b, c, d are the elementary factors of $6a^2b^3cd$; $(x+y)$, $(x-y)$, (x^2+y^2) are elementary factors of x^4-y^4.

The product of *all the elementary* common factors of two or more expressions is called their **Highest Common Factor**; or, in other words, the Highest Common Factor of two or more expressions is that common factor which is formed by the product of the *greatest* number of elementary common factors.

Thus, since $6a^2b(x^2-1)=2\times 3\times a\times a\times b\times(x+1)\times(x-1)$, and $15ab^2(x^2-3x+2)=3\times 5\times a\times b\times b\times(x-1)\times(x-2)$, the elementary common factors of the two expressions on the left are 3, a, b, and $x-1$; hence, their H.C.F $=3ab(x-1)$.

Note 1. *Other common factors of the given expressions are* $3a$, $b(x-1)$, *&c.*, $3(x-1)$, $3ab$, *&c., but none of them is elementary.*

Note 2. *When the expressions considered have no numerical common factor, it is easy to comprehend that the Highest Common Factor is an expression of the higher degree than any other common factor. Hence, when two or more expressions have no numerical factor common, their Highest Common Factor may be defined to be the expression of the highest degree by which each of them is divisible without a remainder.*

Note 3. *If any expression A divides any other expression B without a remainder, then A is evidently the H.C.F. of A and B.*

Note 4. *If H be the H.C.F. of any number of quantities A, B, C, &c., then the quotients of A, B, C, &c., by H have no common factor.*

Note 5. *If an elementary factor occurs more than once in each of two or more given expressions, then the highest power of this factor common to the given expressions, and no higher power, must occur as a factor in the H.C.F. of these expressions.*

Note 6. *If $A = p \times q$, and $B = p' \times q'$, such that q and q' have no common factor, then the H.C.F. of A and B, if any, will be the same as the H.C.F. of p and p'.*

Note 7. *If $A = m \times n$, and $B = m' \times n'$, where m and m' respectively include all the monomial factors of A and B, then the H.C.F. of A and B = (the H.C.F. of m and m') × (the H.C.F. of n and n').*

Note 8. *The H.C.F. of A and B is the same as the H.C.F. of A and mB, if m or any elementary factor of m is not a factor of A.*

101. Highest Common Factors of simple expressions. Such expressions can be at once resolved into their elementary factors, and so there is no difficulty in finding the H.C.F. of any number of them.

Example 1. Find the H.C.F. of $a^2b^4c^5$, $a^4b^3c^7$ and $a^2b^5c^4$.

The elementary common factors are a, b and c; and the *highest* powers of them *common* to the given expressions are respectively a^2, b^3 and c^4.

Hence, the required H.C.F. $= a^2b^3c^4$.

Example 2. Find the H.C.F. of $24ab^2x^3y^4$, $36a^2x^4z^5$ and $240b^3x^6y^2z$.

We have
$$\begin{aligned} 24ab^2x^3y^4 &= 3 \times 2^3 \times ab^2x^3y^4, \\ 36a^2x^4z^5 &= 2^2 \times 3^2 \times a^2x^4z^5, \\ 240b^3x^6y^2z &= 3 \times 5 \times 2^4 \times b^3x^6y^2z. \end{aligned}$$

Evidently then the elementary common factors are 3, 2 and x; and the highest powers of them common to the given expressions are respectively 3, 2^2 and x^3.

Hence, the required H.C.F. $= 3 \times 2^2 \times x^3 = 12x^3$.

Note. *After exhibiting each expression as a product of powers of different elementary factors, the elementary factors* **common** *to the given expressions are at once obtained by writing down in succession such of the elementary factors of the first expression as are also found in every one of the remaining expressions. Thus, in the above example, the elementary factors of the first expression are 3, 2, a, b, x and y, of which 3, 2 and x only are to be found in each of the others.*

EXERCISE 51

Find the H.C.F. of :

1. a^2b^3 and a^3b^2.
2. $12a^3b$ and $20a^2c^3$.
3. $9xy^2z^3$ and $24x^3y^4$.
4. $20a^3x^4y^5$ and $75a^2y^3$.
5. $18m^2n^4$ and $45m^5n^3$.
6. $16a^3x^4y$, $40a^2y^3x$ and $28x^3a$.
7. $24m^2np^5$, $60mn^2p$ and $84m^3p^2$.

8. $45x^3y^2z^4$, $75x^2y^4z^3$ and $90x^4y^3z^2$.
9. $36a^2b^2c^4x^5$, $54a^5c^2x^4$ and $90a^4b^3c^5$.
10. $72a^3b^4c^5$, $96b^3c^4d^5$ and $120c^3d^4a^5$.
11. $48a^5x^4y^3z^2$, $60x^5y^4z^3b^2$, $72y^5z^4b^3a^2$ and $84z^5b^4a^3x^2$.
12. $75m^4n^3p^5q^6$, $90m^3n^5p^6q^4$, $105m^6n^4p^3q^5$ and $135m^5n^6p^4q^3$.
13. $54a^2b^5c^3d^4$, $72a^5b^2c^4d^3$, $108a^3b^4c^5d^2$ and $126a^4b^3c^2d^5$.
14. $18a^3x^4y^5$, $42a^4y^3z^4$, $60x^3y^4z^5$ and $78a^2x^4z^3$.
15. $32a^2b^3x^4y^5z^6$, $40a^3x^5y^4z^3$, $56b^3x^2y^7z^4$, $72x^6a^5y^2z^3$ and $96b^4a^6x^5y^3$.

102. Highest Common Factors of compound expressions whose elementary factors can be easily found.

The method illustrated in the last article will also evidently apply in such cases.

Example 1. Find the H.C.F. of $a^3b^2+2a^2b^3$ and $a^5b-4a^3b^3$.

$$a^3b^2+2a^2b^3=a^2b^2(a+2b);$$

and $$a^5b-4a^3b^3=a^3b(a^2-4b^2)=a^3b(a+2b)(a-2b).$$

Hence, the required H.C.F. $=a^2b(a+2b)$.

Note. *The H.C.F. of two or more algebraic expressions having* ***monomial and compound*** *expressions as factors is the product of the H.C.F. of* ***monomial*** *expressions and that of the compound ones.*

In the preceding example, the H.C.F. of a^2b^2 and a^3b is a^2b and that of $(a+2b)$ and (a^2-4b^2) is $(a+2b)$. Therefore, the required H.C.F. of the expressions $=(a^2b)\times(a+2b)=a^2b(a+2b)$.

Example 2. Find the H.C.F. of

$$x^4y^2+xy^5 \text{ and } x^4y+2x^3y^2+x^2y^3.$$

$$x^4y^2+xy^5=xy^2(x^3+y^3)=xy^2(x+y)(x^2-xy+y^2);$$

and $$x^4y+2x^3y^2+x^2y^3=x^2y(x^2+2xy+y^2)=x^2y(x+y)^2.$$

Hence, the required H.C.F. $=xy(x+y)$.

Example 3. Find the H.C.F. of

$$24(x^4-2ax^3-8a^2x^2) \text{ and } 54(x^5-ax^4-6a^2x^3).$$

The first expression $=3\times8\times x^2(x^2-2ax-8a^2)$
$$=3\times2^3\times x^2(x+2a)(x-4a).$$

The second expression $=6\times9\times x^3(x^2-ax-6a^2)$
$$=2\times3^3\times x^3(x+2a)(x-3a).$$

Hence, the required H.C.F. $=3\times2\times x^2(x+2a)=6x^2(x+2a)$.

EXERCISE 52

Find the H.C.F. of :

1. a^3-ab^2 and $a^4+2a^3b+a^2b^2$.
2. $x^5y^3-x^3y^5$ and $x^5y^4+x^4y^5$.
3. $6(x^2-9)$ and $15(x^3+27)$.
4. $12(a^6-a^2b^2c^2)$ and $20(a^4b^2c^2+a^2b^3c^3)$.
5. $m^6n^3-2m^5n^4+m^4n^5$ and $(m^2n-mn^2)^3$.
6. $4a^4x-9a^2x^3$ and $4a^2x^2+6ax^3$.
7. $18a^4b^3-32a^2b^5$ and $18a^4b^2+24a^3b^3$.
8. $9x^4y^4-36x^2y^6$ and $24x^4y^2-48x^3y^3$.
9. $6a^3b^2-24ab^4$ and $4a^5b+32a^2b^4$.
10. $48x^2a^2(x+a)^2(x^2a^2-xa^3)$ and $64(x^5a^2-x^2a^5)(x^3a+x^2a^2)$.
11. $24(x^3-a^3)$ and $40(x^4+x^2a^2+a^4)$.
12. $56(x^6a^2-x^2a^6)$ and $72(x^5a^3+3a^5x^3+2a^7x)$.
13. $30(a^2+4ab+3b^2)$ and $42(a^2+ab-6b^2)$.
14. $28(x^3-3x^2-10x)$ and $52(x^4-8x^3+15x^2)$.
15. $x^4y+3x^3y^2-18x^2y^3$ and $x^3y^2+10x^2y^3+24xy^4$.
16. $a^4x^3-4a^3x^4-12a^2x^5$ and $a^5x^2+8a^4x^3+12a^3x^4$.
17. $4x^3+12x^2+9x$ and $4x^2-2x-12$.
18. $a^2-ab-2b^2$ and $a^3-a^2b-4ab^2+4b^3$.
19. $(a^3-b^3)(a+b)^2$, a^4-b^4 and $3a^4+2a^3b-5a^2b^2$.
20. $(2x-3)^2(x^2+x-2)$, $4x^2-x-18$ and $2x^2-23x-54$.
21. $8(27a^5b+a^2b^4)$, $12(6a^4b^2-7a^3b^3-3a^2b^4)$ and $40(3a^3b^2+13a^2b^3+ab^4)$.

CHAPTER XV

LOWEST COMMON MULTIPLE

(*By factorisation*)

103. **Definitions.** One expression is said to be a ***multiple*** of another when the former is exactly divisible by the latter.

One expression is said to be a *common multiple* of two or more others when it is exactly divisible by *each* of these latter.

Of the different common multiples of two or more expressions that which consists of the *least* number of elementary factors is called the **Lowest Common Multiple** of those expressions. In other words, a common multiple of two or more expressions is said to be their *Lowest Common Multiple* when it is the product of *just* as many elementary factors as it *must necessarily* have and no more.

Thus, the common multiples of a and b are ab, $2ab$, a^2b, ab^2, a^2b^2, &c.; but of these ab consists of the least number of elementary factors, and hence, it is called the lowest common multiple of the quantities a and b.

Cor. Hence, every common multiple of two or more expressions is divisible by their Lowest Common Multiple.

Note. *The letters L.C.M. are usually written for 'Lowest Common Multiple'.*

104. L.C.M. of simple expressions or such compound expressions as can be easily resolved into their elementary factors.

In such cases the L.C.M. can be written down by inspection. The following examples will illustrate the process :

Example 1. Find the L.C.M. of $4a^2bc$ and $6ab^2d$.

The 1st expression $= 2^2 \times a^2 \times b \times c$.

The 2nd expression $= 2 \times 3 \times a \times b^2 \times d$.

Hence, $2^2 \times 3 \times a^2 \times b^2 \times c \times d$ *must necessarily* be a factor of *every* common multiple of them.

Hence, the required L.C.M.

$$= 2^2 \times 3 \times a^2 \times b^2 \times c \times d$$

$$= 12a^2b^2cd.$$

Example 2. Find the L.C.M. of $24x^2yz$, $18xy^3z^2$ and $27x^4y^2z^2$.

The 1st expression $= 2^3 \times 3 \times x^2 \times y \times z$.

The 2nd expression $= 2 \times 3^2 \times x \times y^3 \times z^2$.

The 3rd expression $= 3^3 \times x^4 \times y^2 \times z^2$.

Hence, $2^3 \times 3^3 \times x^4 \times y^3 \times z^2$ *must necessarily* be a factor of *every* common multiple of them.

Hence, the required L.C.M.

$$=2^3 \times 3^3 \times x^4 \times y^3 \times z^2 = 216x^4y^3z^2.$$

Example 3. Find the L.C.M. of

$$4x^2(x+a)^2,\ 6a^2x(x^2-a^2) \text{ and } 9x^3(x^3-a^3).$$

The 1st expression $=2^2 \times x^2 \times (x+a)^2$.

The 2nd expression $=2 \times 3 \times a^2 \times x \times (x+a)(x-a)$.

The 3rd expression $=3^2 \times x^3 \times (x-a)(x^2+ax+a^2)$.

Hence, $2^2 \times 3^2 \times a^2 \times x^3 \times (x+a)^2(x-a)(x^2+ax+a^2)$ *must necessarily* be a factor of *every* common multiple of them.

Hence, the required L.C.M.

$$=2^2 \times 3^2 \times a^2 \times x^3 \times (x+a)^2(x-a)(x^2+ax+a^2)$$
$$=36a^2x^3(x+a)^2(x^3-a^3).$$

Note. *The L.C.M. of two or more algebraic expressions having monomial and compound expressions as factors is the product of the L.C.M. of the monomial expressions and that of the compound ones.*

In the preceding example, L.C.M. of $4x^2$, $6a^2x$ *and* $9x^3$ *is* $36a^2x^3$ *and that of* $(x+a)^2$, (x^2-a^2) *and* (x^3-a^3) *is* $(x+a)^2(x^3-a^3)$. *Therefore, the required L.C.M. is* $(36a^2x^3) \times (x+a)^2(x^3-a^3) = 36a^2x^3(x+a)^2(x^3-a^3)$.

Example 4. Find the L.C.M. of

$$x^2-3x+2,\ x^3+2x^2-3x \text{ and } x^4+x^3-6x^2.$$

The 1st expression $=(x-1)(x-2)$.

The 2nd expression $=x(x^2+2x-3)=x(x-1)(x+3)$.

The 3rd expression $=x^2(x^2+x-6)=x^2(x-2)(x+3)$.

Hence, $x^2(x-1)(x-2)(x+3)$ *must necessarily* be a factor of *every* common multiple of the given expressions.

Hence, the required L.C.M. $=x^2(x-1)(x-2)(x+3)$.

Example 5. Find the L.C.M. of x^3-3x^2+3x-1, x^3-x^2-x+1 and x^4-2x^3+2x-1.

$$x^3-3x^2+3x-1=(x-1)^3.$$

$$x^3-x^2-x+1=x^2(x-1)-(x-1)$$
$$=(x-1)(x^2-1)=(x-1)^2(x+1).$$

$$x^4-2x^3+2x-1=(x^4-1)-2x(x^2-1)$$
$$=(x^2-1)\{(x^2+1)-2x\}=(x^2-1)(x-1)^2$$
$$=(x-1)^3(x+1).$$

Hence, $(x-1)^3(x+1)$ *must necessarily* be a factor of *every* common multiple of the given expressions.

Hence, the required L.C.M. $=(x-1)^3(x+1)$.

EXERCISE 53

Find the L.C.M. of :

1. a^2b and ab^2.
2. a^3b^2 and a^2bc.
3. $6x^2y^4$ and $10xy^2$.
4. $4m^2n^3$ and $14m^4n^2p$.
5. $8x^2y^3z$ and $12x^3y^2z^2$.
6. $4a^2bc$, $10ab^2c$ and $14abc^3$.
7. $8a^3b^2c$, $12ab^3c^2$ and $20a^2bc^3$.
8. $6x^4y$, $9x^2y^2z$, $12a^2xy^3$ and $15axz^2$.
9. a^3b-ab^3 and $a^3b^2+a^2b^3$.
10. $4(x-y)^2$, $6(x^2-y^2)$ and $8(x+y)^2$.
11. x^2-4x+3 and x^2-5x+6.
12. $a^3+2a^2x-3ax^2$ and $a^4+a^3x-6a^2x^2$.
13. $a^2(a^2-4)$ and $a^4+2a^3-8a^2$.
14. $4a^2x^2$, $2x(x^2-a^2)$ and $6a^3x(a^3+a^3)$.
15. $12(x^2+3x-10)$ and $16(x^2+4x-12)$.
16. $x^2+2x-15$, $x^2+9x+20$ and $x^2+4x-21$.
17. $12a^4-27a^2b^2$, $2a^2+ab-3b^2$ and $2a^2-ab-3b^2$.
18. $8a^3+27b^3$, $8a^3-27b^3$ and $16a^4+36a^2b^2+81b^4$.
19. $8x^4-50x^2y^2$, $12x^3+24x^2y-15xy^2$ and $16x^2-48xy+20y^2$.
20. $4x^2-12ax+9a^2$, $6x^2-7ax-3a^2$ and $6x^2-11ax+3a^2$.
21. $2x^2+6x+9$, $4x^3-12x^2+18x$ and $4x^4+81$.
22. $9a^2-6ax+x^2$, $6a^2+10ax-4x^2$ and $9a^2-21ax+6x^2$.
23. $8x^3-12x^2+6x-1$, $8x^3-4x^2-2x+1$ and $2x^2+5x-3$.
24. $x^2-6xy+8y^2$, $x^2-7xy+12y^2$, $x^2+2xy-15y^2$ and $x^2+xy-20y^2$.
25. $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$. [C. U. 1869]
26. $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$. [C. U. 1871]
27. $9x^4-28x^2+3$, $27x^4-12x^2+1$, $27x^4+6x^2-1$ and x^4-6x^2+9. [C. U. 1886]

CHAPTER XVI

EASY FRACTIONS

105. Definition. The algebraical fraction $\frac{a}{b}$, where a and b may have *any* numerical values, is defined to be a quantity which, when multiplied by b, becomes equal to a. In other words, $\frac{a}{b}$ is defined to be equivalent to $a \div b$. In $\frac{a}{b}$, a is called the **numerator** and b the **denominator.**

Note. *Thus an algebraical fraction is no other than the quotient of one expression by another, expressed by placing the dividend over the divisor with a horizontal line between them ; and the dividend and the divisor so placed are respectively called the numerator and the denominator of the fraction.*

106. The value of a fraction is not altered if both its numerator and denominator are multiplied or divided by any the same quantity.

If a, b and m stand for any quantities whatever, to prove that

$$\frac{a}{b}=\frac{am}{bm}.$$

Let $\qquad x=\frac{a}{b},$

then $\qquad x\times b=\frac{a}{b}\times b=a \qquad$ [by definition] ;

$\therefore \quad x\times b\times m=a\times m, \qquad$ or, $\quad x\times bm=am.$

Hence, $\; x=am\div bm, \;$ *i.e.*, $\; \frac{a}{b}=\frac{am}{bm}.$

Conversely, we have $\frac{am}{bm}=\frac{a}{b}$; *i.e.*, $\frac{am}{bm}=\frac{am\div m}{bm\div m}.$

Thus, the proposition is established.

Cor. $\frac{a}{b}=\frac{a\times(-1)}{b\times(-1)}=\frac{-a}{-b}.$ Thus, the value of a fraction is not altered if the signs of both the numerator and the denominator be changed.

107. Reduction of a fraction to its lowest terms. A fraction is said to be in its lowest terms, when its numerator and denominator have no common factor.

Hence, to reduce a fraction to its lowest terms, or more briefly to *simplify* it, is no other than to find an equivalent fraction whose numerator and denominator have no common factor, and this is evidently done by dividing the numerator and the denominator of the fraction by their highest common factor.

Note. *In all cases where the numerator and the denominator can be factorised by inspection, the reduction is at once effected by simply removing the common factors.*

Example 1. Reduce $\frac{4a^2b^3c^2}{10ab^4c^2}$ to its lowest terms.

$$\frac{4a^2b^3c^2}{10ab^4c^2}=\frac{2\times 2\times a^2\times b^3\times c^2}{2\times 5\times a\times b^4\times c^2}=\frac{2a}{5b}.$$

Example 2. Simplify $\frac{a^2b^3(a^2-b^2)}{3ab^4(a^3+b^3)}$.

$$\frac{a^2b^3(a^2-b^2)}{3ab^4(a^3+b^3)}=\frac{a^2b^3(a+b)(a-b)}{3ab^4(a+b)(a^2-ab+b^2)}=\frac{a(a-b)}{3b(a^2-ab+b^2)}.$$

Example 3. Reduce $\frac{x^2+3x-40}{x^2+4x-32}$ to its lowest terms.

The numerator $=(x+8)(x-5)$.

The denominator $=(x+8)(x-4)$.

Hence, the given fraction $=\frac{(x+8)(x-5)}{(x+8)(x-4)}=\frac{x-5}{x-4}$.

Example 4. Simplify $\frac{2a^2+3ax-2ab-3bx}{3a^2-2ax-3ab+2bx}$.

The numerator $=2a(a-b)+3x(a-b)=(a-b)(2a+3x)$.

The denominator $=3a(a-b)-2x(a-b)$

$=(a-b)(3a-2x)$.

Hence, the given expression $=\frac{(a-b)(2a+3x)}{(a-b)(3a-2x)}=\frac{2a+3x}{3a-2x}$.

EXERCISE 54

Reduce to lowest terms :

1. $\frac{2a^2b^3}{4a^2b^4}$.
2. $\frac{6x^2y^3}{8xy^4}$.
3. $\frac{4a^2xy^2}{10ax^2y^2}$.
4. $\frac{15x^3y^2z^4}{25x^2y^4z^3}$.
5. $\frac{18a^2bc^4d^5}{27a^3b^2c^4d^4}$.
6. $\frac{16x^2a^4y^2z^5}{40a^3z^4x^3y^4}$.
7. $\frac{70a^2b^3c^4d^7}{105c^4d^2a^3b^3}$.
8. $\frac{39m^2n^5p^3q^6}{65p^2m^3n^4q^5}$.
9. $\frac{x^2-a^2}{x^2+ax}$.
10. $\frac{x^2-3x}{9x-x^3}$.
11. $\frac{4x^2-9a^2}{4x^2+6ax}$.
12. $\frac{3a^2-12ab}{48b^2-3a^2}$.

13. $\dfrac{3ax-12a^2}{x^2-16a^2}$.

14. $\dfrac{2x^4-4a^2x^2}{x^4-4a^2x^2+4a^4}$.

15. $\dfrac{4x^2+8x}{x^2+5x+6}$.

16. $\dfrac{x^2+2x-8}{x^2+x-12}$.

17. $\dfrac{x^2+2x-15}{x^2+9x+20}$.

18. $\dfrac{a^2-3ab-4b^2}{a^2-4ab-5b^2}$.

19. $\dfrac{a^4-a^3b+a^2b^2}{a^3+b^3}$.

20. $\dfrac{1-7x+12x^2}{1-8x+15x^2}$.

21. $\dfrac{x^2-6xy+5y^2}{x^2+2xy-35y^2}$.

22. $\dfrac{1-9a^2+14a^4}{1-4a^2-21a^4}$.

23. $\dfrac{x^4-8x^2-65}{x^4+x^2-20}$.

24. $\dfrac{3a^3x+9a^2x^2+27ax^3}{a^3-27x^3}$.

25. $\dfrac{2x^2-x-6}{3x^2-2x-8}$.

26. $\dfrac{3x^2-5ax+2a^2}{3x^2+ax-2a^2}$.

27. $\dfrac{3x^2+16ax+5a^2}{3x^2+22ax+7a^2}$.

28. $\dfrac{6x^2-7x-20}{9x^2+6x-8}$.

29. $\dfrac{2x^2+3ax-20a^2}{3x^2+5ax-28a^2}$.

30. $\dfrac{10-17ax+3a^2x^2}{5-26ax+5a^2x^2}$.

31. $\dfrac{x^2-(a-b)x-ab}{x^3+bx^2+ax+ab}$.

32. $\dfrac{6ac+10bc+9ax+15bx}{6c^2+9cx-2c-3x}$.

33. $\dfrac{8bx+12ab+6xy+9ay}{12bx+8ab+9xy+6ay}$.

34. $\dfrac{2a^2+ab-b^2}{a^3+a^2b-a-b}$.

35. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}$.

108. Reduction of two or more fractions to a common denominator.

Let $\dfrac{a}{b}$, $\dfrac{c}{d}$, $\dfrac{e}{f}$, &c., stand for any number of fractions.

Let L denote the L.C.M. of the denominators, *i.e.*, of b, d, f, &c. Then, since the value of a fraction is not altered when its numerator and denominator are *both* multiplied by the same quantity, we must have

$$\frac{a}{b}=\frac{a\times(L\div b)}{b\times(L\div b)}=\frac{a\times(L\div b)}{L};$$

$$\frac{c}{d}=\frac{c\times(L\div d)}{d\times(L\div d)}=\frac{c\times(L\div d)}{L};$$

$$\frac{e}{f}=\frac{e\times(L\div f)}{f\times(L\div f)}=\frac{e\times(L\div f)}{L};$$

and so on.

Thus, the fractions in the third column are respectively equivalent to the given fractions and they have all got the same denominator, namely, L.

Hence, we have the following rule for reducing fractions to a common denominator: *Find the L.C.M. of the denominators, and multiply the numerator and the denominator of each fraction by the quotient obtained by dividing the L.C.M., thus found, by the denominator of that fraction.*

Example 1. Reduce $\frac{x}{a+b}$, $\frac{x^2}{a(a-b)}$ and $\frac{x^3}{b(a^2-b^2)}$ to a common denominator.

The L.C.M. of the denominators $=ab(a^2-b^2)$; and the quotients obtained by dividing it by the denominators are respectively $ab(a-b)$, $b(a+b)$ and a.

Hence, we have $\frac{x}{a+b}=\frac{x\times ab(a-b)}{(a+b)\times ab(a-b)}=\frac{xab(a-b)}{ab(a^2-b^2)}$;

$$\frac{x^2}{a(a-b)}=\frac{x^2\times b(a+b)}{a(a-b)\times b(a+b)}=\frac{x^2b(a+b)}{ab(a^2-b^2)};$$

$$\frac{x^3}{b(a^2-b^2)}=\frac{x^3\times a}{b(a^2-b^2)\times a}=\frac{x^3a}{ab(a^2-b^2)}.$$

Example 2. Reduce $\frac{x-1}{x^2-5x+6}$, $\frac{x-2}{x^2-4x+3}$ and $\frac{x-3}{x^2-3x+2}$ to a common denominator.

The denominators are respectively

$$(x-2)(x-3),\ (x-1)(x-3) \text{ and } (x-1)(x-2).$$

Hence, their L.C.M. $=(x-1)(x-2)(x-3)$, and the quotients obtained by dividing it by the denominators are respectively $x-1$, $x-2$ and $x-3$. Hence, we have

$$\frac{x-1}{x^2-5x+6}=\frac{(x-1)(x-1)}{(x^2-5x+6)(x-1)}=\frac{x^2-2x+1}{x^3-6x^2+11x-6};$$

$$\frac{x-2}{x^2-4x+3}=\frac{(x-2)(x-2)}{(x^2-4x+3)(x-2)}=\frac{x^2-4x+4}{x^3-6x^2+11x-6};$$

$$\frac{x-3}{x^2-3x+2}=\frac{(x-3)(x-3)}{(x^2-3x+2)(x-3)}=\frac{x^2-6x+9}{x^3-6x^2+11x-6}.$$

EXERCISE 55

Reduce to a common denominator:

1. $\frac{a}{2b}, \frac{3c}{4d}, \frac{e}{f}$.
2. $\frac{x^2}{2bc}, \frac{y^2}{3ca}, \frac{z^2}{4ab}$.
3. $\frac{ab}{4xy^2}, \frac{bc}{6x^2y}, \frac{ca}{10x^3}$.
4. $\frac{a}{a-b}, \frac{b}{a+b}, \frac{c}{a(a+b)}$.

5. $\frac{x^2}{a^2+2ab}, \frac{y^2}{a-2b}$. 6. $\frac{2a}{a-b}, \frac{a-c}{ab-a^2}$.

7. $\frac{2a}{a-b}, \frac{3b}{b-a}, \frac{4c}{a+b}$. 8. $\frac{2x}{a^2(a+x)}, \frac{3y}{b^2(a-x)}, \frac{4z}{c^2(a^2-x^2)}$.

9. $\frac{a^2}{2xy-3y^2}, \frac{b^2}{2x^2+3xy}, \frac{c^2}{4x^3y-9xy^3}$.

10. $\frac{a^2}{x^2+x+1}, \frac{b^2}{x^2-x+1}$. 11. $\frac{3}{x^2-x-2}, \frac{4}{x^2+x-6}$.

12. $\frac{a-2b}{a(a^2-2ab+4b^2)}, \frac{bc}{a^3+8b^3}$. 13. $\frac{a}{a-3b}, \frac{b}{a^2+3ab+9b^2}, \frac{c}{a^3-27b^3}$.

14. $\frac{a}{b(a-b-c)}, \frac{b}{a(a-b+c)}, \frac{c}{a^2+b^2-c^2-2ab}$.

15. $\frac{c-a}{(a-b)(b-c)}, \frac{b-a}{(a-c)(b-c)}, \frac{b-c}{(c-a)(a-b)}$.

109. Addition and Subtraction of Fractions.

From Cor. 3, Art. 47, we know that

$a(b+c+d+e)=ab+ac+ad+ae$, where a, b, c, d, e are any quantities whatever.

Hence, conversely,

$$\frac{ab+ac+ad+ae}{a}=b+c+d+e=\frac{ab}{a}+\frac{ac}{a}+\frac{ad}{a}+\frac{ae}{a}.$$

Hence, putting p, q, r, s respectively for ab, ac, ad, ae, we have

$\frac{p+q+r+s}{a}=\frac{p}{a}+\frac{q}{a}+\frac{r}{a}+\frac{s}{a}$, where p, q, r, s and a are any quantities whatever.

Thus, the sum of any number of fractions which have a common denominator is a fraction whose denominator is the same and whose numerator is the sum of the numerators of the given fractions.

Hence, to obtain the sum of any number of fractions which have not the same denominator, we must first reduce them to equivalent fractions having a common denominator and then proceed as above.

Note. *To subtract a fraction from another fraction we are to follow the above rule; but the numerator of the fraction to be subtracted is to be taken with minus sign.* Thus,

$$\frac{a}{b}-\frac{x}{b}=\frac{a}{b}+\frac{(-x)}{b}=\frac{a+(-x)}{b}=\frac{a-x}{b}.$$

Example 1. Find the value of $\dfrac{a}{a-b}+\dfrac{b}{b-a}$.

Since, $\dfrac{b}{b-a}=\dfrac{b\times(-1)}{(b-a)\times(-1)}=\dfrac{-b}{a-b}$,

we have
$$\frac{a}{a-b}+\frac{b}{b-a}=\frac{a}{a-b}+\frac{-b}{a-b}$$
$$=\frac{a+(-b)}{a-b}=\frac{a-b}{a-b}=1.$$

Example 2. Find the value of $\dfrac{x}{x+a}+\dfrac{a}{x-a}$.

Since the L.C.M. of the denominators $=x^2-a^2$,

we have $\dfrac{x}{x+a}=\dfrac{x(x-a)}{x^2-a^2}$ and $\dfrac{a}{x-a}=\dfrac{a(x+a)}{x^2-a^2}$.

Hence, the required value
$$=\frac{x(x-a)}{x^2-a^2}+\frac{a(x+a)}{x^2-a^2}$$
$$=\frac{x(x-a)+a(x+a)}{x^2-a^2}=\frac{x^2+a^2}{x^2-a^2}.$$

Example 3. Find the value of $\dfrac{a+b}{a^2-b^2}+\dfrac{a-b}{a^3-b^3}-\dfrac{a^2+b^2}{a^3-b^3}$.

In the first and second terms the numerator and the denominator have a common factor. So they are to be reduced to their lowest terms. This is not essential. But this will make the operation easy.

$$\frac{a+b}{a^2-b^2}=\frac{1}{a-b};$$
$$\frac{a-b}{a^3-b^3}=\frac{1}{a^2+ab+b^2}.$$

$\therefore$ the given expression
$$=\frac{1}{a-b}+\frac{1}{a^2+ab+b^2}-\frac{a^2+b^2}{a^3-b^3}$$
$$=\frac{(a^2+ab+b^2)+(a-b)-(a^2+b^2)}{a^3-b^3}=\frac{ab+a-b}{a^3-b^3}.$$

Example 4. Find the value of $\dfrac{1}{a+b}+\dfrac{b}{a^2-b^2}-\dfrac{a}{a^2+b^2}$.

In the present example it is not convenient to reduce all the fractions to a common denominator at once. We can proceed best as follows :

We have
$$\frac{1}{a+b}+\frac{b}{a^2-b^2}=\frac{(a-b)+b}{a^2-b^2}=\frac{a}{a^2-b^2}.$$

Hence, the required value $=\dfrac{a}{a^2-b^2}-\dfrac{a}{a^2+b^2}$

$$=\frac{a(a^2+b^2)-a(a^2-b^2)}{a^4-b^4}=\frac{2ab^2}{a^4-b^4}.$$

Example 5. Simplify $\dfrac{1}{x-2}-\dfrac{1}{x+2}-\dfrac{4}{x^2+4}+\dfrac{32}{x^4+16}.$

We have $\dfrac{1}{x-2}-\dfrac{1}{x+2}=\dfrac{(x+2)-(x-2)}{x^2-4}=\dfrac{4}{x^2-4}$;

$$\frac{4}{x^2-4}-\frac{4}{x^2+4}=\frac{4(x^2+4)-4(x^2-4)}{x^4-16}=\frac{32}{x^4-16}.$$

Lastly, $\dfrac{32}{x^4-16}+\dfrac{32}{x^4+16}=\dfrac{32(x^4+16)+32(x^4-16)}{x^8-256}=\dfrac{64x^4}{x^8-256}$, which is the required result.

Example 6. Simplify $\dfrac{1}{a+b}-\dfrac{1}{a+2b}-\dfrac{1}{a+3b}+\dfrac{1}{a+4b}.$

The given expression $=\left\{\dfrac{1}{a+b}-\dfrac{1}{a+2b}\right\}-\left\{\dfrac{1}{a+3b}-\dfrac{1}{a+4b}\right\}.$

Now, we have

$$\frac{1}{a+b}-\frac{1}{a+2b}=\frac{(a+2b)-(a+b)}{(a+b)(a+2b)}=\frac{b}{(a+b)(a+2b)};$$

and $\dfrac{1}{a+3b}-\dfrac{1}{a+4b}=\dfrac{(a+4b)-(a+3b)}{(a+3b)(a+4b)}=\dfrac{b}{(a+3b)(a+4b)}.$

Lastly, $\dfrac{b}{(a+b)(a+2b)}-\dfrac{b}{(a+3b)(a+4b)}$

$$=\frac{b(a+3b)(a+4b)-b(a+b)(a+2b)}{(a+b)(a+2b)(a+3b)(a+4b)};$$

of which the numerator $=b(a^2+7ab+12b^2)-b(a^2+3ab+2b^2)$

$$=b(4ab+10b^2)=2b^2(2a+5b).$$

Hence, the reqd. result $=\dfrac{2b^2(2a+5b)}{(a+b)(a+2b)(a+3b)(a+4b)}.$

EXERCISE 56

Find the value of :

1. $\dfrac{a+b}{a}+\dfrac{a-b}{b}.$ 2. $\dfrac{x-y}{xy}+\dfrac{y-z}{yz}+\dfrac{z-x}{zx}.$ 3. $\dfrac{a}{a-x}+\dfrac{x}{x-a}.$

4. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}.$ 5. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a-b}{2(a+b)}.$ 6. $\dfrac{4x^2+9y^2}{4x^2-9y^2}-\dfrac{2x-3y}{2x+3y}.$

7. $\frac{a}{(a+b)^2}-\frac{b}{a^2-b^2}$.

8. $\frac{a^2+ab+b^2}{a+b}+\frac{a^2-ab+b^2}{a-b}$.

9. $\frac{1}{(a-b)(a-c)}+\frac{1}{(a-c)(b-c)}$.

10. $\frac{1}{x^2-3x+2}+\frac{1}{x^2-5x+6}$.

11. $\frac{1}{x^2+7x+10}+\frac{1}{x^2+13x+40}$.

12. $\frac{1}{2x+3y}-\frac{(2x-3y)^2}{8x^3+27y^3}$.

13. $\frac{a+b}{a-b}-\frac{a-b}{a+b}+\frac{2ab}{b^2-a^2}$.

14. $\frac{1}{a+2b}+\frac{1}{a-2b}+\frac{2a}{4b^2-a^2}$.

15. $\frac{x+y}{x-y}+\frac{x-y}{x+y}-\frac{2(x^2-y^2)}{x^2+y^2}$.

16. $\frac{a-2x}{a+2x}-\frac{a+2x}{a-2x}+\frac{8ax}{a^2+4x^2}$.

17. $\frac{3x+1}{x-3}-\frac{x-3}{3x+9}-\frac{5x^2+24x}{2x^2-18}$.

18. $\frac{4a-b}{1-4ab}-\frac{4a+b}{1+4ab}-\frac{4b(1-8a^2)}{16a^2b^2-1}$.

19. $\frac{x}{x-2a}+\frac{x}{x+2a}+\frac{2x^2}{x^2+4a^2}$.

20. $\frac{b}{a-b}+\frac{b}{a+b}+\frac{2ab}{a^2+b^2}+\frac{4a^3b}{a^4+b^4}$.

21. $\frac{x}{3x-y}+\frac{x}{3x+y}+\frac{6x^2}{9x^2+y^2}$.

22. $\frac{1}{x-3a}-\frac{1}{2x+6a}-\frac{x-9a}{2x^2+18a^2}$.

23. $\frac{(a^2+b^2)^2}{ab(a-b)^2}-\frac{a}{b}-\frac{b}{a}-2$.

24. $\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}$.

25. $\frac{1}{x-a}-\frac{2}{2x+a}+\frac{1}{x+a}-\frac{2}{2x-a}$.

26. $\frac{3}{a-x}-\frac{1}{x+3a}+\frac{3}{a+x}+\frac{1}{x-3a}$.

27. $\frac{2}{x-1}-\frac{x}{x^2+1}-\frac{1}{x+1}-\frac{3}{x^2-1}$.

28. $\frac{a-c}{(a-b)(x-a)}+\frac{b-c}{(b-a)(x-b)}$.

29. $\frac{x-1}{x^2-3x+2}+\frac{x-2}{x^2-5x+6}+\frac{x-5}{x^2-8x+15}$.

30. $\frac{x+a}{x^2+5ax+4a^2}+\frac{x+7a}{x^2+11ax+28a^2}+\frac{x+13a}{x^2+20ax+91a^2}$.

31. $\frac{1}{x^2+3x+2}+\frac{2x}{x^2+4x+3}+\frac{1}{x^2+5x+6}$.

32. $\frac{1}{1-x+x^2}-\frac{1}{1+x+x^2}-\frac{2x}{1+x^2+x^4}$.

33. $\frac{1}{1+x+x^2}-\frac{1}{1-x+x^2}+\frac{2x}{1-x^2+x^4}$.

34. $\frac{1}{x-2}-\frac{x-2}{x^2+2x+4}+\frac{6x}{x^3+8}$.

35. $\frac{11}{16(2x^2-6ax+9a^2)}-\frac{11}{32x^2+96ax+144a^2}+\frac{33ax}{4(4x^4-81a^4)}$.

110. Multiplication of Fractions.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions; to find the value of $\frac{a}{b}\times\frac{c}{d}$.

Let $x=\frac{a}{b}\times\frac{c}{d}$.

Then, we have $x\times b\times d=\frac{a}{b}\times\frac{c}{d}\times b\times d=\frac{a}{b}\times b\times\frac{c}{d}\times d$

$$=\left(\frac{a}{b}\times b\right)\times\left(\frac{c}{d}\times d\right)=a\times c;$$

or, $x\times bd=ac$, $\therefore\ x=\frac{ac}{bd}$; i.e., $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

Hence, $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ac}{bd}\times\frac{e}{f}=\frac{ace}{bdf}$;

$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}\times\frac{g}{h}=\frac{ace}{bdf}\times\frac{g}{h}=\frac{aceg}{bdfh}$; and so on.

Thus, *the product of any number of fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of their denominators.*

In practice, *factors which are common to numerator and denominator are cancelled.*

Cor. Since, $c=\frac{c}{1}$, we have $\frac{a}{b}\times c=\frac{a}{b}\times\frac{c}{1}=\frac{ac}{b}$.

Example 1. Multiply together $\frac{x^2}{yz}$, $\frac{y^2}{zx}$ and $\frac{z^2}{xy}$.

The required product $=\frac{x^2\times y^2\times z^2}{yz\times zx\times xy}=\frac{x^2\times y^2\times z^2}{y^2\times z^2\times x^2}=1.$

The result is obtained by removing like factors from numerator and denominator.

N. B '*When all the factors of numerator and denominator cancel each other it is a common mistake with the beginners to give the result as 0. A little reflection will shew that the result of such a multiplication can never be zero.*'

Example 2. Multiply $\frac{x(a-x)}{a^2+2ax+x^2}$ by $\frac{a(a+x)}{a^2-2ax+x^2}$.

The required product $=\frac{x(a-x)\times a(a+x)}{(a^2+2ax+x^2)(a^2-2ax+x^2)}$

$$=\frac{ax(a-x)(a+x)}{(a+x)^2(a-x)^2}=\frac{ax}{(a+x)(a-x)}=\frac{ax}{a^2-x^2}.$$

Example 3. Multiply together

$$\frac{1-x^2}{1+y},\ \frac{1-y^2}{x+x^2}\text{ and }1+\frac{x}{1-x}$$

Since, $1+\dfrac{x}{1-x}=\dfrac{1-x+x}{1-x}=\dfrac{1}{1-x}$,

the required product $=\dfrac{(1+x)(1-x)}{1+y}\times\dfrac{(1+y)(1-y)}{x(1+x)}\times\dfrac{1}{1-x}$

$$=\frac{(1+x)(1-x)(1+y)(1-y)}{(1+y)x(1+x)(1-x)}=\frac{1-y}{x}.$$

EXERCISE 57

Multiply together :

1. $\dfrac{2a^2}{3ab}$, $\dfrac{9b^2}{16ac}$ and $\dfrac{8c^2}{9bc}$.
2. $\dfrac{4a^2b^2}{3c^2}$, $\dfrac{9c^2}{16d^2}$ and $\dfrac{4d^2}{27b^2}$.
3. $\dfrac{x^3}{yz}$, $\dfrac{y^3}{zx}$ and $\dfrac{z^3}{xy}$.
4. $\dfrac{7a^2b^2c^2}{12xyz}$ and $\dfrac{4x^3y^3z^3}{21a^4b^4c^4}$.
5. $\dfrac{12m^2n^3}{7xy^2z}$ and $\dfrac{35x^3yz}{96m^3n}$.

Simplify the following :

6. $\dfrac{x+1}{x-1}\times\dfrac{x^2+x-2}{x^2+x}$.
7. $\dfrac{a^2-9b^2}{a^2+3ab}\times\dfrac{3a^2}{a^2-3ab}$.
8. $\dfrac{a^3-b^3}{a^2+ab}\times\dfrac{(a+b)^2}{a^2+ab+b^2}$.
9. $\dfrac{a^3+8x^3}{a^3-2a^2x}\times\dfrac{a^2-4ax+4x^2}{a^2-2ax+4x^2}$.
10. $\dfrac{x^2+4x+3}{x^2-4}\times\dfrac{x^2-3x+2}{x^2-9}$.
11. $\dfrac{x^2-7x+10}{x^2-2x-15}\times\dfrac{x^2-3x-18}{x^2-8x+12}$.
12. $\dfrac{x^2-4x+3}{x^2-6x+5}\times\dfrac{x^2-7x+10}{x^2-5x+6}$.
13. $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a^2+ab}$.
14. $\dfrac{2x^2-5x+2}{3x^2-5x-2}\times\dfrac{3x^2+x}{4x-2}$.
15. $\dfrac{x^2-6x-16}{x^2-4x-21}\times\dfrac{x^2-11x+28}{x^2-12x+32}$.
16. $\dfrac{a^2-x^2}{a+b}\times\dfrac{a^2-b^2}{ax+x^2}\times\left(a+\dfrac{ax}{a-x}\right)$.
17. $\left(\dfrac{x^2}{a^2}-\dfrac{x}{a}+1\right)\left(\dfrac{x^2}{a^2}+\dfrac{x}{a}+1\right)$.
18. $\left(\dfrac{4a}{3x}+\dfrac{3x}{2b}\right)\left(\dfrac{2b}{3x}+\dfrac{3x}{4a}\right)$.
19. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right)\left(\dfrac{c}{d}+\dfrac{d}{c}\right)-\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{c}{d}-\dfrac{d}{c}\right)$.
20. $\dfrac{2x^2-7x+3}{2x^2+7x-4}\times\dfrac{3x^2+11x-4}{3x^2+8x-3}\times\dfrac{2x^2+x-15}{2x^2-11x+15}$.
21. $\dfrac{b^2-c^2-a^2+2ac}{c^2+a^2-b^2+2ac}\times\dfrac{b^2+c^2-a^2-2bc}{a^2-b^2+c^2-2ac}$.
22. $\dfrac{c^2-a^2-b^2+2ab}{b^2-c^2-a^2+2ac}\times\dfrac{a^2-b^2+c^2-2ac}{a^2+b^2-c^2-2ab}$.

111. Division of Fractions.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions ; to find the value of $\frac{a}{b} \div \frac{c}{d}$.

Let $x = \frac{a}{b} \div \frac{c}{d}$.

Then, we have $x \times \frac{c}{d} = \frac{a}{b} \div \frac{c}{d} \times \frac{c}{d} = \frac{a}{b}$.

[$\because$ $m \div n \times n = m$, whatever m and n may be.]

$\therefore$ $x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$; or, $x = \frac{a}{b} \times \frac{d}{c}$. $\left[\because \frac{c}{d} \times \frac{d}{c} = 1.\right]$

Thus, *to divide one fraction by another we have to multiply the former by the reciprocal of the latter.*

Cor. $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}$.

Example 1. Simplify $\frac{a^3+b^3}{a^2-b^2} \div \frac{a^2-ab+b^2}{a-b}$.

$$\text{The required result} = \frac{a^3+b^3}{a^2-b^2} \times \frac{a-b}{a^2-ab+b^2} = \frac{(a^3+b^3)(a-b)}{(a^2-b^2)(a^2-ab+b^2)}$$

$$= \frac{(a+b)(a^2-ab+b^2)(a-b)}{(a+b)(a-b)(a^2-ab+b^2)} = 1.$$

Example 2. Simplify $\frac{x^2+x-2}{x^2+7x+12} \div \frac{x^2-3x-10}{x^2+x-12} \times \frac{x^2-4x-5}{x^2-4x+3}$.

$$\text{The required result} = \frac{x^2+x-2}{x^2+7x+12} \times \frac{x^2+x-12}{x^2-3x-10} \times \frac{x^2-4x-5}{x^2-4x+3}$$

$$= \frac{(x-1)(x+2)}{(x+3)(x+4)} \times \frac{(x+4)(x-3)}{(x-5)(x+2)} \times \frac{(x-5)(x+1)}{(x-3)(x-1)}$$

$$= \frac{(x-1)(x+2)(x+4)(x-3)(x-5)(x+1)}{(x+3)(x+4)(x-5)(x+2)(x-3)(x-1)} = \frac{x+1}{x+3}.$$

Example 3. Simplify $\dfrac{\frac{a}{a-b} - \frac{a}{a+b}}{\frac{b}{a-b} - \frac{b}{a+b}} \div \dfrac{\frac{a+b}{a-b} + \frac{a-b}{a+b}}{\frac{a+b}{a-b} - \frac{a-b}{a+b}} \times \frac{a^2}{a^2+b^2}$.

[C. U. 1876]

$$\text{We have } \dfrac{\frac{a}{a-b} - \frac{a}{a+b}}{\frac{b}{a-b} - \frac{b}{a+b}} = \dfrac{\frac{a(a+b)-a(a-b)}{a^2-b^2}}{\frac{b(a+b)-b(a-b)}{a^2-b^2}} = \frac{2ab}{a^2-b^2} \div \frac{2b^2}{a^2-b^2}$$

$$= \frac{2ab}{a^2-b^2} \times \frac{a^2-b^2}{2b^2} = \frac{a}{b}; \quad \dots \quad \dots \quad (1)$$

and
$$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}=\frac{\dfrac{(a+b)^2+(a-b)^2}{a^2-b^2}}{\dfrac{(a+b)^2-(a-b)^2}{a^2-b^2}}=\frac{2(a^2+b^2)}{a^2-b^2}\div\frac{4ab}{a^2-b^2}$$

$$=\frac{2(a^2+b^2)}{a^2-b^2}\times\frac{a^2-b^2}{4ab}=\frac{a^2+b^2}{2ab}. \quad \text{... (2)}$$

Hence, from (1) and (2),

$$\text{the given expression}=\frac{a}{b}\div\frac{a^2+b^2}{2ab}\times\frac{a^2}{a^2+b^2}$$

$$=\frac{a}{b}\times\frac{2ab}{a^2+b^2}\times\frac{a^2}{a^2+b^2}=\frac{2a^4}{(a^2+b^2)^2}.$$

N. B. 'When several fractions are connected by the signs ×, ÷, each sign applies only to the fraction which immediately follows it.'

EXERCISE 58

Simplify :

1. $\dfrac{4a^2bc}{15xy^2z}\div\dfrac{8ab^2c}{25x^2yz}$

2. $\dfrac{a^2+ab}{a-b}\div\dfrac{ab}{a^2-b^2}.$

3. $\dfrac{x^2-49}{x^2-25}\div\dfrac{x+7}{x+5}$

4. $\dfrac{a^4-b^4}{a^2+2ab+b^2}\div\dfrac{a^2+b^2}{a+b}.$

5. $\dfrac{m^2-9n^2}{m^2+5mn+6n^2}\div\dfrac{m^2-2mn-3n^2}{m^2-n^2}.$

6. $\dfrac{m^3-n^3}{m+n}\div\dfrac{m^2+mn+n^2}{m^2-n^2}.$

7. $\left(\dfrac{2x+y}{x+y}-1\right)\div\left(1-\dfrac{y}{x+y}\right).$

8. $\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right)\div\left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right).$

9. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right).$

10. $\dfrac{x^2-4}{x^2+3x-18}\div\dfrac{x^2-5x-14}{x^2-36}.$

11. $\left(1-\dfrac{2pq}{p^2+q^2}\right)\div\left(\dfrac{p^3-q^3}{p-q}-3pq\right).$

12. $\dfrac{a^3+b^3+3ab(a+b)}{(a+b)^2-4ab}\div\dfrac{(a-b)^2+4ab}{a^3-b^3-3ab(a-b)}.$

13. $\dfrac{x^3+y^3}{(x-y)^2+3xy}\div\dfrac{(x+y)^2-3xy}{x^3-y^3}\times\dfrac{xy}{x^2-y^2}.$

14. $\dfrac{a(a-b)^2+4a^2b}{ab+b^2}\div\dfrac{a^2-b^2}{ab}\times\dfrac{b(a+b)^2-4ab^2}{a^2-ab}.$

15. $\dfrac{x^2-x-30}{x^2-36}\div\dfrac{x^2+3x-10}{x^2+2x-8}\div\dfrac{x+4}{2x^2+12x}.$

16. $\dfrac{x^2+3x-108}{x^2-64} \div \dfrac{x^2+6x-72}{x^2+x-56} \div \dfrac{x^2-16x+63}{x^2-14x+48}.$

17. $\left(\dfrac{x^2+y^2}{x^2-y^2} - \dfrac{x^2-y^2}{x^2+y^2}\right) \div \left(\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}\right).$

18. $\left(\dfrac{a+b}{a-b} + \dfrac{a^2+b^2}{a^2-b^2}\right) \div \left(\dfrac{a-b}{a+b} - \dfrac{a^3-b^3}{a^3+b^3}\right).$ [C. U. 1868]

19. $\dfrac{a^4-b^4}{(a+b)^3-3ab(a+b)} \div \dfrac{(a+b)^2-4ab}{(a+b)^2-3ab} \times \dfrac{a}{(a+b)^2-2ab}.$

20. $\dfrac{(a-b)\{(a+b)^2-ab\}}{(a-b)^2+2ab} \div \dfrac{(a-b)^2+3ab}{(a+b)\{(a-b)^2+ab\}} \times \dfrac{(a+b)^2-2ab}{(a+b)^2-3ab}.$

21. $\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)} \times \dfrac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}.$ [C. U. 1874]

CHAPTER XVII

SIMPLE EQUATIONS AND PROBLEMS

I. Simple Equations

112. In Chapter V, we have explained the process of solving easy simple equations. We propose to consider the subject more fully here.

We have stated that the process of solving any equation is primarily based upon certain axioms [Art. 63] from which it has been noticed that an equation is not altered,

(i) if any term be transposed from one side of the equation to the other ; and (ii) if both the sides be multiplied or divided by any the *same* quantity.

Hence, the general rule for solving a simple equation involving one unknown quantity may be put as follows :

(1) *Simplify the two sides separately by clearing of fractions and brackets, if any, and by performing operations indicated by the symbols.*

(2) *Transpose all the terms involving the unknown quantity to the left-hand side of the equation and the remaining terms to the right-hand side.*

(3) *Next, simplify the two sides again.*

(4) ***Finally, divide both the sides by the coefficient of the unknown quantity.***

The value of the unknown quantity, thus obtained, is the required solution.

Note. *The student should verify for his own satisfaction that this value does really satisfy the given equation.*

Example 1. Solve $(6x+9)^2+(8x-7)^2=(10x+3)^2-71$. [C. U. 1882]

The left side $=(36x^2+108x+81)+(64x^2-112x+49)$

$$=100x^2-4x+130;$$

and the right side $=(100x^2+60x+9)-71$

$$=100x^2+60x-62.$$

Hence, the equation stands thus:

$$100x^2-4x+130=100x^2+60x-62.$$

Removing $100x^2$ from both sides, we have

$$-4x+130=60x-62.$$

Hence, by transposition,

$$-4x-60x=-130-62,$$

or, $\qquad -64x=-192;$

and therefore, dividing both sides by -64, we have $x=3$.

Thus, the required root is 3.

Example 2. Given $\dfrac{x-6}{8}-\dfrac{2x-15}{9}+1=\dfrac{x}{15}-\dfrac{x-12}{6}$; find x.

Multiplying both sides by $8\times9\times5$ or 360, which is the L.C.M. of the denominators, we have

$$\frac{360(x-6)}{8}-\frac{360(2x-15)}{9}+360=\frac{360x}{15}-\frac{360(x-12)}{6},$$

or, $\qquad 45(x-6)-40(2x-15)+360=24x-60(x-12),$

or, $\qquad 45x-270-80x+600+360=24x-60x+720,$

or, $\qquad -35x+690=-36x+720.$

Hence, by transposition, $-35x+36x=720-690,$

or, $\qquad x=30.$

Example 3. Solve $\frac{1}{3}\{4a(1+x)-\frac{9}{4}(a-x)\}=\frac{1}{4}\{3a(1-x)-\frac{10}{3}(a+x)\}$.

The left side $=\dfrac{4a}{3}(1+x)-\frac{3}{4}(a-x)=\left(\dfrac{4a}{3}-\dfrac{3a}{4}\right)+\left(\dfrac{4a}{3}+\dfrac{3}{4}\right)x$

$$=\frac{7a}{12}+\frac{16a+9}{12}\cdot x;$$

and the right side

$$=\frac{3a}{4}(1-x)-\tfrac{4}{3}(a+x)=\left(\frac{3a}{4}-\frac{4a}{3}\right)-\left(\frac{3a}{4}+\frac{4}{3}\right)x$$

$$=-\frac{7a}{12}-\frac{9a+16}{12}.x.$$

Hence, the equation stands thus :

$$\frac{7a}{12}+\frac{16a+9}{12}.x=-\frac{7a}{12}-\frac{9a+16}{12}.x.$$

Multiplying both sides by 12,

$$7a+(16a+9)x=-7a-(9a+16)x.$$

Hence, by transposition,

$$\{(16a+9)+(9a+16)\}x=-14a,$$

or, $25(a+1)x=-14a$;

$\therefore$ dividing both sides by $25(a+1)$, we have

$x=\dfrac{-14a}{25(a+1)}$, which is the required root.

Example 4. Given $\dfrac{x}{a+b}+1=\dfrac{x}{a-b}+\dfrac{a-b}{a+b}$; find x.

Multiplying both sides by a^2-b^2, which is the L.C.M. of the denominators, we have,

$$(a-b)x+(a^2-b^2)=(a+b)x+(a-b)^2.$$

Hence, by transposition,

$$(a-b)x-(a+b)x=(a-b)^2-(a^2-b^2),$$

or, $\{(a-b)-(a+b)\}x=-2ab+2b^2,$

or, $-2bx=-2b(a-b).$

Therefore, dividing both sides by $-2b$, we have $x=a-b$.

EXERCISE 59

Find the value of x, when

1. $3(x-4)^2+5(x-3)^2=(2x-5)(4x-1)+24.$
2. $(12x+9)^2+(5x+3)^2=(13x+9)^2+33.$
3. $5(x+1)^2+7(x+3)^2=12(x+2)^2.$
4. $(3x-14)^2+(4x-19)^2-(5x-23)^2=22.$
5. $(5x-8)^2+(12x-7)^2=(13x-10)^2+37.$
6. $(x-1)^3+(x+1)^3=2x(x^2-1)+4.$
7. $(x-2)^3+2x^3+(x+2)^3=4x^2(x+2).$

8. $(x+2)(x+3)(x+4)+96=x^2(x+9)+5(3x+13)$.

9. $3(x^2-14)=(x+1)^2+(x-2)^2+(x-5)^2$.

10. $a(x-a)=b(x-b)$. 11. $3(x-a)+5(2x-3a)=8a$.

Solve the following equations :

12. $(x+a)(x+b)-(a+b)^2=(x-a)(x-b)$.

13. $a^2(x-a)+b^2(x-b)=abx$. 14. $m^2(x-m)+n^2(x+n)+mnx=0$.

15. $b(x-2a)+a(x-2b)=(a-b)^2$. 16. $a(4x-a)+b(4x-b)-2ab=0$.

17. $x(x-a)+x(x-b)-2(x-a)(x-b)=0$.

18. $(x+3a)(x-3b)+3(x-3a)(x+3b)=4(x-3a)(x-3b)$.

19. $(2b+2c-x)^2+(2b-2c+x)^2=(2b+2d-x)^2+(2b-2d+x)^2$.

20. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.

21. $(x+a)^3+(x+b)^3+(x+c)^3=3(x+a)(x+b)(x+c)$.

22. $\frac{x}{a}+a=\frac{x}{b}+b$. 23. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$.

24. $\frac{1}{2}(x+1)+\frac{1}{3}(x+2)+\frac{1}{4}(x+3)=16$.

25. $\frac{x-6}{5}+\frac{x-4}{3}=8-\frac{x-2}{7}$. 26. $\frac{x}{10}+\frac{2x-13}{9}=8-\frac{4x-35}{15}$.

27. $\frac{x+7}{2}+\frac{x+13}{5}+\frac{x+17}{7}=\frac{x+27}{4}$. 28. $6\frac{1}{3}-\frac{x-7}{3}=\frac{4x-2}{5}$.

[C. U. 1861]

29. $\frac{x-1}{3}-\frac{x-9}{2}+\frac{3x-2(x-2)}{7}=4\frac{1}{2}$.

30. $\frac{2x-9}{27}+\frac{x}{18}-\frac{x-3}{4}=8\frac{1}{3}-x$. 31. $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36$.

32. $\frac{7x+9}{4}-\left(x-\frac{2x-1}{9}\right)=7$. 33. $\frac{x+7}{3}-5\frac{3}{4}=\frac{2x+5}{7}+\frac{10-5x}{8}$.

34. $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}\left(2x-57\right)-\frac{5}{3}$. [C. U. 1889]

35. $\frac{4x-21}{7}+7\frac{5}{6}+\frac{7x-28}{3}=x+3\frac{3}{4}-\frac{9-7x}{8}+\frac{1}{12}$.

36. $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0$. [C. U. 1866]

37. $\frac{x-3}{7}-\frac{\frac{1}{2}x-3}{3}=\frac{\frac{1}{6}x+2}{2}-\frac{x-6}{3}+\frac{x}{8}$. [C. U. 1866]

38. $\frac{1}{3}(x-2)-\frac{1}{4}(x-4)=\frac{1}{12}(2x-3)-2\frac{3}{4}$. [C. U. 1869]

39. $\frac{a-x}{a}+\frac{2a-x}{2a}=\frac{3a-x}{3a}$. [C. U. 1870]

40. $\frac{2x-13}{9}-\frac{x-1}{11}=\frac{x}{8}+\frac{x}{7}-9.$ [C. U. 1876]

41. $\frac{2x-3}{6}+\frac{3x-8}{11}=\frac{4x+15}{33}+\frac{1}{2}.$ [C. U. 1877]

42. $\frac{4x+3}{9}+\frac{13x}{108}=\frac{8x+19}{18}.$ [C. U. 1878]

43. $\frac{x^2-2\frac{1}{2}}{4}-\frac{x-3\frac{1}{2}}{5}=\frac{2x^2-3}{8}-\frac{x-5\frac{1}{2}}{3}.$ [C. U. 1883]

44. $\frac{a-x^2}{bx}-\frac{b-x}{c}=\frac{c-x}{b}-\frac{b-x^2}{cx}.$ [C. U. 1886]

45. $\frac{x+2\frac{1}{2}}{15}+\frac{x+3\frac{1}{3}}{25}=\frac{x+4\frac{1}{6}}{55}.$ [C. U. 1888]

46. $\frac{11x-13}{25}+\frac{19x+3}{7}-\frac{5x-25\frac{1}{3}}{4}=28\frac{1}{7}-\frac{17x+4}{21}.$

47. $\frac{x-1\frac{25}{26}}{2}-\frac{2-6x}{13}=x-\frac{5x-\frac{1}{4}(10-3x)}{39}.$

48. $\frac{3x-\frac{2}{3}(1+x)}{4}+\frac{1-\frac{1}{3}x}{5\frac{1}{2}}=\frac{2\frac{2}{3}+\frac{1}{25}(x-1)}{2\frac{1}{3}}.$

49. $\frac{1}{3}(x-a)-\frac{1}{4}(2x-3b)-\frac{1}{2}(a-x)=10a+11b.$

50. $\frac{2x+a}{b}-\frac{x-b}{a}=\frac{3ax+(a-b)^2}{ab}.$ 51. $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}.$

52. $\frac{15-\frac{2}{3}x}{5}-\frac{2x+5}{2\frac{1}{2}}=\frac{17-\frac{14}{3}x}{3}.$ [C. U. 1874]

113. Equations involving Decimals.

The decimals, if necessary, may be converted into vulgar fractions.

Example 1. Solve $\frac{x-2}{\cdot 0\dot{5}}-\frac{x-4}{\cdot 0625}=56.$

Since, $\cdot 0\dot{5}=\frac{5}{90}=\frac{1}{18}$, and $\cdot 0625=\frac{625}{10000}=\frac{1}{16}$,

we have $\frac{x-2}{\frac{1}{18}}-\frac{x-4}{\frac{1}{16}}=56,$

or, $18(x-2)-16(x-4)=56,$ or, $2x+28=56;$

$\therefore$ $2x=28,$ or, $x=14.$

Example 2. Solve ${\cdot}65x+\frac{{\cdot}585x-{\cdot}975}{{\cdot}6}=\frac{1{\cdot}56}{{\cdot}2}-\frac{{\cdot}39x-{\cdot}78}{{\cdot}9}$.

Since, $\frac{{\cdot}585x-{\cdot}975}{{\cdot}6}=\frac{5{\cdot}85x-9{\cdot}75}{6}=\frac{1{\cdot}95x-3{\cdot}25}{2}$,

$\frac{1{\cdot}56}{{\cdot}2}=\frac{15{\cdot}6}{2}=7{\cdot}8$,

and $\frac{{\cdot}39x-{\cdot}78}{{\cdot}9}=\frac{3{\cdot}9x-7{\cdot}8}{9}=\frac{1{\cdot}3x-2{\cdot}6}{3}$,

the equation stands thus :

$${\cdot}65x+\frac{1{\cdot}95x-3{\cdot}25}{2}=7{\cdot}8-\frac{1{\cdot}3x-2{\cdot}6}{3}.$$

Hence, multiplying both sides by 6, we have

$$3{\cdot}9x+(5{\cdot}85x-9{\cdot}75)=46{\cdot}8-(2{\cdot}6x-5{\cdot}2).$$

By transposition, $(3{\cdot}9+5{\cdot}85+2{\cdot}6)x=46{\cdot}8+5{\cdot}2+9{\cdot}75$,

or, $12{\cdot}35x=61{\cdot}75$;

$\therefore \quad x=\frac{61{\cdot}75}{12{\cdot}35}=5$.

EXERCISE 60

Solve the following equations :

1. ${\cdot}5x-{\cdot}\dot{2}x={\cdot}\dot{8}x-1{\cdot}5$.

2. $3{\cdot}75x+{\cdot}5=2{\cdot}25x+8$.

3. $1{\cdot}2x-\frac{{\cdot}18x-{\cdot}05}{{\cdot}5}={\cdot}4x+8{\cdot}9$.

4. $\frac{x+{\cdot}75}{{\cdot}125}-\frac{x-{\cdot}25}{{\cdot}25}=15$.

5. $\frac{x}{{\cdot}5}-\frac{1}{{\cdot}05}+\frac{x}{{\cdot}005}-\frac{1}{{\cdot}0005}=0$. [C. U. 1883]

6. ${\cdot}5x+\frac{{\cdot}45x-{\cdot}75}{{\cdot}6}=\frac{1{\cdot}2}{{\cdot}2}-\frac{{\cdot}3x-{\cdot}6}{{\cdot}9}$.

7. ${\cdot}7x+{\cdot}\dot{4}={\cdot}6\dot{7}x+{\cdot}\dot{5}$.

8. ${\cdot}15x+\frac{{\cdot}135x-{\cdot}225}{{\cdot}6}=\frac{{\cdot}36}{{\cdot}2}-\frac{{\cdot}09x-{\cdot}18}{{\cdot}9}$.

9. ${\cdot}5x+\frac{{\cdot}02x+{\cdot}07}{{\cdot}03}-\frac{x+2}{9}=9{\cdot}5$.

10. ${\cdot}011x+\frac{{\cdot}001x-{\cdot}125}{{\cdot}6}=\frac{5-x}{{\cdot}03}-{\cdot}145$. [C. U. 1886]

114. Solution of equations facilitated by suitable transposition and combination of terms.

Example 1. Solve $\frac{23x-29}{12}+\frac{19x+13}{7}=\frac{97x+72\frac{1}{2}}{35}+\frac{7x-8\frac{1}{3}}{4}$.

By transposition, we have

$$\frac{23x-29}{12}-\frac{7x-8\frac{1}{3}}{4}=\frac{97x+72\frac{1}{2}}{35}-\frac{19x+13}{7};$$

or, $$\frac{(23x-29)-(21x-25)}{12}=\frac{(97x+72\frac{1}{2})-(95x+65)}{35};$$

or, $$\frac{x-2}{6}=\frac{2x+7\frac{1}{2}}{35}.$$

Hence, multiplying both sides by 6 × 35,

$$35x-70=12x+45.$$

Hence, $23x=115$, or, $x=5$.

Example 2. Solve $\frac{x-a(b+c)}{bc}+\frac{x-b(c+a)}{ca}+\frac{x-c(a+b)}{ab}=3$.

The equation may be written as

$$\frac{x-a(b+c)}{bc}+\frac{x-b(c+a)}{ca}+\frac{x-c(a+b)}{ab}=1+1+1.$$

By transposition, we have

$$\left\{\frac{x-a(b+c)}{bc}-1\right\}+\left\{\frac{x-b(c+a)}{ca}-1\right\}+\left\{\frac{x-c(a+b)}{ab}-1\right\}=0;$$

or, $$\frac{x-a(b+c)-bc}{bc}+\frac{x-b(c+a)-ca}{ca}+\frac{x-c(a+b)-ab}{ab}=0;$$

or, $$\frac{x-(ab+ac+bc)}{bc}+\frac{x-(bc+ba+ca)}{ca}+\frac{x-(ca+cb+ab)}{ab}=0;$$

or, $$\left\{x-(ab+bc+ca)\right\}\left\{\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right\}=0.$$

When the product of two quantities equals to 0, at least one of them must be equal to 0. Since the sum of three known quantities, *viz.*, $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}$ cannot be zero, the other must be equal to 0.

$\therefore \quad x-(ab+bc+ca)=0$.

Hence, $x=ab+bc+ca$.

EXERCISE 61

Solve the following equations :

1. $\frac{5x+6}{4}+\frac{64x-35}{15}=\frac{20x+23}{16}+\frac{13x-7}{3}.$

2. $\frac{17x-13}{9}+\frac{108x+75}{32}=\frac{27x+19}{8}+\frac{50\frac{7}{18}x-39}{27}.$

3. $\frac{29x-18}{8}+\frac{189x-93}{49}=\frac{86\frac{18}{21}x-54}{24}+\frac{27x-13}{7}.$

4. $\frac{16x-17}{9}-\frac{23x-15}{16}=\frac{142\frac{7}{18}x-153}{81}-\frac{92x-65}{64}.$

5. $\frac{18x-19}{7}+\frac{135x+62\frac{1}{2}}{65}=\frac{27x+14}{13}+\frac{106\frac{5}{18}x-114}{42}.$

6. $\frac{33-19x}{15}-\frac{41+27x}{28}+\frac{164+107\frac{11}{18}x}{112}-\frac{164\frac{13}{25}-95x}{75}=0.$

7. $\frac{18-41x}{9}-\frac{17-16x}{8}+\frac{9\frac{16}{21}-10x}{5}-\frac{14-32x}{7}=0.$

8. $\frac{x-a^2}{b^2+c^2}+\frac{x-b^2}{c^2+a^2}+\frac{x-c^2}{a^2+b^2}=3.$

9. $\frac{3x-bc}{b+c}+\frac{3x-ca}{c+a}+\frac{3x-ab}{a+b}=a+b+c.$

10. $\frac{ax-b^2+c^2}{c-b}+\frac{bx-c^2+a^2}{a-c}+\frac{cx-a^2+b^2}{b-a}=2(a+b+c).$

11. $\frac{x-(b^3+c^3)}{a^2-3bc}+\frac{x-(c^3+a^3)}{b^2-3ca}+\frac{x-(a^3+b^3)}{c^2-3ab}=a+b+c.$

12. $\frac{p^2x+(l^3+m^3)}{l^2-lm+m^2}+\frac{q^2x+(m^3+n^3)}{m^2-mn+n^2}+\frac{r^2x+(n^3+l^3)}{n^2-ln+l^2}=2(l+m+n).$

II. Problems

115. We have already explained in Chapter VI how simple algebraic problems can be expressed symbolically and solved. We proceed now to consider examples of a harder type.

As pointed out before, the chief difficulty in the solution of a problem lies in constructing its symbolical expression. The student should, therefore, become proficient in it by constant and varied practice.

No general rule for solution can be stated. The following advice can, however, be offered :

Read the problem several times and consider its meaning carefully.

See what quantity is required to be found out in the problem. Represent it by x.

Next, express the conditions of the problem in terms of the symbol x and obtain a simple equation in x.

Finally, solving this equation, find the value of x.

The process is explained by the following examples. For further illustrations, the student is referred to Chapter VI.

Example 1. How old is a man now, who, 20 years ago, was five times as old as his son who will be 41 years old 16 years after ?

The present age of the man is to be found out. Let it be x years.

$\therefore$ 20 years ago, the man's age $=(x-20)$ years,

16 years after, the son's age will be 41 years ;

$\therefore$ the son's present age $=41-16=25$ years.

Hence, 20 years ago, the son's age $=25-20=5$ years.

$\therefore$ from the condition of the problem,

$$x-20=5\times 5,$$

$$\text{or, } x=20+5\times 5=20+25=45 \text{ years.}$$

Thus, the man's present age $=45$ years.

Example 2. The sum of five consecutive odd numbers is 1185. What are the numbers ?

[In solving problems, $2x$ and $2x+1$ are taken as even and odd number respectively ; because for any integral value of x, the value of $2x$ is even and that of $2x+1$ is odd.]

Let $2x+1=$ the smallest of the consecutive odd numbers. Since consecutive odd numbers differ from each other by 2, the numbers after $2x+1$ are $2x+3$, $2x+5$, $2x+7$, $2x+9$, etc. In the present problem, the five consecutive odd numbers are, therefore, $2x+1$, $2x+3$, $2x+5$, $2x+7$ and $2x+9$.

By the condition of the problem, their sum $=1185$;

or, $(2x+1)+(2x+3)+(2x+5)+(2x+7)+(2x+9)=1185$,

or, $10x+25=1185$, or, $10x=1185-25=1160$;

$\therefore$ $x=\frac{1160}{10}=116$.

Thus, the smallest of the consecutive odd numbers is **233**.

Hence, the five required consecutive odd numbers are **233**, 235. **237, 239, 241**.

Example 3. Two persons started at the same time from A. One rode on horseback at the rate of $7\frac{1}{2}$ kilometres an hour and arrived at B, 30 minutes later than the other who travelled the same distance by train at the rate of 30 kilometres an hour. Find the distance between A and B.

Let x be the distance in kilometres between A and B. Then the time taken by the first man to travel the distance $=\frac{x}{7\frac{1}{2}}$ hours $=\frac{2x}{15}$ hours and the time taken by the other $=\frac{x}{30}$ hours.

But the time taken by the former is half an hour more than that taken by the latter.

Hence, $\frac{2x}{15}=\frac{x}{30}+\frac{1}{2}$; or, $4x=x+15$;

$\therefore$ $3x=15$; $\therefore$ $x=5$.

Thus, the distance between A and $B=5$ kilometres.

Example 4. A person being asked his age, replied, "Ten years ago I was 5 times as old as my son, but 20 years hence I shall be only twice as old as he." What is his age ?

Let the present age of the person be x years

Then 10 years ago his age was $(x-10)$ years, and $\therefore$ that of his son was $\frac{1}{5}(x-10)$ years.

Hence, the present age of his son $=\{\frac{1}{5}(x-10)+10\}$ years, and $\therefore$ the son's age 20 years hence will be $\{\frac{1}{5}(x-10)+30\}$ years ; and the age of the person 20 years hence will evidently be $(x+20)$ years.

Hence, by the second condition of the problem, we must have

$$x+20=2\{\tfrac{1}{5}(x-10)+30\}$$
$$=\tfrac{2}{5}(x-10)+60 ;$$
$$\therefore\ 5x+100=2x-20+300 ;$$
$$\therefore\ 3x=180 ; \quad \therefore\ x=60.$$

Note. *Fractions might have been avoided by assuming the present age of the person to be $5x$ years. The student can easily proceed on this assumption.*

Example 5. A and B have the same annual income. A lays by a *fifth* of his, but B, by spending annually Rs. 80 more than A, at the end of 4 years finds himself Rs. 220 in debt. What was their income ?

Let Rs. x be the income of each.

Then A spends Rs. $\frac{4}{5}x$ annually. Hence, B spends annually Rs. $(\frac{4}{5}x+80)$.

But spending at this rate B contracts a debt of Rs. 220 in 4 years, or a debt of Rs. 55 per year. His annual income, therefore, falls short of his annual expenses by Rs. 55.

Hence, we must have $x=(\frac{4}{5}x+80)-55$;

$\therefore$ $\frac{1}{5}x=25$; $\therefore$ $x=125$.

Thus, A and B had each an income of Rs. 125.

Example 6. A market woman bought a certain number of eggs at 5 a rupee, and as many at 7 a rupee, and sold them at the rate of 12 for two rupees, losing rupee one by her bargain. What number of eggs did she buy ?

Let x = the number of eggs bought.

Then, since one half of them were bought at 5 a rupee, and the other half at 7 a rupee, the whole cost in buying the eggs

$$=\left(\frac{x}{2}\cdot\frac{1}{5}+\frac{x}{2}\cdot\frac{1}{7}\right)\text{ rupees}=\left(\frac{x}{10}+\frac{x}{14}\right)\text{ rupees.}$$

By selling the eggs at 12 for two rupees,

the amount realised $=x\times\frac{2}{12}$ rupees.

Hence, by the equation, $\frac{2x}{12}=\left(\frac{x}{10}+\frac{x}{14}\right)-1$;

or, $35x=21x+15x-210$; $\therefore$ $x=210$.

Thus, altogether 210 eggs were bought.

Example 7. Divide 28 into two such parts that the difference between their squares is equal to 112.

Suppose, x is the greater part, so the other is $(28-x)$.

By the condition of the problem,

$$x^2-(28-x)^2=112 \text{ ;}$$

or, $(x+28-x)(x-28+x)=112$;

or, $28(2x-28)=112$;

or, $(x-14)=2$; [dividing both sides by 28×2]

or, $x=14+2$;

$\therefore$ $x=16$; $\therefore$ the parts are 16 and 12.

Example 8. There is a number consisting of two digits, the digit in the units' place is twice that in the tens' place, and if 2 be subtracted from the sum of the digits, the difference is equal to $\frac{1}{6}$th of the number. Find the number.

Let x = the digit in the tens' place.

Then $2x$ = " " " " units' " .

Clearly, therefore, the number $=10x+2x$.

[See Example 4 worked out in Art. 65.]

Hence, by the second condition of the problem,

$$(x+2x)-2=\frac{10x+2x}{6} \text{ ;}$$

whence, $18x-12=12x$;

or, $6x=12$; $\therefore$ $x=2$.

Hence, the required number = 24.

Example 9. Divide 127 into 4 parts, such that if the first be increased by 18, the second diminished by 5, the third multiplied by 6 and the fourth divided by $2\frac{1}{2}$, the results will all be equal. [B. U. 1883]

Let x be the result in all the cases.

By the condition of the problem,

The 1st part $+18=x$; $\therefore$ the 1st part $=x-18$.
The 2nd part $-\ 5=x$; $\therefore$ the 2nd part $=x+5$.
The 3rd part $\times\ 6=x$; $\therefore$ the 3rd part $=\frac{x}{6}$.
The 4th part $\div 2\frac{1}{2}=x$; $\therefore$ the 4th part $=\frac{5}{2}x$.

Therefore, $(x-18)+(x+5)+\frac{x}{6}+\frac{5}{2}x=127$;

or, $6(x-18)+6(x+5)+6\cdot\frac{x}{6}+6\cdot\frac{5}{2}x=6\times 127$;
or, $6x-108+6x+30+x+15x=762$;
or, $28x=762+108-30$;
or, $28x=840$;
$\therefore$ $x=\frac{840}{28}=30$.

$\therefore$ the parts are $(30-18)$, $(30+5)$, $(30\div6)$ and $30\times\frac{5}{2}$,

i.e., 12, 35, 5, 75.

EXERCISE 62

1. The length of a field is twice its breadth ; another field which is 50 metres longer and 10 metres broader, contains 6800 square metres more than the former ; find the size of each.

2. The length of a room exceeds its breadth by 3 metres ; if the length had been increased by 3 metres, and the breadth diminished by 2 metres, the area would not have been altered ; find the dimensions.

3. A and B began to play with equal sums, and when B has lost $\frac{5}{11}$ths of what he had to begin with, A has gained Rs. 6 more than half of what B has been left with ; what had they at first ?

4. The ages of a father and his son together are 80 years ; and if the age of the son be doubled, it will exceed the father's age by 10 years. Find the age of each.

5. A person distributed Rs. 100 among 36 persons, men and women, giving rupees three to each man and two rupees and a half to each woman. How many were there of each ?

6. The sum of four consecutive odd numbers is 488. What are the numbers ?

7. The sum of six consecutive even numbers is 1362. What are the numbers ?

8. There are two places, 154 kilometres distant from each other, from which two persons *A* and *B* set out at the same instant with a desire to meet on the road, *A* travelling at the rate of 3 kilometres in 2 hours and *B* at the rate of 5 kilometres in 4 hours. How long and how far did each travel before they met ?

9. A labourer was engaged for 36 days, upon the condition that he should receive two rupees and fifty paise for every day he worked, but should pay rupee one and fifty paise for every day he was idle. At the end of the time he received fifty-eight rupees. How many days did he work ?

10. A person bought a picture at a certain price and paid the same price for the frame ; if the frame had cost rupees twenty less and the picture rupees fifteen more, the price of the frame would have been only half that of the picture. Find the cost of the picture.

11. A post has a fourth of its length in the mud, a third of its length in the water and 10 metres above the water, what is its length ?

12. Divide 20 into two such parts that the difference between their squares is 160. [G. U. 1950]

13. A labourer is engaged for 30 days on condition that he receives two rupees and fifty paise for each day he works, and loses rupee one for each day he is idle ; he receives rupees forty-seven in all. How many days does he work, and how many days is he idle ?

14. *A* can do a piece of work in 9 days, *B* in twice that time ; *C* can do only $\frac{3}{4}$ as much as *A*, in a day ; how long would *A*, *B* and *C*, working together, require to do the same piece of work ? [C. U. 1876]

15. Two sums of money are together equal to rupees fifty-seven and twenty paise and there are as many rupees in the one as 10 paise in the other. What are the sums ?

16. A certain sum is to be divided among *A*, *B* and *C*. *A* is to have Rs. 30 less than the half, *B* is to have Rs. 10 less than the third part, and *C* is to have Rs. 8 more than the fourth part. What does each receive ?

17. A farmer wishing to purchase a number of sheep, found that if they cost him Rs. 42 a head, he would not have money enough by Rs. 28 ; but if they cost him Rs. 40 a head, he would then have Rs. 40 more than he required. Find the number of sheep, and the money which he had

18. Two coaches start at the same time from York and London, a distance of 320 kilometres travelling, one at 9 kilometres an hour, the other at 11. Where will they meet and in what time from starting ?

19. I bought a certain number of apples at three a rupee, and five-sixths of that number at four a rupee; by selling them at sixteen for six rupees I gained rupees three and a half. How many apples did I buy?

20. A number consists of two digits; the sum of the digits is 5, and if the left digit be increased by 1, it will be equal to $\frac{1}{7}$th of the number. Find the number.

21. A number consists of two digits, the digit in the tens' place exceeds that in the units' place by 5, and if 5 times the sum of the digits be subtracted from the number, the digits will be inverted. Find the number.

22. There is a number consisting of two digits, the sum of whose digits is 5, and if 10 times the digit in tens' place be added to 4 times the digit in the units' place, the number will be inverted. What is the number?

23. Divide the number 39 into four parts, such that if the first be increased by 1, the second diminished by 2, the third multiplied by 3, and the fourth divided by 4, the results will all be equal.

24. Divide 60 into 4 parts, such that if the first be diminished by 3, the second increased by 11, the third multiplied by 4, and the fourth divided by 2, the results will all be equal.

25. Divide the number 116 into four such parts that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same.

CHAPTER XVIII

SIMPLE SIMULTANEOUS EQUATIONS AND PROBLEMS

I. Simple Simultaneous Equations

116. Introductory remarks. The equation $x-y=2$, in which x and y are both unknown, evidently admits of an infinite number of solutions; for any pair of numbers, whose difference is 2 will satisfy it. [For instance, the equation will be satisfied if $x=3$, $y=1$; if $x=4$, $y=2$; if $x=5$, $y=3$; if $x=6$, $y=4$; and so on.] If, however, x and y be such that they must *also* satisfy the equation $x+y=8$, then of the different pairs of numbers whose difference is 2, we shall have to reject all excepting that of which the sum is 8. Thus the two equations,

$$\left.\begin{array}{l} x-y=2 \\ x+y=8 \end{array}\right\}$$

will *both* be satisfied by the *same* values of x and y, *only when* $x=5$ and $y=3$.

Again, it may be seen that the three equations,

$$\left.\begin{aligned} x+y+z&=6 \\ x-y+z&=4 \\ x+y-z&=2 \end{aligned}\right\}$$

will be satisfied by the *same* values of x, y, z *only when* $x=3$, $y=1$, $z=2$. The equations may be *individually* satisfied by innumerable sets of values of the unknown quantities, but there is *only one* set which will satisfy them *all.*

Two or more equations (like those just referred to) which are *all* satisfied by the same values of the unknown quantities involved in them are called **simultaneous equations.** They are said to be **simple or of the first degree** when each unknown quantity occurs only in the first power and the product of the unknown quantities does not occur.

We shall consider first of all simultaneous equations involving two unknown quantities, and later on, those that involve more than two. There are three general methods for solving such equations and we shall treat them successively in the next three articles.

117. First Method : Method of Substitution : From either equation find the value of one of the unknown quantities in terms of the other and substitute the value thus found in the other equation.

Example 1. Solve $\left.\begin{aligned} 5x-24y&=16 \\ 4x-y&=31 \end{aligned}\right\}$

From the 2nd equation, we have

$$y=4x-31 \qquad \cdots \qquad \cdots \quad (1)$$

Substituting this value of y in the 1st equation, we have

$$5x-24(4x-31)=16,$$

or, $\quad 5x-96x+744=16$;

$\therefore \quad -91x=-728$; $\quad \therefore\ x=8.$

Hence, from (1), $y=4\times8-31=1.$

Thus, we have $\quad x=8$ and $y=1.$

Note. *The student is recommended to verify for his own satisfaction that these values of x and y do really satisfy* **both** *of the given equations.*

Example 2. Solve $\dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5}$; $8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3}$.

Multiplying both sides of the 1st equation by 10, we have

$$5(3x-5y)+30=2(2x+y),$$

or, $\quad 15x-25y+30=4x+2y$;

$\therefore \quad 11x=27y-30. \qquad \cdots \qquad \cdots \quad (1)$

Multiplying both sides of the 2nd equation by 12, we have

$$96-3(x-2y)=6x+4y,$$

or, $\quad 96-3x+6y=6x+4y$;

$\therefore \quad 2y-9x+96=0. \qquad \ldots \qquad \ldots \quad (2)$

From (1), we have $\quad x=\dfrac{27y-30}{11}. \qquad \ldots \qquad \ldots \quad (3)$

Substituting this value of x in (2), we have

$$2y-\frac{9(27y-30)}{11}+96=0\,;$$

$\therefore\ 22y-9(27y-30)+1056=0$;

$\therefore\ 22y-243y+270+1056=0$;

$\therefore\ 221y=1326$; $\quad \therefore\ y=6.$

Hence, from (3), $x=\dfrac{27\times 6-30}{11}=\dfrac{132}{11}=12.$

Thus, we have $x=12$ and $y=6$.

EXERCISE 63

Solve the following equations :

1. $\left.\begin{aligned} x+4y&=14\\ 7x-3y&=5 \end{aligned}\right\}$ 2. $\left.\begin{aligned} 5x-8y&=9\\ 13x+7y&=79 \end{aligned}\right\}$ 3. $\left.\begin{aligned} 2x+3y&=32\\ 11y-9x&=3 \end{aligned}\right\}$

4. $\left.\begin{aligned} 9x-4y&=8\\ 13x+7y&=101 \end{aligned}\right\}$ 5. $\left.\begin{aligned} x+ay&=b\\ ax-by&=c \end{aligned}\right\}$ 6. $\left.\begin{aligned} 2x-\tfrac{1}{5}(y-3)&=4\\ 3y+\tfrac{1}{8}(x-2)&=9 \end{aligned}\right\}$

7. $\left.\begin{aligned} \tfrac{1}{2}(x+y)&=\tfrac{1}{3}(2x+4)\\ \tfrac{1}{3}(x-y)&=\tfrac{1}{2}(x-24) \end{aligned}\right\}$ 8. $\left.\begin{aligned} \tfrac{1}{3}(x-y)&=\tfrac{1}{4}(y-1)\\ \tfrac{1}{7}(4x-5y)&=x-7 \end{aligned}\right\}$ [C. U. 1872]

9. $\left.\begin{aligned} \tfrac{1}{2}(3x-2y)-3&=\tfrac{1}{4}(2x-y)\\ \tfrac{1}{2}(5x-4y)-3&=\tfrac{1}{3}(4x-3y) \end{aligned}\right\}$ 10. $\left.\begin{aligned} \tfrac{1}{6}(2x+3y)+\tfrac{1}{3}x&=8\\ \tfrac{1}{2}(7y-3x)-y&=11 \end{aligned}\right\}$

118. Second Method : Method of Comparison : From each equation find the value of the same unknown quantity in terms of the other and equate the values thus found.

Example 1. Solve $\left.\begin{aligned} 6x-5y&=11\\ 2x+3y&=27 \end{aligned}\right\}$

From the 1st equation, we have

$$5y=6x-11.$$

$\therefore \quad y=\dfrac{6x-11}{5}. \qquad \ldots \qquad \ldots \quad (1)$

From the 2nd equation, we have

$$3y = 27 - 2x.$$

$$\therefore \quad y = \frac{27-2x}{3}. \qquad \dots \qquad \dots \quad (2)$$

Hence, from (1) and (2), we have

$$\frac{6x-11}{5} = \frac{27-2x}{3}.$$

$$\therefore \quad 3(6x-11) = 5(27-2x),$$

$$\text{or,} \quad 18x - 33 = 135 - 10x;$$

$$\therefore \quad 28x = 168; \quad \therefore \quad x = 6.$$

Hence, from (1), $y = \frac{6 \times 6 - 11}{5} = 5.$

Thus, we have $x = 6$ and $y = 5$.

Example 2. Solve $\left.\begin{aligned} \frac{7+x}{5} - \frac{2x-y}{4} &= 3y - 5 \\ \frac{5y-7}{2} + \frac{4x-3}{6} &= 18 - 5x \end{aligned}\right\}$ [C. U. 1880]

Multiplying both sides of the 1st equation by 20, we have

$$4(7+x) - 5(2x-y) = 20(3y-5),$$

$$\text{or,} \quad 28 - 6x + 5y = 60y - 100;$$

$$\therefore \quad 55y + 6x = 128. \qquad \dots \qquad \dots \quad (1)$$

Multiplying both sides of the 2nd equation by 6, we have

$$3(5y-7) + (4x-3) = 6(18-5x),$$

$$\text{or,} \quad 15y + 4x - 24 = 108 - 30x;$$

$$\therefore \quad 15y + 34x = 132. \qquad \dots \qquad \dots \quad (2)$$

From (1), $y = \frac{128-6x}{55}.$ ... ... (3)

From (2), $y = \frac{132-34x}{15}.$ ... ... (4)

Hence, from (3) and (4), we have

$$\frac{128-6x}{55} = \frac{132-34x}{15}, \quad \text{or,} \quad \frac{64-3x}{11} = \frac{66-17x}{3};$$

[Multiplying both sides by $\frac{5}{2}$]

$$\therefore \quad 3(64-3x) = 11(66-17x),$$

$$\text{or,} \quad 192 - 9x = 726 - 187x;$$

$$\therefore \quad 178x = 534; \quad \therefore \quad x = 3.$$

Hence, from (3), $y = \frac{128-18}{55} = \frac{110}{55} = 2.$

Thus, we have $x = 3$ and $y = 2$.

EXERCISE 64

Solve the following equations :

1. $\left.\begin{aligned} 5x-3y&=9 \\ 5y+2x&=16 \end{aligned}\right\}$ 2. $\left.\begin{aligned} 3y-4x&=1 \\ 3x+4y&=18 \end{aligned}\right\}$ 3. $\left.\begin{aligned} 3x-7y&=7 \\ 11x+5y&=87 \end{aligned}\right\}$

4. $\left.\begin{aligned} y(3+x)&=x(7+y) \\ 4x+9&=5y-14 \end{aligned}\right\}$ 5. $\left.\begin{aligned} 32x-25y&=28 \\ 14x+15y&=116 \end{aligned}\right\}$

6. $\left.\begin{aligned} \tfrac{1}{7}(3x+y)&=\tfrac{1}{5}(2x+y+1) \\ 8-\tfrac{1}{5}(x-y)&=6 \end{aligned}\right\}$ 7. $\left.\begin{aligned} 1\tfrac{1}{3}(5x-6y)+3x&=4y-2 \\ \tfrac{1}{6}(5x+6y)-\tfrac{1}{4}(3x-2y)&=2y-2 \end{aligned}\right\}$

8. $\left.\begin{aligned} 2x-\tfrac{1}{4}(y+3)&=7+\tfrac{1}{5}(3y-2x) \\ 4y+\tfrac{1}{3}(x-2)&=26\tfrac{1}{2}-\tfrac{1}{2}(2y+1) \end{aligned}\right\}$

9. $\left.\begin{aligned} 2x-\tfrac{1}{3}(2y-1)&=3\tfrac{5}{24}+\tfrac{1}{4}(3x-2y) \\ 4y-\tfrac{1}{4}(5-2x)&=6-\tfrac{1}{5}(3-2y) \end{aligned}\right\}$ [C. U. 1873]

10. $\dfrac{x}{3}-\dfrac{2}{y}=1,\ \dfrac{x}{4}+\dfrac{3}{y}=3.$ [A. U. 1923]

119. **Third Method : Method of Elimination :** "Multiply the equations by such numbers as will make the coefficient of one of the unknown quantities the same in the two resulting equations ; then by addition or subtraction we can form an equation containing only the other unknown quantity."

Example 1. Solve $\left.\begin{aligned} 3x-4y&=5 \\ 5x+2y&=17 \end{aligned}\right\}$

Multiplying the 2nd equation by 2, we have

$$\left.\begin{aligned} &10x+4y=34 \\ \text{and the 1st equation is}\quad &3x-4y=5 \end{aligned}\right\}$$

Hence, by addition, $13x=39$; $\therefore\ x=3$.

Substituting this value of x in the 1st equation, we have

$$4y=9-5=4 ;\qquad \therefore\ y=1.$$

Thus, we have $x=3,\ y=1$.

Example 2. Solve $\left.\begin{aligned} 5x+9y&=89 \\ 2x-17y&=15 \end{aligned}\right\}$

Multiplying the 1st equation by 2, and the 2nd by 5, we have

$$\left.\begin{aligned} &10x+18y=178 \\ \text{and}\quad &10x-85y=75 \end{aligned}\right\}$$

Hence, by subtraction, we have

$$103y=103 ;\qquad \therefore\ y=1.$$

Substituting this value of y in the 2nd equation, we have

$$2x=15+17=32 ;\qquad \therefore\ x=16.$$

Thus, we have $x=16,\ y=1$.

Note. *We might as well have multiplied the 1st equation by* 17 *and the 2nd equation by* 9 *and added the two resulting equations; this would have given us the value of x. But we have preferred the other alternative because, the coefficients of x being smaller, the required multiplications have been more easily effected.*

Example 3. Solve $\left.\begin{aligned}23x-24y&=21\\25x-16y&=43\end{aligned}\right\}$

Multiplying the 1st equation by 2, and the 2nd by 3, we have

$$\left.\begin{aligned}46x-48y&=42\\ \text{and}\quad 75x-48y&=129\end{aligned}\right\}$$

Hence, by subtraction, we have

$$29x=87; \qquad \therefore \quad x=3.$$

Substituting this value of x in the 2nd equation, we have

$$16y=75-43=32; \qquad \therefore \quad y=2.$$

Thus, we have $x=3$, $y=2$.

Note. *It may be noticed that the coefficient of y in each of the resulting equations is the* **least common multiple** *of* 24 *and* 16 *and this is all that is required. The process would have been unnecessarily tedious if the 1st equation were multiplied by* 16 *and the 2nd by* 24.

Example 4. Solve $\left.\begin{aligned}\frac{x-2}{2}-\frac{x+y}{14}&=\frac{x-y-1}{8}-\frac{y+12}{4}\\ \frac{x+7}{3}+\frac{y-5}{10}&=1-x-\frac{5(y+1)}{7}\end{aligned}\right\}$ [C. U. 1882]

From the 1st equation, we have

$$\frac{7(x-2)-(x+y)}{14}=\frac{(x-y-1)-2(y+12)}{8},$$

$$\text{or,}\quad \frac{6x-y-14}{7}=\frac{x-3y-25}{4},$$

$$\text{or,}\quad 24x-4y-56=7x-21y-175,$$

$$\text{or,}\quad 17x+17y=-119,$$

$$\text{or,}\quad x+y=-7. \qquad \cdots \qquad \cdots \quad (1)$$

From the 2nd equation, we have

$$\frac{10(x+7)+3(y-5)}{30}=\frac{7(1-x)-5(y+1)}{7},$$

$$\text{or,}\quad \frac{10x+3y+55}{30}=\frac{2-7x-5y}{7},$$

$$\text{or,}\quad 70x+21y+385=60-210x-150y,$$

$$\text{or,}\quad 280x+171y=-325. \qquad \cdots \qquad \cdots \quad (2)$$

Multiplying (1) by 171, we have

$$\left.\begin{array}{rl} & 171x+171y=-1197 \\ \text{also} & 280x+171y=-\ \ 325 \end{array}\right\}$$

Hence, by subtraction,

$$109x=872\,; \qquad \therefore \quad x=8.$$

Substituting this value of x in (1), we have

$$y=-7-8=-15.$$

Thus, we have $x=8$, $y=-15$.

Example 5. Solve $\left.\begin{array}{l} \dfrac{2}{x}+\dfrac{3}{y}=1 \\ \dfrac{7}{x}+\dfrac{4}{y}=1\frac{7}{8} \end{array}\right\}$

Multiplying the 1st equation by 4, and the 2nd by 3, we have

$$\frac{8}{x}+\frac{12}{y}=4 \text{ and } \frac{21}{x}+\frac{12}{y}=\frac{45}{8}.$$

Hence, by subtraction,

$$\frac{13}{x}=\frac{13}{8}\,; \qquad \therefore \quad x=8.$$

Substituting this value of x in the 1st equation, we have

$$\frac{3}{y}=1-\frac{1}{4}=\frac{3}{4}\,; \qquad \therefore \quad y=4.$$

Thus, we have $x=8$, $y=4$.

Alternative Method :

Supposing $\dfrac{1}{x}=u$, $\dfrac{1}{y}=v$,

we get

$$2u+3v=1 \qquad \cdots \qquad (1)$$

$$\text{and} \quad 7u+4v=\tfrac{15}{8} \qquad \cdots \qquad (2)$$

Multiplying the equation (1) by 4 and (2) by 3, we have

$$8u+12v=4$$

$$\text{and } 21u+12v=\tfrac{45}{8}.$$

Hence, by subtraction,

$$13u=\tfrac{13}{8}\,;$$

$$\therefore \quad u=\tfrac{1}{8}.$$

$$\therefore \quad \frac{1}{x}=u=\frac{1}{8}\,; \qquad \therefore \quad x=8.$$

Substituting the value of u in equation (1),

$$\tfrac{2}{8}+3v=1\,;$$

or, $3v=1-\tfrac{2}{8}\,;$

or, $3v=\tfrac{3}{4}\,;$

$\therefore\quad v=\tfrac{1}{4}.$

$\therefore\ \dfrac{1}{y}=v=\dfrac{1}{4}\,;\qquad \therefore\ y=4.$

Thus, we have $x=8,\ y=4.$

Example 6. Solve $\left.\begin{aligned}\frac{12}{x+y}+\frac{8}{x-y}&=8\\ \frac{27}{x+y}-\frac{12}{x-y}&=6\end{aligned}\right\}$ [A. U. 1927]

Supposing $u=\dfrac{1}{x+y}$ and $v=\dfrac{1}{x-y}$,

we get, $12u+\ 8v=8 \quad \cdots \quad (1)$

$27u-12v=6 \quad \cdots \quad (2)$

Multiplying the equation (1) by 3 and (2) by 2, we have

$36u+24v=24$

and $54u-24v=12.$

Hence by addition,

$$90u=36\,;$$

$\therefore\quad u=\tfrac{36}{90}=\tfrac{2}{5}\,;$

$\therefore\quad \dfrac{1}{x+y}=\dfrac{2}{5}\,;$

$\therefore\quad x+y=\tfrac{5}{2} \quad \cdots \quad (3)$

Substituting the value of u in the equation (1), we have

$$12.\tfrac{2}{5}+8v=8\,;$$

or, $8v=8-\tfrac{24}{5}=\tfrac{16}{5}\,;$

or, $v=\tfrac{2}{5}\,;$

$\therefore\ \dfrac{1}{x-y}=\dfrac{2}{5}\,;$

$\therefore\ x-y=\tfrac{5}{2} \quad \cdots \quad (4)$

Hence, by adding (3) and (4), we have

$$2x=5\,;$$

$\therefore\quad x=\tfrac{5}{2}.$

Substituting the value of x in the equation (3), we have

$$\frac{5}{8}+y=\frac{5}{8};$$

or, $y=0.$

$\therefore$ $x=\frac{5}{8},\ y=0.$

EXERCISE 65

Solve the following equations :

1. $\left.\begin{aligned}7x-5y&=11\\3x+2y&=13\end{aligned}\right\}$ 2. $\left.\begin{aligned}13x+6y&=58\\5x-11y&=9\end{aligned}\right\}$ 3. $\left.\begin{aligned}8x-9y&=20\\7x-10y&=9\end{aligned}\right\}$

4. $\left.\begin{aligned}25x-14y&=8\\12x+7y&=45\end{aligned}\right\}$ 5. $\left.\begin{aligned}12x+11y&=70\\8x-7y&=18\end{aligned}\right\}$ 6. $\left.\begin{aligned}13x-14y&=22\\17x-21y&=18\end{aligned}\right\}$

7. $\left.\begin{aligned}28x-15y&=41\\21x+13y&=55\end{aligned}\right\}$ 8. $\left.\begin{aligned}19x+24y&=34\\23x+36y&=62\end{aligned}\right\}$ 9. $\left.\begin{aligned}47x-56y&=123\\25x+84y&=293\end{aligned}\right\}$

10. $\left.\begin{aligned}51x-16y&=3\\68x+23y&=137\end{aligned}\right\}$ 11. $\left.\begin{aligned}52x-9y&=34\\39x+14y&=67\end{aligned}\right\}$ 12. $\left.\begin{aligned}12x+85y&=-49\\19x-34y&=91\end{aligned}\right\}$

13. $\left.\begin{aligned}65x-14y&=9\\91x-15y&=31\end{aligned}\right\}$ 14. $\left.\begin{aligned}15x+46y&=17\\13x+69y&=73\end{aligned}\right\}$ 15. $\left.\begin{aligned}14x+81y&=53\\17x+135y&=101\end{aligned}\right\}$

16. $\left.\begin{aligned}5x+11y&=146\\11x+5y&=110\end{aligned}\right\}$ 17. $\left.\begin{aligned}ax+by&=c\\a^2x+b^2y&=c^2\end{aligned}\right\}$ [C. U. 1881]

18. $\left.\begin{aligned}\frac{x+y}{2}+\frac{3x-5y}{4}&=2\\\frac{x}{14}+\frac{y}{18}&=1\end{aligned}\right\}$ [C. U. 1876]

19. $\left.\begin{aligned}\frac{4x+5y}{40}&=x-y\\\frac{2x-y}{3}+2y&=\frac{1}{2}\end{aligned}\right\}$ 20. $\left.\begin{aligned}\frac{4x-3y-7}{5}&=\frac{3x}{10}-\frac{2y}{15}-\frac{5}{6}\\\frac{y-1}{3}+\frac{x}{2}-\frac{3y}{20}&=\frac{y-x}{15}+\frac{x}{6}+\frac{11}{10}\end{aligned}\right\}$

21. $\left.\begin{aligned}\frac{5x-3y}{12}+\frac{7x-5y}{15}&=1-\frac{25x+3y}{60}\\\frac{(3\frac{1}{2})x+2y-5}{16}+\frac{11x-(4\frac{1}{3})y+17}{11}&=\frac{19}{22}+\frac{17x-10y+2}{3}\end{aligned}\right\}$

22. $\left.\begin{aligned}\frac{3x-5y}{3}-\frac{2x-8y-33}{12}&=\frac{y}{2}+\frac{x}{3}+\frac{1}{4}\\3\frac{1}{2}\left(\frac{x}{7}+\frac{y}{4}+1\frac{1}{3}\right)&=3\frac{1}{3}\left(4x-\frac{y}{8}-24\right)\end{aligned}\right\}$

23. $\left.\begin{aligned}2{\cdot}4x+{\cdot}32y-\frac{{\cdot}18x-{\cdot}025}{{\cdot}25}&={\cdot}8x+\frac{5{\cdot}2+{\cdot}01y}{{\cdot}5}\\\frac{{\cdot}2y+{\cdot}5}{1{\cdot}5}&=\frac{{\cdot}49x-{\cdot}7}{4{\cdot}2}\end{aligned}\right\}$

24. $\left.\begin{array}{l}\dfrac{4}{x}+\dfrac{10}{y}=2\\[2ex]\dfrac{3}{x}+\dfrac{2}{y}=\dfrac{19}{20}\end{array}\right\}$ [C. U. 1879]

25. $\left.\begin{array}{l}\dfrac{2}{x}+\dfrac{3}{y}=2\\[2ex]\dfrac{5}{x}+\dfrac{10}{y}=5\frac{5}{6}\end{array}\right\}$ [C. U. 1887]

26. $\left.\begin{array}{l}\dfrac{a}{x}+\dfrac{b}{y}=m\\[2ex]\dfrac{b}{x}+\dfrac{a}{y}=n\end{array}\right\}$

27. $\left.\begin{array}{l}\dfrac{1}{3x}+\dfrac{1}{5y}=1\\[2ex]\dfrac{1}{5x}+\dfrac{1}{3y}=1\frac{2}{15}\end{array}\right\}$

28. $\left.\begin{array}{l}\dfrac{3}{y}-\dfrac{1}{x}=1\\[2ex]\dfrac{2}{5x}+\dfrac{5}{2y}=7\end{array}\right\}$

29. $\left.\begin{array}{l}\dfrac{x}{4}+\dfrac{2}{y}=2\\[2ex]\dfrac{2x}{5}+\dfrac{3}{2y}=2\frac{7}{20}\end{array}\right\}$

30. $\left.\begin{array}{l}\dfrac{1}{5x}+\dfrac{y}{9}=5\\[2ex]\dfrac{1}{3x}+\dfrac{y}{2}=14\end{array}\right\}$

[C. U. 1870]

31. $\left.\begin{array}{l}\dfrac{14}{x+y}+\dfrac{3}{x-y}=5\\[2ex]\dfrac{21}{x+y}-\dfrac{1}{x-y}=2\end{array}\right\}$

32. $\left.\begin{array}{l}\dfrac{7}{4x+3y}+\dfrac{4}{4x-3y}=\dfrac{5}{4}\\[2ex]\dfrac{8}{4x-3y}-\dfrac{14}{4x+3y}=\dfrac{3}{2}\end{array}\right\}$

II. Problems leading to simple equations with more than one unknown quantity

120. Easy Problems.

Example 1. The present age of the father is double of that of the son. 16 years ago the father's age was thrice that of the son. Find their present ages.

Let x = the present age of the father,

and y = the present age of the son.

By the given condition of the problem,

$$x=2y \quad \cdots \quad (1)$$

$$\text{and} \quad (x-16)=3(y-16) \quad \cdots \quad (2)$$

Substituting $2y$ for x in equation (2),

$$2y-16=3(y-16),$$

$$\text{or,} \quad 2y-16=3y-48,$$

$$\text{or,} \quad 2y-3y=16-48,$$

$$\text{or,} \quad -y=-32.$$

$$\therefore \quad y=32.$$

Therefore, the son's age is 32 years and that of the father is
$(32 \times 2 =)$ 64 years.

Example 2. *A* and *B* each had a number of mangoes. *A* said to *B*, "If you give me 30 of your mangoes, my number will be *twice* yours." *B* replied, "If you give me 10, my number will be *thrice* yours." How many had each ?

Let $x =$ the number of mangoes *A* had,

and $y =$ " " " " *B* ".

Then, in accordance with what *A* said, we must have the equation

$$x + 30 = 2(y - 30) ; \qquad \cdots \quad (1)$$

and in accordance with *B*'s reply, we must have the equation

$$y + 10 = 3(x - 10). \qquad \cdots \quad (2)$$

From (2), $3x - y = 40$, or, $6x - 2y = 80$; $\cdots$ (3)

and from (1), $x - 2y = -90$. $\cdots$ $\cdots$ (4)

Hence, by subtraction, $5x = 170$; $\therefore$ $x = 34$.

Substituting this value of x in (4), we have

$$2y = 34 + 90 = 124 ; \qquad \therefore \; y = 62.$$

Thus, *A* had 34 mangoes and *B* had 62.

Example 3. A certain fraction becomes 2 when 7 is added to its numerator, and 1 when 2 is subtracted from the denominator. What is the fraction ?

Let $\dfrac{x}{y}$ represent the fraction.

Then, we have $\dfrac{x + 7}{y} = 2$; $\cdots$ $\cdots$ (1)

and $\dfrac{x}{y - 2} = 1$. $\cdots$ $\cdots$ (2)

$$\left.\begin{array}{ll} \text{From (1),} & x + 7 = 2y ; \;\; \therefore \; x = 2y - 7 \\ \text{From (2),} & \qquad\qquad\qquad\quad\; x = y - 2 \end{array}\right\}$$

Therefore, $2y - 7 = y - 2$, whence $y = 5$.

Hence, $x = 5 - 2 = 3$.

Thus, the fraction is $\frac{3}{5}$.

Example 4. 2 men and 7 boys can do in 4 days a piece of work which would be done in 3 days by 4 men and 4 boys. How long would it take one man or one boy to do it ?

Let $x =$ the number of days in which one man would do the work,

and $y =$ the number of days in which one boy would do it.

Then, in one day a man does $\frac{1}{x}$th of the work and a boy does $\frac{1}{y}$th of it.

Hence, since 2 men and 7 boys do $\frac{1}{4}$th of the work in one day, we must have

$$\frac{2}{x}+\frac{7}{y}=\frac{1}{4}. \qquad \cdots \qquad \cdots \quad (1)$$

Again, since 4 men and 4 boys do $\frac{1}{3}$rd of the work in one day, we must have

$$\frac{4}{x}+\frac{4}{y}=\frac{1}{3}. \qquad \cdots \qquad \cdots \quad (2)$$

Multiplying (1) by 2, and subtracting (2) from the resulting equation, we must have

$$\frac{10}{y}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}; \qquad \therefore \ y=60.$$

Hence, from (2), $\frac{4}{x}=\frac{1}{3}-\frac{1}{15}=\frac{4}{15}$; $\quad \therefore \ x=15.$

Thus, one man would do the work in 15 days and one boy in 60 days.

Example 5. Cost of 3 doors and 5 windows is Rs. 487 and that of 5 doors and 3 windows is Rs. 561. Find the value of a door and a window.

Let $x=$ the cost of a door and $y=$ the cost of a window.

By the given conditions of the problem,

$$3x+5y=487 \qquad \cdots \qquad \cdots \quad (1)$$

$$\text{and} \quad 5x+3y=561 \qquad \cdots \qquad \cdots \quad (2)$$

Multiplying (1) by 3 and (2) by 5, we have

$$9x+15y=1461 \qquad \cdots \qquad \cdots \quad (3)$$

$$25x+15y=2805 \qquad \cdots \qquad \cdots \quad (4)$$

Subtracting (3) from (4),

$$16x=1344;$$

$$\therefore \quad x=84.$$

Substituting the value of x in (1), we have

$$y=47.$$

Thus, the value of a door is Rs. 84 and that of a window Rs. 47.

Example 6. Two plugs are opened in the bottom of a cistern containing 192 litres of water; after 3 hours one of them becomes stopped, and the cistern is emptied by the other in 11 hours; had

6 hours elapsed before the stoppage, it would have only required 6 hours more to empty it. How many litres will each plug-hole discharge in one hour, supposing the discharge to be uniform ?

Let x, y be the numbers of litres of water which the plugs can respectively discharge in an hour.

In the first case, the first plug remains opened for 3 hours, and the second for 3 + 11 or 14 hours.

Hence, $3x+14y=192$. ... ... (1)

In the second case, the first plug remains opened for 6 hours, and the second for 6 + 6 or 12 hours.

Hence, $6x+12y=192$. ... ... (2)

Multiplying (1) by 2 and subtracting (2) from the resulting equation, we have

$$16y=2\times192-192$$
$$=192; \qquad \therefore\ y=12.$$

Hence, from (2), $6x=192-144=48$; $\therefore\ x=8$.

Thus, the plug-holes respectively discharge 8 and 12 litres in an hour.

Example 7. The dimension of a rectangular court is such that if the length were increased by 3 metres, and the breadth diminished by the same, its area would be diminished by 18 square metres ; and if its length were increased by 3 metres, and its breadth increased by the same, its area would be increased by 60 square metres, find the dimensions.

Let x metres = length of the court,

and y metres = its breadth.

Then, from the first condition of the problem, we have

$$(x+3)(y-3)=xy-18; \qquad \cdots (1)$$

and from the second condition,

$$(x+3)(y+3)=xy+60. \qquad \cdots (2)$$

From (1), $3y-3x=-9$, or, $y-x=-3$. ... (3)

From (2), $3y+3x=51$, or, $y+x=17$. ... (4)

From (3) and (4), by addition,

$$2y=14; \qquad \therefore\ y=7;$$

and by subtraction, $2x=20$; $\therefore\ x=10$.

Thus, the length of the court is 10 metres, and the breadth is 7 metres.

Example 8. There is a certain number consisting of two digits, to the sum of whose digits if you add 7, the result will be three times the left-hand digit ; and if from the number itself you subtract 18, the digits will be inverted. Find the number.

Let x and y be the left and right-hand digits respectively; then the required number is represented by $10x+y$, and the number with inverted digits $=10y+x$.

Hence, by the conditions of the problem,

$$x+y+7=3x, \qquad \cdots (1)$$

and $$(10x+y)-18=10y+x. \qquad \cdots (2)$$

From (1), $$2x-y=7; \qquad \cdots (3)$$

and from (2), $$9x-9y=18, \quad \text{or, } x-y=2. \qquad \cdots (4)$$

Subtracting (4) from (3), we have

$$x=7-2=5.$$

Hence, from (4), $y=5-2=3$.

Thus, the required number is 53.

Example 9. *A* and *B* play at bowls, and *A* bets *B* three shillings to two upon every game; after a certain number of games it appears that *A* has won three shillings; but if *A* had bet five shillings to two and lost one game more out of the same number, he would have lost thirty shillings. How many games did each win?

Let x = number of games that *A* won,

and y = " " " " *B* ".

Then, the total number of games played is evidently $x+y$.

Now, since *A* receives from *B*, 2*s*. for every game that he wins and gives *B*, 3*s*. for every game that he loses (*i.e.*, for every game that *B* wins), his total gain must be equal to $(2x-3y)$ shillings.

Hence, $$2x-3y=3. \qquad \cdots \qquad \cdots (1)$$

According to the other condition, *A* would have gained $2(x-1)$ shillings and lost $5(y+1)$ shillings; and therefore, his total loss would have been $[5(y+1)-2(x-1)]$ shillings.

Hence, $5(y+1)-2(x-1)=30$,

or, $$5y-2x=23. \qquad \cdots \qquad \cdots (2)$$

From (1) and (2), by addition, $2y=26$; $\therefore$ $y=13$.

Hence, from (1), $x=\dfrac{3+39}{2}=21$.

Thus, *A* won 21 games and *B* won 13 games.

EXERCISE 66

1. What fraction is that whose numerator being doubled and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

2. Find two numbers such that if the first be added to 5 times the second, the sum is 52 ; and if the second be added to 8 times the first, the sum is 65.

3. Find two numbers such that five times the greater exceeds four times the less by 22, and three times the greater together with seven times the less is 32.

4. What numbers are those whose difference is 45, and the quotient of the greater by the less is 4 ?

5. The age of the father exceeds twice that of his son by 10 years. Twenty years ago, the age of the father was five times that of his son. Find their present ages.

6. Ten years ago the age of the father was seven times that of his son. Two years hence twice the age of the father will be equal to five times that of his son. Find their present ages.

7. There are two numbers such that one-fourth of the greater added to one-third of the less is 11 ; and if one-fifth of the less be taken from one-eighth of the greater, the remainder is nothing ; find the numbers.

8. A certain fraction becomes $\frac{1}{3}$ when 1 is subtracted from its denominator, and 1 when 7 is added to its numerator. What is the fraction ?

9. What fraction is that which, if 1 be added to the numerator, becomes 1, and if 1 be added to the denominator, becomes $\frac{1}{2}$? [C. U. 1862]

10. A certain fraction becomes $\frac{1}{2}$ when its numerator is increased by unity, and $\frac{1}{3}$ when its denominator is increased by unity. What is the fraction ?

11. The denominator of a fraction exceeds the numerator by 4 and if 5 be taken from each, the sum of the reciprocal of the new fraction and 4 times the original fraction is 5. Find the original fraction.

[Solving the problem, we will find the fraction to be $\frac{8}{12}$. Students should note that the fraction should not be reduced to its lowest terms as is generally done. If reduced to lowest terms, it will not satisfy the given conditions.]

12. *A* and *B* have 39 rupees between them, but if *A* were to lose two-thirds of his money, and *B* three-fourths of his, they would then have only 11 rupees. How much has each ?

13. Two numbers are such that if 7 be added to the less, the sum is twice the greater, and if 4 be added to the greater, the sum is 3 times the less. Find the numbers.

14. Two persons, 27 kilometres apart, setting out at the same time, meet together in 9 hours, if they walk in the same direction, but in 3 hours if they walk in opposite directions ; find their rates of walking.

15. A banker was asked to pay Rs. 50 in 50 paise and 25 paise so that the number of the latter should be exactly twice that of the former. How must he do it ?

16. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys. How long would it take one man to do it ?

17. A rectangle is of the same area as another which is 6 metres longer and 4 metres narrower ; it is also of the same area as a third, which is 8 metres longer and 5 metres narrower. What is its area ?

18. If 15 kgs. of tea and 17 kgs. of coffee together cost Rs. 183 and 25 kgs. of tea and 13 kgs. of coffee together cost Rs. 213, find the price of each per kilogram.

19. *A* takes 3 hours longer than *B* to walk 30 kilometres ; but if he doubles his pace he takes 2 hours less time than *B* ; find their rates of walking.

20. Says Charles to William, "If you give me 10 of your marbles, I shall then have just *twice* as many as you" ; but says William to Charles, "If you give me 10 of yours, I shall then have *three times* as many as you." How many had each ?

21. If a certain number be divided by the sum of its two digits the quotient is 6 and the remainder is 3. If the digits be inverted and the resulting number be divided by the sum of the digits, the quotient is 4 and the remainder 9. Find the number.

22. Find that number of 2 figures to which, if the number formed by changing the places of the digits, be added, the sum is 121 ; and if the smaller number be subtracted from the larger, the remainder is 9.

23. A bill of 25 guineas was paid with crowns and half-guineas ; and twice the number of half-guineas exceeded 3 times that of the crowns by 17. How many were there of each ?

24. A person sells to one person 9 horses and 7 cows for Rs. 3000 ; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each ?

25. *A* and *B* received £5. 17*s*. for their wages, *A* having been employed for 15 and *B* for 14 days ; and *A* received, for working 4 days, 11*s*. more than *B* did for three days. What were their daily wages ?

26. *A* and *B* can do a piece of work in 16 days ; they work together for 4 days, when *A* leaves, and *B* finishes it in 36 days more. In what time would each do the work separately ?

27. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$; and if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{1}{2}$. Find the fraction.

28. A traveller walks a certain distance ; had he gone half a kilometre an hour faster, he would have walked it in four-fifths of the time ; had he gone half a kilometre an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance.

29. A certain number between 10 and 100 is eight times the sum of its digits, and if 45 be subtracted from it, the digits will be reversed; find the number.

30. *A* and *B* lay a wager of 10*s*. If *A* loses, he will have twenty-five shilling less than twice as much as *B* will then have; but if *B* loses, he will have five-seventeenths of what *A* will then have; find how much money each of them has.

31. A farmer wishing to purchase a number of sheep found that if they cost him Rs. 42 a head, he would not have money enough by Rs. 28; but if they cost him Rs. 40 a head, he would then have Rs. 40 more than he required; find the number of sheep, and the money which he had.

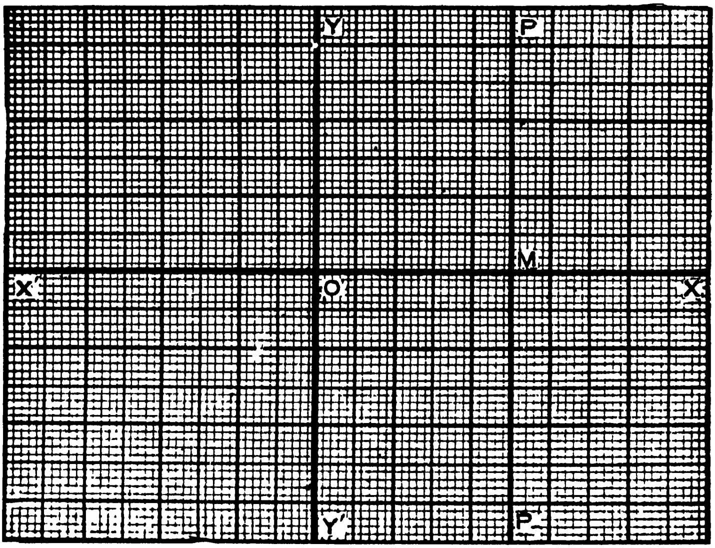
32. There is a number consisting of two digits; the number is equal to three times the sum of its digits, and if it be multiplied by 3, the result will be equal to the square of the sum of its digits. Find the number.

CHAPTER XIX

GRAPHS OF SIMPLE EQUATIONS

121. In Chapter VII, we have discussed representations of numbers by geometric points. We now propose to show how simple equations are represented graphically. The following examples will make the subject clear.

Example 1. If a point moves in such a manner that its abscissa is always equal to 5 units of length, find the path along which the point will move.



Let five times the side of a small square represent the unit of length.

On OX take the point M such that $OM=5$ units of length; through M draw the straight line PMP' parallel to YOY'.

Now, if any point be taken on the straight line PMP', its x will evidently be equal to 5 units of length; but this will *not* be so if the point be taken on either side of the line PMP'.

Hence, the moving point will always be on the line PMP'.

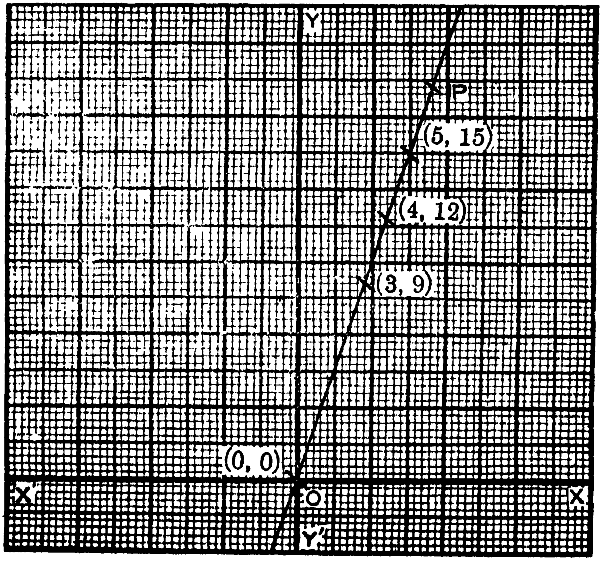
We see, therefore, that if a point moves in such a manner that its x is always equal to 5 units of length, the path along which the point will move is the straight line PMP'. This fact is briefly expressed by saying that the straight line PMP′ is the graph of the equation $x=5$.

Note 1. *From the above it is clear that the graph of the equation* $y=5$ *is a straight line parallel to* XOX'.

Note 2. *Generally speaking, the graph of the equation* $x=a$ *is a straight line parallel to the axis of* y, *and passing through a point on the axis of* x *which is at a distance of* a *units of length from the origin; and the graph of the equation* $y=b$ *is a straight line parallel to the axis of* x, *and passing through a point on the axis of* y, *which is at a distance of* b *units of length from the origin.*

Note 3. *Evidently, therefore, the graph of the equation* $x=0$ *is the axis of* y *itself, and the graph of the equation* $y=0$ *is the axis of* x *itself.*

Example 2. If a point moves in such a manner that its x and y are always connected by the relation $y=3x$, find the path along which the point will move, *i.e.*, draw the graph of the equation $y=3x$.



Giving different values of x in the given equation, we get different values of y. They may be tabulated as follows :

x	0	3	4	5
y	0	9	12	15

Take three times the length of a side of a small square as the unit of length and plot the points tabulated above.

Join the points (0, 0), (3, 9), (4, 12) and (5, 15) and produce the straight line both ways. Then this straight line will be the required path.

Take any point P on this straight line. The co-ordinates of P are found to be 6 and 18, which evidently satisfy the given relation. Similarly, the co-ordinates of any other point on this straight line may be shown to satisfy the given relation. But the co-ordinates of a point which is outside the line OP will *not* satisfy the given relation, as can be easily verified.

Hence, the moving point will always be on the line OP and never stray out of it.

Thus, it is found that if a point moves in such a way that its x and y are invariably connected by the relation $y=3x$, the path along which the point will move is the straight line OP. In other words, **the line OP is the graph of the equation $y=3x$.**

Note 1. *Generally speaking, the graph of the equation $y=mx$, where m is any given number, is a straight line passing through the origin.*

Note 2. *It should be observed that the greater the number of points plotted and closer their positions to each other, the more accurately the graph will be drawn. No graph should be drawn without plotting at least three points.*

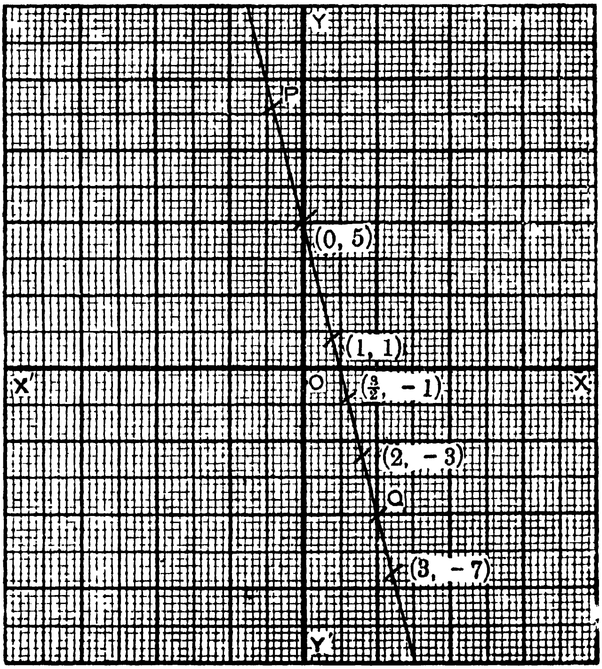
Example 3. If a point moves in such a way that its x and y are invariably connected by the relation $y=-4x+5$, find the path along which the point will move, *i.e.*, draw the graph of the equation $y=-4x+5$.

The corresponding values of x and y in the equation $y=-4x+5$ may be tabulated as follows :

x	0	1	$\frac{3}{2}$	2	3
y	5	1	-1	-3	-7

The points indicated by the tabulated values of x and y are some of the different positions of the moving point.

Let four times the side of a small square represent the unit of length. Plot the points and join them. Produce the straight line both ways. Then this straight line will be the required path.



Take a point P on this straight line. The co-ordinates of P, which are found to be -1 and 9, satisfy the given relation. Take another point Q on the straight line; its co-ordinates which are found to be $\frac{5}{2}$ and -5, also satisfy the given relation. Similarly, the co-ordinates of any other point on this straight line may be shown to satisfy the given relation. But if a point be taken outside the line PQ, its co-ordinates will *not* satisfy the given relation, as can be easily seen. Hence, the moving point will always be on the line PQ and never stray out of it.

Thus, it is found that if a point moves in such a manner that its co-ordinates always satisfy the equation $y = -4x + 5$, the path along which the point will move is the straight line PQ. In other words, **the straight line PQ is the graph of the equation $y = -4x + 5$.**

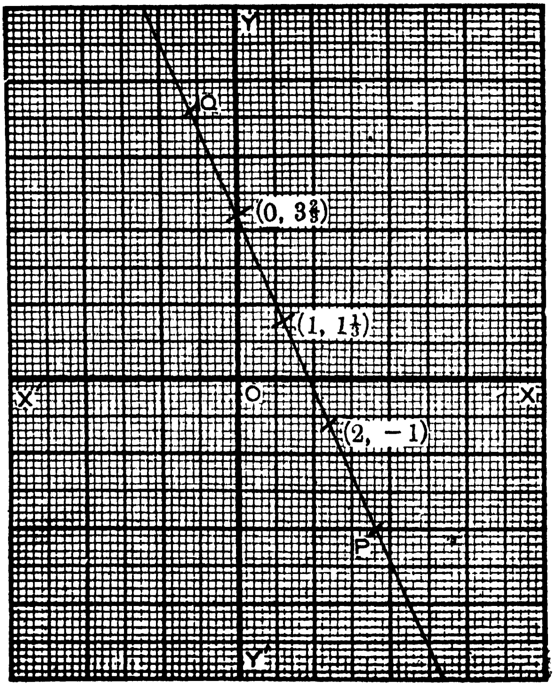
Note 1. *Generally speaking, the graph of the equation $y = mx + c$, where m and c are any given numbers, is a straight line passing through the point $(0, c)$.*

Note 2. *As every equation of the first degree in x and y can be reduced to the form $y = mx + c$, it is clear that* ***graphs of all simple equations are straight lines.*** *Suppose, $ax + by + c = 0$ is a simple equation with two unknown quantities. By transposition, $by = -ax - c$. Dividing both sides by b, we get $y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$.*

Note 3. *The graph of the equation $y = mx + c$ is also said to be* ***the graph of the expression $mx + c$.***

Note 4. *The graph of any given equation may be defined to be the path described by a point which moves in such a manner that in every position of the point its co-ordinates satisfy the given equation.*

Example 4. Draw the graph of the equation $7x + 3y = 11$.



The corresponding values of x and y in the equation $7x+3y=11$, may be tabulated as follows :

x	0	1	2
y	$3\frac{2}{3}$	$1\frac{1}{3}$	-1

Evidently, therefore, $(0, 3\frac{2}{3})$, $(1, 1\frac{1}{3})$ and $(2, -1)$ are points on the graph.

Let 6 times the side of a small square represent the unit of length. Join the points $(0, 3\frac{2}{3})$, $(1, 1\frac{1}{3})$ and $(2, -1)$ and produce the straight line both ways. Then this straight line will be the required graph. [See the diagram of page 216.]

Take any point P on the line ; its co-ordinates, which are found to be 3 and $-3\frac{1}{3}$, satisfy the given relation. Take any other point Q on the line ; its co-ordinates, which are found to be -1 and 6, also satisfy the given relation. Similarly, it may be shown that the co-ordinates of any point that may be taken on the line PQ will satisfy the given relation ; but the co-ordinates of any point which is outside PQ will *not*. Hence, the line PQ is the required graph.

Note 1. *The equation* $7x+3y=11$ *may be written as* $y=\frac{11-7x}{3}$ *after transposiion and division of both sides by the coefficient of* y. *The graph of the equation* $x+3y=11$ *is also said to be* ***the graph of the expression*** $\frac{11-7x}{3}$.

Note 2. *The straight line PQ being the graph of the equation* $7x+3y=11$, ***his equation is said to be the equation of the straight line PQ.***

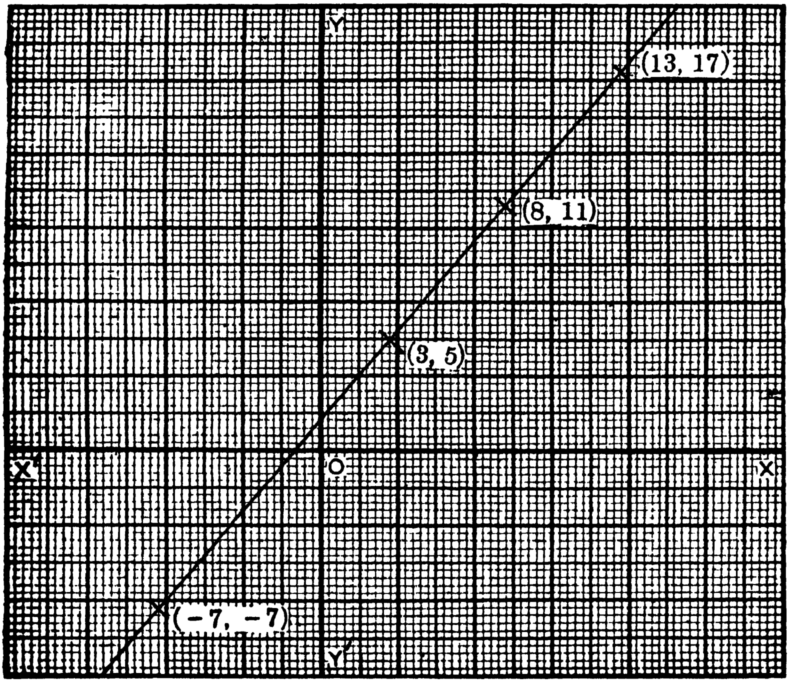
Note 3. ***The equation of a given straight line means the equation which s satisfied by the co-ordinates of every point on that straight line.***

Example 5. Draw the graph of $\frac{6x+7}{5}$.

From Note 1 of Example 4, the given quantity is equal to another variable y. Find some corresponding values of x and y in the equation $y=\frac{6x+7}{5}$ and plot them. The straight line formed by joining them will be the graph of the given quantity.

x	-7	3	8	13
y	-7	5	11	17

Let 3 times the side of a small square represent the unit of length. Join the points $(-7, -7)$, $(3, 5)$, $(8, 11)$ and $(13, 17)$, and produce the straight line both ways. This straight line is the graph of the given quantity.



Example 6. Find the equation of the straight line which passes through the points $(1, 1)$ and $(3, -\frac{1}{2})$.

Let $y = mx + c$ be the required equation.

This equation being satisfied by $(1, 1)$ and also by $(3, -\frac{1}{2})$, we must have

$$\left.\begin{aligned} 1 &= m + c \\ \text{and } -\tfrac{1}{2} &= 3m + c \end{aligned}\right\} \qquad \text{Hence, } 2m = -\tfrac{3}{2}, \text{ and } \therefore m = -\tfrac{3}{4}; \text{ whence } c = 1 + \tfrac{3}{4} = \tfrac{7}{4}.$$

Thus, the required equation is $y = -\frac{3}{4}x + \frac{7}{4}$; or, $3x + 4y = 7$.

Example 7. Draw the graph of the equation $\frac{x}{3}+\frac{y}{4}=1$ and find the length of its portion intercepted between the two axes.

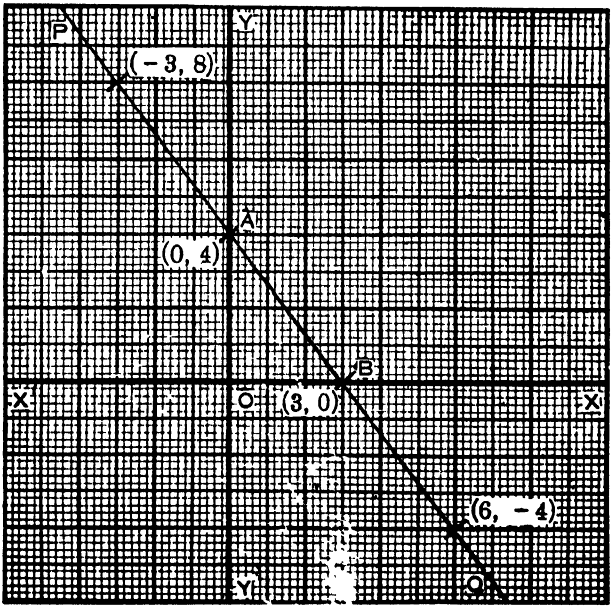
$$\frac{x}{3}+\frac{y}{4}=1,$$

or, $4x+3y=12$. [Multiplying both sides by 12]

From the equation $4x+3y=12$, we get

x	3	0	6	−3
y	0	4	−4	8

Taking 5 times the side of a small square as unit of length the graph *PQ* is drawn



PQ cuts the axes at *A* and *B*.

(i) By measurement with a scale *AB* is found to be equal to 5 units of length.

(ii) AOB is a right-angled triangle, of which $AO=4$, $OB=3$ and $\angle AOB$ is a right angle.

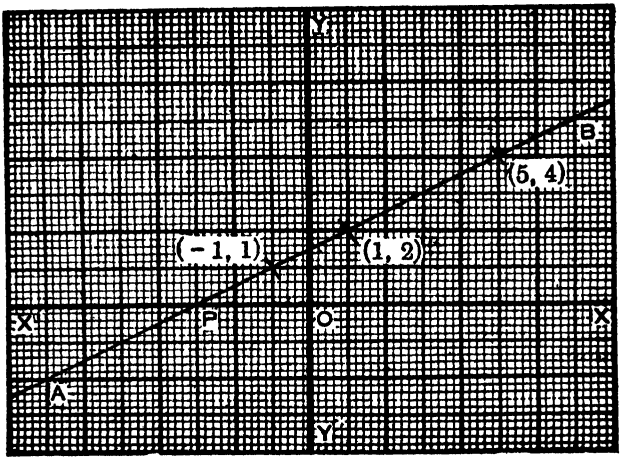
$$\therefore\quad AB=\sqrt{AO^2+OB^2}=\sqrt{4^2+3^2}=5.$$

Example 8. Draw the graph of $\frac{x+3}{2}$. Find the value of the function from the graph when $x=3$. [D. B. 1934]

Find the value of x when the value of the given function is 0.

The graph of $\frac{x+3}{2}$ is that of the equation $y=\frac{x+3}{2}$. From the equation $y=\frac{x+3}{2}$, we get

x	1	5	-1
y	2	4	1



Taking 5 times the side of a small square as unit of length the straight line AB is drawn. [See the figure above.]

From the figure $y=3$, when $x=3$.

The straight line AB cuts the XOX' axis at P. The ordinate of P is 0. Therefore, we are to find out that abscissa of P for which the value of the function $=0$. Counting from O, $OP=-3$. $\therefore\ x=-3$.

EXERCISE 67

1. Draw the graphs of the following equations :

(1) $x=8$. (2) $x=13$. (3) $x+11=0$.
(4) $y=-7$. (5) $y-9=0$. (6) $y+10=0$.

2. Draw the graphs of the following equations :

(1) $y=x$. (2) $y=-x$. (3) $y=2x$.
(4) $y+2x=0$. (5) $y=-3x$. (6) $3y=5x$.
(7) $7y+8x=0$. (8) $6y+13x=0$.

3. Draw the graphs of the following equations :

(1) $y=3x+4$. (2) $y=7x-8$. (3) $y=-5x+9$.
(4) $y=-8x-11$. (5) $3y=7x+4$. (6) $-6y=7x-10$.

4. Draw the graphs of the following equations :

(1) $2x+7y=10$. (2) $4x-5y-7=0$.
(3) $5x+6y+8=0$. (4) $-3x+7y+8=0$.
(5) $10y-9x=13$. (6) $8x-11y+13=0$.

5. Draw the graphs of the following equations :

(1) $\frac{x}{5}+\frac{y}{4}=1$. (2) $\frac{x}{7}+\frac{y}{-9}=1$. (3) $\frac{x}{-8}+\frac{y}{13}=1$.

(4) $y=\frac{5-7x}{6}$. (5) $y=\frac{9x-13}{4}$. (6) $\frac{3x}{4}-\frac{4y}{3}=1$.

6. Draw the graphs of the following expressions :

(1) $x-3$. (2) $3x+4$. (3) $-7x+8$.

(4) $\frac{7-4x}{3}$. (5) $\frac{5x-9}{4}$. (6) $\frac{8x+11}{5}$.

7. Find the equation of the straight line which passes through each of the following pairs of points :

(1) $(0, 0), (5, 6)$. (2) $(0, 5), (7, 0)$. (3) $(6, -8), (-7, 5)$.
(4) $(-4, 8), (-9, -13)$. (5) $(-11, 0), (7, -10)$.

8. Draw the graph of the equation $3x-2y-4=0$. Find, from the graph, the value of y when $x=2$. [D. B. 1936]

9. Draw the graph of the equation $3x+4y-12=0$ and find the length of the graph intercepted by the axes.

CHAPTER XX

EASY QUADRATIC EQUATIONS AND PROBLEMS

122. Definition. Any equation which contains the square of the unknown quantity, but no higher power, is called a **quadratic equation or an equation of the second degree.**

If an equation contains *only* the second power of the unknown quantity (and *not* the *first*), it is called a **pure** quadratic ; if it contains the second *as well as* the first power, it is called an **adfected** quadratic.

Thus, $3x^2 = 75$ is a pure quadratic ;

and $3x^2 - 7x = 6$ is an adfected quadratic.

123. Solution of a Pure Quadratic. In solving a Pure Quadratic we have to find the *square of the unknown quantity* just in the same way as simple equations are solved and then to extract the square root of the value so found.

Example 1. Solve $5(x^2+1)-2=3(x^2+7)$.

We have $5x^2+3=3x^2+21$;

hence, $2x^2=18$; [by transposition]

$\therefore$ $x^2=9$;

now, since the unknown quantity is one of which the square is 9, it must be *either* $+3$ *or* -3. (Thus there are *two* values of x satisfying the given equation. as the student can easily verify.)

Note. *The student should carefully observe that the last step of the above solution amounts to answering the following question : 'What quantity is that of which the square is 9 ?'*

Example 2. Solve $\frac{1}{3}(x-2)(x-3)-\frac{1}{21}(x-21)(x-14)=2$.

Multiplying both sides by 21, we have

$$7(x-2)(x-3)-(x-21)(x-14)=42.$$

The left side $=(7x^2-35x+42)-(x^2-35x+294)$

$$=7x^2-35x+42-x^2+35x-294$$

$$=6x^2-252.$$

Hence, the equation reduces to

$$6x^2-252=42,$$

or, $6x^2=252+42$, [by transposition]

i.e., $=294$.

Dividing both sides by 6, we have

$$x^2=49.$$

Now, the unknown quantity is such that its square is 49 ;

$\therefore$ it must be *either* $+7$ *or* -7.

Hence, $x=$ *either* $+7$ *or* -7.

Example 3. Find the side of a square whose area is equal to that of a rectangle of length 9 metres and breadth 4 metres.

Let the side of the square $=x$ metres.

$\therefore$ the area of the square $=x\times x$ sq. metres

$=x^2$ sq. metres.

Again, the area of the rectangle $=4\times 9$ sq. metres

$=36$ sq. metres.

Hence, by the condition of the problem,

x^2 sq. metres $=36$ sq. metres,

or, $x^2=36$; $\therefore$ $x=6$, or, -6.

Since, the actual length of the side of a square is a positive quantity, the solution $x=-6$ is inadmissible.

$\therefore$ the required side $=6$ metres.

N. B. In problems leading to quadratic equations, the solutions which are found inadmissible by the condition of the problem should be rejected.

EXERCISE 68

Find the values of x in each of the following equations :

1. $3x^2=27.$
2. $a^2x^2=a^4.$
3. $\frac{1}{7}x^2=28.$
4. $8x+\frac{7}{x}=\frac{65}{7}x.$
5. $2(x^2-5)+x(3-x)=3(x+5).$
6. $(x-7)(x-10)+(x-3)(x-2)=(x-17)(x-5).$
7. $\frac{2x^2+10}{15}=7-\frac{50+x^2}{25}.$
8. $(x+a)^2-2a(a+x)=3a^2.$
9. $x^2+2bx-b^2=a^2-b(b-2x).$
10. $2x(3x+5)-5x(x+2)=36.$
11. $\frac{3x^2+15}{7}+\frac{2x^2+9}{3}=\frac{2x^2+87}{21}+2.$

12. Find the number four times which is equal to sixteen times its reciprocal.

13. Find the side of a square three times the area of which is equal to four times the area of a rectangle whose length and breadth are respectively 9 metres and 3 metres.

14. A has got a square plot of land which he exchanges with a rectangular garden of area 91 sq. metres, belonging to B and gains by the transaction an area of 10 sq. metres. Find a side of the square plot.

15. Divide a straight line of length 10 cm. into two portions such that five times the square on one exceeds the square on the other by twenty times the former portion.

124. Solution of a Quadratic by the method of resolution into factors. Reducing a Quadratic to the form $ax^2+bx+c=0$, if we know the factors of which the left-hand side is the product, then by equating to zero either of these factors, we get a solution of the quadratic.

Example 1. Solve $x^2-5x+6=0$.

Evidently the left-hand side $=(x-2)(x-3)$.

Hence, we have $(x-2)(x-3)=0$.

$$\therefore \left.\begin{array}{ll}\text{either} & x-2=0\\ \text{and}\ \therefore & x=2\end{array}\right\} \text{ or, } \left.\begin{array}{ll} & x-3=0\\ \text{and}\ \therefore & x=3\end{array}\right\}$$

Thus, 2 and 3 are the roots of the equation, as the student can easily verify.

Example 2. Solve $2x^2-10x=3x-15$.

We have $\quad 2x(x-5)=3(x-5)$.

If $x-5\neq 0$ ($\neq$ stands for 'is not equal to'), we may remove the factor from each side of the equation.

Thus $\quad 2x=3; \qquad \therefore\ x=\frac{3}{2}$.

But if $x-5=0$, each side of the equation reduces to zero and the equation is satisfied.

Hence from $x-5=0$ we get another root *viz.*, $x=5$.

Thus the roots are $\frac{3}{2}$, 5.

If in course of simplification any factor which contains the unknown is found to be common to both sides of the equation, it must not be rejected, since every such linear factor equated to zero will give one root of the equation.

Example 3. Solve $\quad 10(2x+3)(x-3)+(7x+3)^2=20(x+3)(x-1)$.

We have, $10(2x^2-3x-9)+(49x^2+42x+9)=20(x^2+2x-3)$.

$$\therefore \quad 49x^2-28x-21=0;$$

$$\therefore \quad 7x^2-4x-3=0, \text{ or, } (7x^2-7x)+(3x-3)=0,$$

$$\text{or,} \quad (7x+3)(x-1)=0.$$

$$\text{Hence, } \left.\begin{array}{ll}\text{either} & 7x+3=0\\ \text{and}\ \therefore & x=-\frac{3}{7}\end{array}\right\} \text{ or, } \left.\begin{array}{ll} & x-1=0\\ \text{and}\ \therefore & x=1\end{array}\right\}.$$

Thus, $-\frac{3}{7}$ and 1 are the roots of the equation.

Example 4. Find the number which exceeds sixty-five times its reciprocal by 64.

Let x be the required number.

Then, by the condition of the problem,

$$x-\frac{65}{x}=64.$$

Multiplying both sides by x, we have

$$x^2-65=64x,$$

or, $x^2-64x-65=0$, [by transposition]

or, $(x-65)(x+1)=0$; [by factorisation]

$\therefore$ either $\left.\begin{array}{l} x-65=0 \\ \textit{i.e.,}\quad x=65 \end{array}\right\}$ or, $\left.\begin{array}{l} x+1=0 \\ \textit{i.e.,}\quad x=-1 \end{array}\right\}$

Hence, the required number is either 65, or, -1.

EXERCISE 69

Solve the following equations :

1. $3x^2-12x+1=6x-23$.
2. $4x^2-4x=80$.
3. $x+2-\dfrac{6}{x+2}=1$.
4. $x^2+9x-52=0$.
5. $x^2-\frac{5}{3}x-4=0$.
6. $6x^2+5x-4=0$.
7. $3(x-2)^2=18+(8x+1)$.
8. $x-\dfrac{x^3-8}{x^2+5}=2$.
9. $\dfrac{21x^3-16}{3x^2-4}-7x=5$.
10. $x^2-(a+b)x+ab=0$.

11. Find two numbers whose product is equal to 399 and sum is equal to 40.

12. The sum of a number and its square is eight times the next higher number ; find the number.

13. Find the number whose square exceeds ten times itself by 96.

14. Find the number which exceeds 12 by as much as thirty-nine times its reciprocal falls short of 4.

15. The difference between the ages of a man and his son is 25 years now. If the product of the numbers denoting their ages, ten years back, be 150, find the present age of the father.

16. The length of a rectangular garden of area 100 sq. metres exceeds its breadth by 15 metres. Find the cost of fencing it by wire-net the price of which is Re. 1 50 P. per metre.

MISCELLANEOUS EXERCISES IV

I

1. Define *Highest Common Factor* and *Lowest Common Multiple* of two or more algebraical expressions. Find the H.C.F. and L.C.M. of $36x^2a^4c^5$, $24xy^2a^3b^4$ and $240y^3a^6b^2c$.

2. Factorise the following expressions and find their H.C.F. :

$$x^2-6x+9 \text{ and } 4x^2-11x-3.$$

3. Find the L.C.M. of

$$ab-ac-b^2+bc \text{ and } b^2-12ac-4a^2-9c^2.$$

4. Solve the equation :

$$\frac{2(x-1)}{5}+\frac{15}{2}\left(1-\frac{x}{3}\right)+\frac{19}{10}=\frac{9}{5}\left(\frac{x}{6}-\frac{1}{3}\right).$$

5. If $2s=a+b+c$, show that

$$\frac{2bc+(b^2+c^2-a^2)}{2bc-(b^2+c^2-a^2)}=\frac{s(s-a)}{(s-b)(s-c)}.$$

6. Reduce the following to its simplest form :

$$\frac{x^6}{x^2-1}-\frac{x^4}{x^2+1}-\frac{1}{x^2-1}+\frac{1}{x^2+1}.$$

7. Solve $ax+1=by+1=ay+bx$.

8. One pipe can fill a cistern in a hours ; another can do it in b hours ; in what time could the two running together fill it ? And if a third pipe could empty the cistern in c hours, how long would it take to do this if the first two were running at the same time ?

II

1. Find the H.C.F. of $7x^2-26x+15$ and $5x(x-1)+3(3x-11)-24$.

2. Find the L.C.M. of $x^3+bx^2+ax+ab$ and $x^2-(a-b)x-ab$.

3. Reduce the following to their simplest forms :

(i) $\dfrac{(3x^4y^2-3x^2y^4)^2}{(2x^3y-2xy^3)^2}$; (ii) $\dfrac{3(x^2-x-30)(x^2-9x+14)}{(x^2-13x+42)(x^2+3x-10)}$.

4. Find the value of

$$\frac{x+y}{x-y}+\frac{x-y}{x+y}, \text{ when } x=a^2+b^2 \text{ and } y=a^2-b^2.$$

5. Simplify $\dfrac{(2x-9)^2-(x-6)^2}{3(x^2-10x+25)}+\dfrac{2(x-3)^2}{3(x^2-8x+15)}$.

6. What value of x will make the product of $2x+1$ and $x+1$ less than the product of $2x+3$ and $x+3$ by 20?

7. Find the value of x, when

$$\frac{5}{7}(2x-11)-\frac{3}{4}(x-5)=\frac{x}{3}-(10-x).$$

8. Solve $ax+by=c^2$ and $\frac{a+x}{b}-\frac{b+y}{a}=0$.

III

1. Find the H.C.F. of $a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b$ and $2a^2x^4-5a^4x^2+3a^6-2b^2x^4+5a^2b^2x^2-3a^4b^2$.

2. Find the L.C.M. of $x^5+x^4+x^3+x^2+x+1$ and $x^5-x^4+x^3-x^2+x-1$.

3. Find the H.C.F. of x^2-9, $(x+3)^2$ and x^2+x-6. [C. U. 1910]

4. State and prove the rule for finding the Lowest Common Multiple of two algebraical expressions. [B. U. 1902]

Find the L.C.M. of $x^2+(a+b)x+ab$, x^2-b^2 and $x^2+(a-b)x-ab$.

5. Simplify $\frac{1}{4}\cdot\left(\frac{x+3}{x^2+x-6}-\frac{x-5}{x^2-3x-10}\right)-\frac{1}{x^2+4}$.

6. Solve $ax+y=x+by=\frac{1}{2}(x+y)+1$.

7. An income of Rs. 196 is derived from two sums invested, one at 4 per cent., the other at 7 per cent. per annum; if the interest on the former had been 5 per cent., and on the latter 6 per cent., the income derived would have been Rs. 212. Find the sums invested.

8. Find the value of x, when $3(x^2-4)=15$.

IV

1. Define H.C.F. and L.C.M. of two or more algebraical expressions.

2. Find the H.C.F. of x^2-y^2, $x^2-2xy+y^2$ and x^3-y^3; and show that when their L.C.M. is divided by x^2+xy+y^2, the quotient is $(x-y)(x^2-y^2)$.

3. Find the defect of $\frac{x+6}{x^2+5x-6}$ from $\frac{x+5}{x^2+3x-10}$.

4. Simplify $\frac{1}{m^2+m+1}+\frac{2m}{m^4+m^2+1}$.

5. Show that $(x+y)^3-(y+z)^3=3(x-z)\{(x+y)(y+z)+\frac{1}{3}(x-z)^2\}$.

6. A number of three digits has 5 in the units' place and the middle figure is half the sum of the other two; if 108 be added to the number, the hundreds' figure will take the units' place, and the units' the tens'. Find the number.

7. If 3 be added to the numerator and denominator of a certain fraction, the fraction becomes $\frac{3}{4}$; if 5 be subtracted from the numerator and denominator, it becomes $\frac{1}{2}$. Find the fraction.

8. Solve $5(x^2-3x+11)+3(x^2+2x+4)=3(3x^2-3x+1)$.

V

1. Find the H.C.F. of $x^4-(a^2+b^2)x^2+a^2b^2$ and $x^4-(a+b)^2x^2+2ab(a+b)x-a^2b^2$.

2. Find the L.C.M. of $35x^2-11x-6$ and $40x^2-29x+3$.

3. Reduce to simplest form :

$$\left(\frac{2x}{x+y}-\frac{x^2}{x^2-y^2}+\frac{2y}{x-y}\right)\times\left(\frac{1}{x}+\frac{1}{y}\right)+\left(\frac{3}{x-y}-\frac{2}{x}+\frac{1}{y}\right).$$

4. Simplify $\frac{a^2+bc+ca+ab}{a^2+2bc+2ca+ab}\times\frac{a^3+8c^3}{a^4+a^2c^2+6ac^3+4c^4}$.

5. Show that $\frac{x+2}{1+x+x^2}-\frac{x-2}{1-x+x^2}-\frac{2x^2-4}{1-x^2+x^4}=\frac{4x^4+8}{x^8+x^4+1}$.

6. *A* and *B* travel together 120 kilometres by rail. *A* takes a return ticket for which he has to pay one fare and a half. Coming back they find that *A* has travelled cheaper than *B* by 50 paise for every 100 kilometres. Show that the fare per kilometre is 2 paise.

7. The expression $ax+b$ is equal to 13 when x is 5, and to 29 when x is 13. Show that the value of the expression is 4 when x is ·5.

8. The defect of 4 from twice the square of a number is 28. Find the number.

VI

1. Find the H.C.F. of

$$2x^2+x-10,\ x^2-5x+6 \text{ and } x^2-3x+2.$$

2. Find the L.C.M. of $ax^2-(a^2+ab)x+a^2b$, $bx^2-(b^2+bc)x+b^2c$ and $cx^2-(c^2+ac)x+c^2a$.

3. There are two quantities a and b of which the L.C.M. is x, and the G.C.M. is y; if $x+y=ma+\frac{b}{m}$, show that $x^3+y^3=m^3a^3+\frac{b^3}{m^3}$.

4. Simplify $\frac{z(x^3-y^3)}{x^2+xy+y^2}+\frac{x(y^3-z^3)}{y^2+yz+z^2}+\frac{y(z^3-x^3)}{z^2+zx+x^2}$.

5. If $x=\frac{a}{a+b}$ and $y=\frac{b}{a+b}$, show that

(i) $\frac{x^2+y^2}{x^2-y^2}=\frac{a^2+b^2}{a^2-b^2}$; (ii) $\frac{x^3-y^3}{x^3+y^3}=\frac{a^3-b^3}{a^3+b^3}$.

6. Solve $\frac{1}{3}(7x-5)+\frac{1}{39}(34x+10)-\frac{(3x-2)(5x-3)}{4}=\frac{(4-x)(2+15x)}{4}-18$.

7. A market-woman bought apples at three for a rupee and as many more at four for a rupee; and thinking to make her money again, she sold them at seven for two rupees. She lost, however, three rupees by the business. How much did she sell them for?

8. Solve $(2x+3)(x-5)+(x+5)(3x+1)=34+(x+4)(x+5)$.

VII

1. Find the H.C.F. of $a(a+1)x^2+x-a(a-1)$ and $a(a+2)x^2+2x-a^2+1$.

2. Find the L. C. M. of $ab-ac+bc-b^2$, $bc-ab+ac-c^2$ and $ac-bc+ab-a^2$.

3. The H.C.F. and L.C.M. of two numbers x and y are respectively 3 and 105; if $x+y=36$, prove that

$$\frac{1}{x}+\frac{1}{y}=\frac{4}{35}.$$

4. Simplify $\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}+\frac{1}{(x-3)(x-1)}$.

5. Find the value of $\frac{x+y}{x-y}$, when $x=\frac{a+b}{a-b}$ and $y=\frac{a-b}{a+b}$.

6. Show that if a number formed by two digits is four times the sum of its digits, the number formed by interchanging the digits is seven times their sum.

7. Solve $\left.\begin{aligned}3x+20&=4y-10\\4(x-1)-3(y-3)&=0\end{aligned}\right\}$ [C. U. 1895]

8. Find the number, the square of which exceeds 7 by as much as the square of half the number falls short of 13.

CHAPTER XXI

HARDER FORMULÆ

We shall now consider some important formulæ of a somewhat harder type than those treated of in Chapter IV.

125. Formula $(x+a)(x+b)(x+c)$ $=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc.$

Note. *The student can easily verify this. It is also evident that the following results are included in it :*

$$(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc\ ;$$
$$(x+a)(x+b)(x-c)=x^3+(a+b-c)x^2-(bc+ca-ab)x-abc\ ;$$
$$(x+a)(x-b)(x-c)=x^3+(a-b-c)x^2+(bc-ca-ab)x+abc.$$

For instance,

$$\begin{aligned}(x-a)(x-b)(x-c)&=\{x+(-a)\}\{x+(-b)\}\{x+(-c)\}\\&=x^3+\{(-a)+(-b)+(-c)\}x^2+\{(-b)(-c)\\&\qquad+(-c)(-a)+(-a)(-b)\}x+(-a)(-b)(-c\\&=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc.\end{aligned}$$

Similarly, the other two results can be established, which is left as an exercise for the student.

Example 1. Write down the product of $x+2$, $x+4$ and $x+6$.

$$2+4+6=12,$$
$$4\times6+6\times2+2\times4=24+12+8=44,$$
$$2\times4\times6=48.$$

Hence, the required product $=x^3+12x^2+44x+48$.

Example 2. Write down the product of $x-3$, $x-5$ and $x-7$.

$$(-3)+(-5)+(-7)=-15,$$
$$(-5)(-7)+(-7)(-3)+(-3)(-5)=35+21+15=71,$$
$$(-3)(-5)(-7)=-105.$$

Hence, the required product $=x^3-15x^2+71x-105$.

Example 3. Write down the product of $x-4$, $x+5$ and $x-3$.

$$(-4)+5+(-3)=-2,$$
$$(5)(-3)+(-3)(-4)+(-4)(5)=-15+12-20=-23,$$
$$(-4)\times5\times(-3)=60.$$

Hence, the required product $=x^3-2x^2-23x+60$

Example 4. Write down the product of $x+3$, $x+5$ and $x-8$.

$$3+5+(-8)=0,$$
$$(5)(-8)+(-8)(3)+(3)(5)=-40-24+15=-49,$$
$$3\times5\times(-8)=-120.$$

Hence, the reqd. product $=x^3-0.x^2-49x-120=x^3-49x-120.$

EXERCISE 70

Write down the product of :

1. $x+1$, $x+2$ and $x+3$.
2. $x+2$, $x+5$ and $x+7$.
3. $x+3$, $x-6$ and $x+2$.
4. $x+4$, $x+5$ and $x-10$.
5. $x-8$, $x+3$ and $x+1$.
6. $x-5$, $x-2$ and $x+8$.
7. $x-3$, $x+7$ and $x-4$.
8. $x+6$, $x-5$ and $x-7$.
9. $x-5$, $x-7$ and $x-11$.
10. $x-3$, $x-6$ and $x-9$.
11. $x+4$, $x-5$ and $x-12$.
12. $x+5$, $x+9$ and $x+11$.
13. $x-6$, $x+8$ and $x-2$.
14. $x-3$, $x-7$ and $x-13$.
15. $x-3$, $x+12$ and $x+4$.
16. $x-9$, $x-10$ and $x+12$.
17. $x+9$, $x-5$ and $x-7$.
18. $x+8$, $x+12$ and $x+15$.
19. $x-14$, $x+8$ and $x+6$.
20. $x-5$, $x-10$ and $x-16$.

126. Squares of multinomials. It has been respectively shown in examples 10 and 11 of Art. 54 that $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$ and $(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.

Thus, in each of these cases we may observe that the square of the whole expression is obtained by taking the sum of the squares of the different terms and of twice the product of each term by every term which *follows* it. The results are best remembered when put as follows :

$$(a+b+c)^2=a^2+b^2+c^2+2a(b+c)+2bc\,;$$
$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2a(b+c+d)+2b(c+d)+2cd.$$

The same rule may be shown to hold good in every other case ; for instance, let us find the square of $a+b+c+d+e$.

We have,

$$\begin{aligned}(a+b+c+d+e)^2&=\{(a+b+c)+(d+e)\}^2\\&=(a+b+c)^2+2(a+b+c)(d+e)+(d+e)^2\\&=\{a^2+b^2+c^2+2a(b+c)+2bc\}+\{2a(d+e)+2b(d+e)\\&\qquad+2c(d+e)\}+(d^2+e^2+2de)\\&=a^2+b^2+c^2+d^2+e^2+2a(b+c+d+e)\\&\qquad+2b(c+d+e)+2c(d+e)+2de.\end{aligned}$$

Hence, we conclude that the square of any multinomial is equal to the sum of the squares of its different terms together with twice the product of each term by every term which *follows* it.

It is needless to add that the above rule will also hold good when the multinomial under consideration contains one or more negative terms, for the symbols used above are perfectly general in character and any of them may stand either for a positive or a negative quantity.

Note. *Since* $(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+bc)$, *we have*

$$2(ab+ac+bc)=\{a^2+b^2+c^2+2(ab+ac+bc)\}-(a^2+b^2+c^2)$$
$$=(a+b+c)^2-(a^2+b^2+c^2).$$

Similarly, $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$.

Example 1. Write down the square of $x-y+z-v$.

$$(x-y+z-v)^2=x^2+y^2+z^2+v^2+2x(-y+z-v)+2(-y)(z-v)+2z(-v)$$
$$=x^2+y^2+z^2+v^2-2xy+2xz-2xv-2yz+2yv-2zv.$$

Example 2. Write down the square of $-a+2b-3c-d$.

$$(-a+2b-3c-d)^2=a^2+4b^2+9c^2+d^2+2(-a)(2b-3c-d)$$
$$+2(2b)(-3c-d)+2(-3c)(-d)$$
$$=a^2+4b^2+9c^2+d^2-4ab+6ac+2ad$$
$$-12bc-4bd+6cd.$$

Example 3. Find the value of $a^2+b^2+c^2+2ab-2ac-2bc$, when $a=19$, $b=18$ and $c=32$.

The given expression $=a^2+b^2+c^2+2a(b-c)+2b(-c)=(a+b-c)^2$.

Hence, the required value $=(19+18-32)^2=(5)^2=25$.

Example 4. If $x=b+c$, $y=c-a$, $z=a-b$, prove that

$$x^2+y^2+z^2-2xy-2xz+2yz=4b^2. \qquad \text{[C. U. 1883]}$$

$$x^2+y^2+z^2-2xy-2xz+2yz$$
$$=x^2+y^2+z^2+2x(-y-z)+2(-y)(-z)=(x-y-z)^2$$
$$=\{(b+c)-(c-a)-(a-b)\}^2=(2b)^2=4b^2.$$

EXERCISE 71

Write down the square of :

1. $x+y-z$.
2. $x-y+z$.
3. $-x+y+z$.
4. $-x-y+z$.
5. $x-y-z$.
6. $a-x+y-z$.
7. $a-x-y-z$.
8. $m+n+p+q+r$.
9. $p-q+r-x-y$.
10. $-a+b-c+x-y-z$.
11. $a-2x-3y-4z$.
12. $2a-b+2c-d$.

Find the value of :

13. $l^2+m^2+n^2-2lm+2ln-2mn$, when $l=17$, $m=23$ and $n=13$.
14. $p^2+q^2+r^2+2pq-2pr-2qr$, when $p=16$, $q=12$ and $r=25$.
15. $a^2+b^2+c^2-2ab-2ac+2bc$, when $a=28$, $b=13$ and $c=15$.
16. $x^2+y^2+1+2xy-2x-2y$, when $x=6$ and $y=7$.

17. $x^2+y^2+2xy-2x-2y+36$, when $x=23$ and $y=18$.
18. $x^2+4y^2+1-4xy-2x+4y$, when $x=26$ and $y=12$.
19. $x^2+9y^2-6xy-2x+6y+64$, when $x=49$ and $y=16$.
20. $9x^2+y^2-6xy+6x-2y-24$, when $x=14$ and $y=38$.
21. If $a+b+c=12$ and $a^2+b^2+c^2=50$, find the value of $ab+ac+bc$.
22. If $a+b+c=13$ and $ab+ac+bc=50$, find the value of $a^2+b^2+c^2$.

127. Powers of Binomials : Involution.

By actual multiplication it may be seen that

$$\left.\begin{aligned}(a+b)^2&=a^2+2ab+b^2\\(a-b)^2&=a^2-2ab+b^2\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^3&=a^3+3a^2b+3ab^2+b^3\\(a-b)^3&=a^3-3a^2b+3ab^2-b^3\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^4&=a^4+4a^3b+6a^2b^2+4ab^3+b^4\\(a-b)^4&=a^4-4a^3b+6a^2b^2-4ab^3+b^4\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^5&=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\(a-b)^5&=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5\end{aligned}\right\}$$

$$\left.\begin{aligned}(a+b)^6&=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\\(a-b)^6&=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\end{aligned}\right\}$$

Note. *On examining the above cases we observe that :*

(1) *The total number of terms in the resulting expression is one more than the index of the binomial. Thus, in the* **fifth** *power the number of terms is* **six**, *in the* **sixth** *power the number of terms is* **seven** *; and so on.*

(2) *Any power of a − b differs from the same power of a + b only in this that the signs of the terms of the former are* **alternately** *+ and −, whilst those of the latter are all +.*

(3) *The first term is a raised to a power equal to that of the binomial, and the last term is b raised to the same power. Thus, in the* **fourth** *power, the first term is* a^4 *and the last* b^4 *; in the* **fifth** *power, the first term is* a^5, *and the last* b^5 *; and so on. As to the other terms the power of a in any term is one less, whilst the power of b is one greater than that in the preceding term.*

(4) *The coefficient of the second term is the same as the index of the power to which the binomial is raised ; and if the coefficient of any term be multiplied by the index of a in that term, and divided by the number* **indicating the position** *of that term, the result gives the coefficient of the next term. Thus, if we multiply the coefficient of the second term by the index of a in it and divide the product by two we get the coefficient of the third term ; if the coefficient of the third term be*

multiplied by the index of a in it and the product divided by three, we obtain the coefficient of the 4th term ; and so on.

(5) *The coefficient of the terms equidistant from the beginning and the end are the same ; in other words, the coefficient of the term which has any number of terms before it, is equal to that of the term which has the same number of terms after it.*

The laws observed above, a proof of the universal truth of which is beyond the scope of our limits, furnish us with a ready means of raising a binomial to any power without the process of actual multiplication. The following examples are intended to illustrate the application of those laws.

[*The resulting expression in each case is called the* ***expansion*** *of the corresponding power of the binomial.*]

The operation of raising any expression to any power is called ***Involution.***

Example 1. Raise $a+b$ to the *seventh* power.

The total number of terms in the expansion $=8$.

$$\left.\begin{array}{lll} \text{The first term} & & =a^7 \\ \quad\text{,, 2nd} & \text{,,} & =7a^6b \\ \quad\text{,, 3rd} & \text{,,} & =\frac{7\times6}{2}a^5b^2=21a^5b^2 \\ \quad\text{,, 4th} & \text{,,} & =\frac{21\times5}{3}a^4b^3=35a^4b^3 \end{array}\right\} \quad \text{[Laws (3) and (4)]}$$

Now, since the four terms from the end will have respectively the same coefficients as the four terms from the beginning [law (5)], the next four terms of the expansion will respectively be $35a^3b^4$, $21a^2b^5$, $7ab^6$ and b^7.

Hence, we have

$$(a+b)^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

Example 2. Expand $(x-y)^8$.

The total number of terms in the expansion $=9$.

$$\left.\begin{array}{lllll} \text{The first term} & & =x^8 & & \\ \quad\text{,, 2nd} & \text{,,} & =-8x^7y & & \\ \quad\text{,, 3rd} & \text{,,} & =\frac{8\times7}{2}x^6y^2 & = & 28x^6y^2 \\ \quad\text{,, 4th} & \text{,,} & =-\frac{28\times6}{3}x^5y^3 & = & -56x^5y^3 \\ \quad\text{,, 5th} & \text{,,} & =\frac{56\times5}{4}x^4y^4 & = & 70x^4y^4 \end{array}\right\} \quad \text{[Laws (2), (3) and (4)]}$$

The coefficients of the remaining four terms need not be calculated as the coefficients of the first four terms only will now reappear in the reverse order.

Hence, we have

$$(x-y)^8=x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5 \\ +28x^2y^6-8xy^7+y^8.$$

Example 3. Expand $(2x-3y)^7$.

The total number of terms in the expansion $=8$.

As we have $2x$ for a and $3y$ for b, we must have

$$\left.\begin{array}{llll} \text{The first term} & =(2x)^7 & & \\ \quad ,, \ \text{2nd} \ ,, & =-7(2x)^6(3y) & & \\ \quad ,, \ \text{3rd} \ ,, & =\frac{7\times 6}{2}(2x)^5(3y)^2 & = & 21(2x)^5(3y)^2 \\ \quad ,, \ \text{4th} \ ,, & =-\frac{21\times 5}{3}(2x)^4(3y)^3 & = & -35(2x)^4(3y)^3 \end{array}\right\}$$

We can now write down the remaining four terms which will respectively be $35(2x)^3(3y)^4$, $-21(2x)^2(3y)^5$, $7(2x)(3y)^6$ and $-(3y)^7$.

Hence, we have

$$\begin{aligned}(2x-3y)^7 &= (2x)^7-7(2x)^6(3y)+21(2x)^5(3y)^2-35(2x)^4(3y)^3\\ &\qquad +35(2x)^3(3y)^4-21(2x)^2(3y)^5+7(2x)(3y)^6-(3y)^7\\ &= 128x^7-7(64x^6)(3y)+21(32x^5)(9y^2)-35(16x^4)(27y^3)\\ &\qquad +35(8x^3)(81y^4)-21(4x^2)(243y^5)+7(2x)(729y^6)-2187y^7\\ &= 128x^7-1344x^6y+6048x^5y^2-15120x^4y^3+22680x^3y^4\\ &\qquad -20412x^2y^5+10206xy^6-2187y^7.\end{aligned}$$

Example 4. Find the value of

$$x^6+6x^5+15x^4+20x^3+15x^2+6x-8, \text{ when } x=\sqrt[3]{3}-1.$$

The given expression

$$\begin{aligned}&=(x^6+6x^5+15x^4+20x^3+15x^2+6x+1)-9\\ &=(x+1)^6-9\\ &=(\sqrt[3]{3})^6-9=9-9=0.\end{aligned}$$

EXERCISE 72

Expand :

1. $(x+1)^5$.
2. $(x+1)^6$.
3. $(a+b)^8$.
4. $(a+b)^9$.
5. $(x-y)^5$.
6. $(m-n)^7$.
7. $(x+2)^4$.
8. $(x+2)^5$.
9. $(x+1)^8$.
10. $(x+3)^4$.
11. $(x-1)^5$.
12. $(2-z)^6$.
13. $(2x-1)^4$.
14. $(x-y)^9$.
15. $(3x-2)^5$.
16. $(1-a)^8$.
17. $(1-c)^7$.
18. $(1-3x)^6$.
19. $(1-2x)^7$.
20. $(2x-a)^8$.
21. $(x-a)^{10}$.
22. $(3x-2a)^5$.

Simplify :

23. $(x+1)^5-(x-1)^5$.
24. $(x-1)^6+(x+1)^6$.
25. $(x+a)^7-(x-a)^7$.

Find the sum of the coefficients in the expansion of :

26. $(x+a)^4$.
27. $(x+a)^5$.
28. $(x+a)^6$.
29. $(x+a)^7$.
30. $(x+a)^8$.

Find the value of :

31. $x^5+5x^4+10x^3+10x^2+5x+32$, when $x=-2$.
32. $x^6-6x^5+15x^4-20x^3+15x^2-6x$, when $x=\sqrt[3]{2}+1$.
33. $16x^4-32x^3+24x^2-8x-80$, when $x=2$.
34. $x^4+12x^3+54x^2+108x+81$, when $x=-5$.
35. $x^4+8x^3+24x^2+32x-609$, when $x=-7$.

128. Formula $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$$\begin{aligned}&=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}\\&=a^3+b^3+c^3-3abc.\end{aligned}$$

[$(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$$\begin{aligned}&=(a+b+c)\{(a^2+b^2-ab)-(ac+bc)+c^2\}\\&=(a+b+c)\{\overline{(a+b)^2-3ab}-c(a+b)+c^2\}\\&=(\overline{a+b}+c)\{\overline{(a+b)^2-c(a+b)+c^2}-3ab\}\\&=\overline{(a+b)^3+c^3}-3ab(\overline{a+b}+c)\\&=(a+b)^3-3ab(a+b)+c^3-3abc\\&=a^3+b^3+c^3-3abc.\ ]\end{aligned}$$

Cor. Conversely, $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$. Hence, we can always resolve an expression into factors whenever it is of the form $a^3+b^3+c^3-3abc$.

Note. *Since* $a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}$, *we have* $a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}$.

Example 1. Multiply $x^2+y^2+z^2+xy+xz-yz$ by $x-y-z$.

Putting a for x, b for $-y$ and c for $-z$, we have

$$\begin{aligned}&(x-y-z)(x^2+y^2+z^2+xy+xz-yz)\\&\quad=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)\\&\quad=a^3+b^3+c^3-3abc=x^3-y^3-z^3-3xyz.\end{aligned}$$

Example 2. Resolve $m^3-n^3+1+3mn$ into factors.

Putting a for m, b for $-n$ and c for 1, we have

$$\begin{aligned}m^3-n^3+1+3mn&=a^3+b^3+c^3-3abc\\&=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)\\&=(m-n+1)(m^2+n^2+1+mn-m+n).\end{aligned}$$

Example 3. Show that $(x-y)^3+(y-z)^3+(z-x)^3$
$$=3(x-y)(y-z)(z-x).$$

Putting a for $x-y$, b for $y-z$ and c for $z-x$, we have

$$a+b+c=(x-y)+(y-z)+(z-x)=0.$$

Hence, $\{(x-y)^3+(y-z)^3+(z-x)^3\}-3(x-y)(y-z)(z-x)$

$$=a^3+b^3+c^3-3abc$$
$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$
$$=0\times(a^2+b^2+c^2-ab-ac-bc)=0\,;$$

$\therefore\quad (x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x).$

EXERCISE 73

Multiply :

1. $x^2+y^2+z^2-xy+xz+yz$ by $x+y-z$.
2. $p^2+4q^2+r^2+2pq+pr-2qr$ by $p-2q-r$.
3. $4x^2+9y^2+z^2+6xy+2xz-3yz$ by $2x-3y-z$.
4. $a^2+4b^2+2ab-3a+6b+9$ by $a-2b+3$.
5. $9a^2+25b^2+15ab+12a-20b+16$ by $3a-5b-4$.

Resolve into factors :

6. $x^3-y^3-1-3xy$. 7. $x^3-y^3+6xy+8$. 8. $x^3-8y^3-27z^3-18xyz$.

Find the value of :

9. $x^3+y^3+18xy-216$, when $x+y=6$.
10. $a^3-8b^3-24ab-64$, when $a-2b=4$.
11. $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$, when $3s=a+b+a$.
12. Show that $(a-2b)^3+(2b-3c)^3+(3c-a)^3$
$$=3(a-2b)(2b-3c)(3c-a).$$
13. Show that $(x+y-2z)^3+(y+z-2x)^3+(z+x-2y)^3$
$$=3(x+y-2z)(y+z-2x)(z+x-2y).$$
14. Show that $(a+2b-3c)^3+(b+2c-3a)^3+(c+2a-3b)^3$
$$=3(a+2b-3c)(b+2c-3a)(c+2a-3b).$$
15. Show that $(2p-5q+3r)^3+(2q-5r+3p)^3+(2r-5p+3q)^3$
$$=3(2p-5q+3r)(2q-5r+3p)(2r-5p+3q).$$
16. Find the value of $x^6+y^6-z^6+3x^2y^2z^2$, when $x=a^2-b^2$, $y=2ab$, $z=a^2+b^2$.
17. Find the value of $x^3+y^3+z^3-3xyz$, when $x=658$, $y=668$, $z=674$.

129. Formula $(a-b)(a-c)(b-c)$

$$=a^2(b-c)+b^2(c-a)+c^2(a-b)$$
$$=bc(b-c)+ca(c-a)+ab(a-b).$$

[$(a-b)(a-c)(b-c)=\{a^2-a(b+c)+bc\}(b-c)$

$$=a^2(b-c)-a(b^2-c^2)+bc(b-c)$$
$$=a^2(b-c)+b^2(c-a)+c^2(a-b).$$]

Cor. 1. Conversely, $a^2(b-c)+b^2(c-a)+c^2(a-b)$

$$=(a-b)(a-c)(b-c).$$

Hence, we know at once the factors of an expression which is of the form $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

Cor. 2. Since $a-c=-(c-a)$, we have

$$(a-b)(a-c)(b-c)=-(a-b)(b-c)(c-a).$$

Hence, the above relation can also be put in the form

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a).$$

Cor. 3. Since $a^2(b-c)+b^2(c-a)+c^2(a-b)$ can be put in the form $ab(a-b)+bc(b-c)+ca(c-a)$, we have also

$$ab(a-b)+bc(b-c)+ca(c-a)=-(a-b)(b-c)(c-a).$$

Example. Simplify $(a+2b+3c)^2(a-2b+c)+(b+2c+3a)^2(b-2c+a)$
$+(c+2a+3b)^2(c-2a+b)+(a-2b+c)(b-2c+a)(c-2a+b)$.

$$\left.\begin{aligned}\text{Putting } x \text{ for } & a+2b+3c,\\ y \text{ for } & b+2c+3a,\\ \text{and } z \text{ for } & c+2a+3b,\end{aligned}\right\}\qquad \text{we have}\quad \left.\begin{aligned}y-z&=a-2b+c\\ z-x&=b-2c+a\\ x-y&=c-2a+b\end{aligned}\right\}$$

Hence, the given expression

$$=x^2(y-z)+y^2(z-x)+z^2(x-y)+(y-z)(z-x)(x-y)$$
$$=-(y-z)(z-x)(x-y)+(y-z)(z-x)(x-y)=0.$$

EXERCISE 74

1. Show that $(x-2y+z)(2x-y-z)(y-2z+x)$
$=(x-y)^2(y-2z+x)+(y-z)^2(z-2x+y)+(z-x)^2(x-2y+z)$.

2. Show that $(a+b)^2(b-a)+(b+c)^2(c-b)+(c+a)^2(a-c)$
$+(b-a)(c-b)(a-c)=0$.

3. Resolve into factors
$2(a-b+c)^2(a-c)+2(b-c+a)^2(b-a)+2(c-a+b)^2(c-b)$.

4. Resolve into factors
$(x+y)^2(y-x)+(y+z)^2(z-y)+(z+x)^2(x-z)$.

5. Simplify $2(a-b-c)^2(b-c)+2(b-c-a)^2(c-a)$
$+2(c-a-b)^2(a-b)+8(a-b)(b-c)(c-a)$.

6. Simplify $(x-y)(y-z)(x-2y+z)$
$+(y-z)(z-x)(y-2z+x)+(z-x)(x-y)(z-2x+y)$
$+(x-2y+z)(y-2z+x)(z-2x+y)$.

130. Formula $(b+c)(c+a)(a+b)$

$$\begin{aligned}&= a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\\&= a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc\\&= bc(b+c)+ca(c+a)+ab(a+b)+2abc\\&= (a+b+c)(bc+ca+ab)-abc.\end{aligned}$$

[$(b+c)(c+a)(a+b)$

$$\begin{aligned}&=(b+c)\{(a+b)(a+c)\}=(b+c)\{a^2+a(b+c)+bc\}\\&=a^2(b+c)+a(b+c)^2+bc(b+c)\\&=a^2(b+c)+a(b^2+2bc+c^2)+b^2c+bc^2\\&=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc \quad \text{[re-arranging the terms].}\end{aligned}$$

But, $a^2(b+c)+b^2(c+a)+c^2(a+b)$

$$\begin{aligned}&=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2) \qquad \text{[re-arranging the terms]}\\&=(b^2c+bc^2)+(c^2a+ca^2)+(a^2b+ab^2)\\&=bc(b+c)+ca(c+a)+ab(a+b)\\&=bc(a+b+c-a)+ca(a+b+c-b)+ab(a+b+c-c)\\&=bc(a+b+c)+ca(a+b+c)+ab(a+b+c)-bca-cab-abc\\&=(a+b+c)(bc+ca+ab)-3abc.\end{aligned}$$

Hence, the result follows.]

131. Formula $(a+b+c)(bc+ca+ab)=P+3abc$, where P stands for any of the equivalent forms

(i) $a^2(b+c)+b^2(c+a)+c^2(a+b)$;

(ii) $bc(b+c)+ca(c+a)+ab(a+b)$;

(iii) $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$.

[From Art. 130, we have by transposition, or by direct multiplication, $(a+b+c)(bc+ca+ab)$

$$\begin{aligned}&=a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc\\&=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc\\&=bc(b+c)+ca(c+a)+ab(a+b)+3abc. \text{]}\end{aligned}$$

Example 1. Find the product of

$$(2x+3y+5z)(15yz+10zx+6xy).$$

Putting a, b and c for $2x$, $3y$ and $5z$ respectively, we have

$$\begin{aligned}a+b+c&=2x+3y+5z,\\bc+ca+ab&=15yz+10zx+6xy.\end{aligned}$$

$$\therefore \quad (2x+3y+5z)(15yz+10zx+6xy)$$
$$=(a+b+c)(bc+ca+ab)$$
$$=a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$$
$$=4x^2(3y+5z)+9y^2(5z+2x)+25z^2(2x+3y)+3.2x.3y.5z$$
$$=12x^2y+20x^2z+45y^2z+18y^2x+50z^2x+75z^2y+90xyz.$$

Example 2. Show that $(x+3y+12z)(12yz+4zx+xy)-12xyz$
$$=(y+4z)(12z+x)(x+3y).$$

Putting a, b and c for x, $3y$ and $12z$ respectively, we have
$$a+b+c=x+3y+12z,$$
$$bc+ca+ab=36yz+12zx+3xy$$
$$=3(12yz+4zx+xy),$$
$$abc=36xyz.$$

$\therefore$ the left-hand side $=\frac{1}{3}\{(a+b+c)(bc+ca+ab)-abc\}$
$$=\tfrac{1}{3}(b+c)(c+a)(a+b) \qquad \text{[Art. 130]}$$
$$=\tfrac{1}{3}(3y+12z)(12z+x)(x+3y)$$
[restoring values of a, b, c]
$$=(y+4z)(12z+x)(x+3y)$$

EXERCISE 75

Write down the products of the following :

1. $(x+2y)(2y+3z)(3z+x)$.
2. $(8x+y)(y+5z)(5z+8x)$.
3. $(a+2b)(2b+3c)(3c+a)$.
4. $(3x+y+10z)(10yz+30zx+3xy)$.
5. $(x+2y+z)(2x+y+z)(x+y+2z)$.
6. $(a-2b)(2b-3c)(3c+a)$.

Simplify the following :

7. $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$+(b+c-a)(c+a-b)(a+b-c)$.

8. $c(b+c-a)(c+a-b)+a(c+a-b)(a+b-c)$
$+b(a+b-c)(b+c-a)+(b+c-a)(c+a-b)(a+b-c)$.

9. $(y+z)^2(2x+y+z)+(z+x)^2(x+2y+z)+(x+y)^2(x+y+2z)$
$-(2x+y+z)(x+2y+z)(x+y+2z)+2(y+z)(z+x)(x+y)$.

10. $2a(b+c-a)^2+2b(c+a-b)^2+2c(a+b-c)^2-3abc$
$+2(a+b+c)\{(c+a-b)(a+b-c)+(a+b-c)(b+c-a)$
$+(b+c-a)(c+a-b)\}$.

11. Prove that $(x+y-z)\{(y+z-x)^2+(z+x-y)^2\}+(y+z-x)$
$\times\{(z+x-y)^2+(x+y-z)^2\}+(z+x-y)\{(x+y-z)^2$
$+(y+z-x)^2\}+2(y+z-x)(z+x-y)(x+y-z)=8xyz$.

132. Formula $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$.

$$\begin{aligned}\Big[\ (a+b+c)^3&=\{(a+b)+c\}^3\\ &=(a+b)^3+c^3+3(a+b)c\{(a+b)+c\} &&[\text{ Art. 57 }]\\ &=\{a^3+b^3+3ab(a+b)\}+c^3+3(a+b)c(a+b+c)\\ &=a^3+b^3+c^3+\{3ab(a+b)+3(a+b)c(a+b+c)\}\\ &=a^3+b^3+c^3+3(a+b)\{ab+c(a+b+c)\}\\ &=a^3+b^3+c^3+3(a+b)\{c^2+c(a+b)+ab\}\\ &=a^3+b^3+c^3+3(a+b)(c+b)(c+a) &&[\text{ Art. 61 }]\\ &=a^3+b^3+c^3+3(b+c)(c+a)(a+b).\ \Big]\end{aligned}$$

Cor. $(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b)$.

Example 1. Factorise $8(x+y+z)^3-(y+z)^3-(z+x)^3-(x+y)^3$.

Put a, b and c for $y+z$, $z+x$ and $x+y$ respectively.

We have $a+b+c=2(x+y+z)$.

∴ the given expression

$$\begin{aligned}&=\{2(x+y+z)\}^3-(y+z)^3-(z+x)^3-(x+y)^3\\ &=(a+b+c)^3-a^3-b^3-c^3\\ &=3(b+c)(c+a)(a+b) &&[\text{ Cor. }]\\ &=3(2x+y+z)(x+2y+z)(x+y+2z). &&[\text{ restoring the values of } a,\ b,\ c\ ]\end{aligned}$$

Example 2. Show that

$$(x+y+z)^3=(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+24xyz.$$

Put a, b and c for $y+z-x$, $z+x-y$ and $x+y-z$ respectively.

We have $a+b+c=(y+z-x)+(z+x-y)+(x+y-z)=x+y+z$,

$$\begin{aligned}b+c&=(z+x-y)+(x+y-z)=2x,\\ c+a&=(x+y-z)+(y+z-x)=2y,\\ a+b&=(y+z-x)+(z+x-y)=2z\ ;\end{aligned}$$

$$\begin{aligned}\therefore\quad (x+y+z)^3=(a+b+c)^3&=a^3+b^3+c^3+3(b+c)(c+a)(a+b)\\ &=(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+3.2x.2y.2z\\ &=(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+24xyz.\end{aligned}$$

EXERCISE 76

1. If $a+b+c=0$, show that $a^3+b^3+c^3=3a(c+a)(a+b)$
$=3b(b+c)(b+a)=3c(c+a)(c+b)=3abc$.

2. If $2s=x+y+z$, prove that $(s-x)^3+(s-y)^3+(s-z)^3+3xyz=s^3$.

3. Prove that $(2x-y-z)^3+(2y-z-x)^3+(2z-x-y)^3$
$=3(2x-y-z)(2y-z-x)(2z-x-y)$.

4. Simplify $(3x-y-z)^3+(3y-z-x)^3+(3z-x-y)^3$
$+24(y+z-x)(z+x-y)(x+y-z)-x^3-y^3-z^3$
$-3(y+z)(z+x)(x+y)$.

5. Show that $(2x-y-z)^3+y^3+z^3+3(y+z)(2x-y)(2x-z)$
$=(2x-y-3z)^3+y^3+27z^3+3(y+3z)(2x-y)(2x-3z)$.

6. If $2s=x+y+z$, prove that
$s^3+(s-2x)^3+(s-2y)^3+(s-2z)^3-24(s-x)(s-y)(s-z)=0$.

7. If $3s=2(x+y+z)$, show that $(s-y-z)^3+(s-z-x)^3$
$+(s-x-y)^3+3(y+z-s)(z+x-s)(x+y-s)=0$.

8. Simplify $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3-(a+b+c)^3+108abc$.

9. Simplify $(x+y+z)^3-(y+z)^3-(z+x)^3-(x+y)^3+x^3+y^3+z^3$.

10. Factorise $x^3-(2x-y-z)^3-(2y-z-x)^3+(y-2z)^3$.

11. Resolve into factors
$64(x+y+z)^3-(2x+y+z)^3-(x+2y+z)^3-(x+y+2z)^3$.

Find the value of :

12. $a^3+b^3+c^3$, when $b+c=10$, $c+a=16$ and $a+b=20$.

13. $x^3+y^3+z^3$, when $x=32$, $y=-25$ and $z=-7$.

14. $(x+y+z)^3-(x+z-y)^3-(y+z-x)^3-(x+y-z)^3-23xyz$,
when $x=10$, $y=64$ and $z=2$.

15. $(6x-y-z)^3+y^3+z^3+3(y+z)(6x-y)(6x-z)$,
when $x=\frac{11}{6}$, $y=\frac{115}{116}$ and $z=17$.

133. Recapitulation of the Formulæ. The different formulæ treated of in Chapter IV as well as in the present one are grouped below to facilitate any reference to them. It is desired, however, that the student should commit them so fully to memory that the necessity even for occasional references may be altogether done away with.

I. $(a+b)^2=a^2+2ab+b^2$.

II. $(a-b)^2=a^2-2ab+b^2$.

III. $(a+b)(a-b)=a^2-b^2$.

IV. $\left.\begin{aligned}(a+b)^3&=a^3+3a^2b+3ab^2+b^3\\&=a^3+b^3+3ab(a+b).\end{aligned}\right\}$

V. $\left.\begin{aligned}(a-b)^3&=a^3-3a^2b+3ab^2-b^3\\&=a^3-b^3-3ab(a-b).\end{aligned}\right\}$

VI. $\left.\begin{aligned}a^3+b^3&=(a+b)^3-3ab(a+b)\\&=(a+b)(a^2-ab+b^2).\end{aligned}\right\}$

VII. $\left.\begin{aligned}a^3-b^3&=(a-b)^3+3ab(a-b)\\&=(a-b)(a^2+ab+b^2).\end{aligned}\right\}$

VIII. $(x+a)(x+b)=x^2+(a+b)x+ab$.

IX. $(x-a)(x+b)=x^2+(b-a)x-ab$.

X. $(x-a)(x-b)=x^2-(a+b)x+ab$.

XI. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc.$

XII. $(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc.$

XIII.
$$\left.\begin{aligned}a^3+b^3+c^3-3abc&=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)\\&=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}.\end{aligned}\right\}$$

XIV.
$$\left.\begin{aligned}(a-b)(a-c)(b-c)&=-(b-c)(c-a)(a-b)\\&=a^2(b-c)+b^2(c-a)+c^2(a-b)\\&=bc(b-c)+ca(c-a)+ab(a-b)\\&=-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}.\end{aligned}\right\}$$

XV.
$$\left.\begin{aligned}(b+c)(c+a)(a+b)&=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\\&=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc\\&=bc(b+c)+ca(c+a)+ab(a+b)+2abc\\&=(a+b+c)(bc+ca+ab)-abc.\end{aligned}\right\}$$

XVI.
$$\left.\begin{aligned}&(a+b+c)(bc+ca+ab)\\&\qquad=(b+c)(c+a)(a+b)+abc\\&\qquad=a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc\\&\qquad=bc(b+c)+ca(c+a)+ab(a+b)+3abc\\&\qquad=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc.\end{aligned}\right\}$$

XVII.
$$\left.\begin{aligned}&(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)\\&\quad=a^3+b^3+c^3+3\{a^2(b+c)+b^2(c+a)+c^2(a+b)\}+6abc,\\\text{or,}\quad&(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b).\end{aligned}\right\}$$

The following useful results are deserving of notice. They can be deduced from the above formulæ or verified by actual multiplication.

XVIII. $(a+b)^2+(a-b)^2=2(a^2+b^2).$

XIX. $(a+b)^2-(a-b)^2=4ab,$

or, $ab=\left(\dfrac{a+b}{2}\right)^2-\left(\dfrac{a-b}{2}\right)^2.$

XX. $(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.$

XXI. $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c).$

XXII. $(b-c)+(c-a)+(a-b)=0.$

XXIII. $a(b-c)+b(c-a)+c(a-b)=0.$

XXIV. $\tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}=a^2+b^2+c^2-bc-ca-ab.$

XXV. $(a+b)^3+(a-b)^3=2a^3+6ab^2.$

XXVI. $(a+b)^3-(a-b)^3=6a^2b+2b^3.$

XXVII. $(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4.$

XXVIII. $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

XXIX. $(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$

CHAPTER XXII

HARDER FACTORS AND IDENTITIES

I. Factors

We have already explained in Chapter XII how simple expressions of the types a^2-b^2, a^3+b^3, a^3-b^3, x^2+px+q and px^2+qx+r can be resolved into factors, and shall in this section consider factorisations of a harder type.

134. To factorise expressions of the form

$$a^3+b^3+c^3-3abc.$$

Since, $b^3+c^3=(b+c)^3-3bc(b+c)$, we have

$$\begin{aligned}
&a^3+b^3+c^3-3abc\\
&=a^3+\{(b+c)^3-3bc(b+c)\}-3abc\\
&=\{a^3+(b+c)^3\}-3bc\{(b+c)+a\}\\
&=\{a+(b+c)\}\{a^2-a(b+c)+(b+c)^2\}-3bc(a+b+c)\\
&=\{a+b+c\}\{a^2-ab-ac+b^2+2bc+c^2-3bc\}\\
&=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)\\
&=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}.
\end{aligned}$$

Example 1. Factorise $a^3-b^3+c^3+3abc$.

The given expression

$$\begin{aligned}
&=a^3+(-b)^3+c^3-3a(-b)c\\
&=\{a+(-b)+c\}\{a^2+(-b)^2+c^2-(-b)c-ca-a(-b)\}\\
&=(a-b+c)(a^2+b^2+c^2+bc-ca+ab).
\end{aligned}$$

Example 2. Factorise $x^3-y^3+6xy+8$.

The given expression

$$\begin{aligned}
&=x^3+(-y)^3+(2)^3-3x(-y).2\\
&=\{x+(-y)+2\}\{x^2+(-y)^2+2^2-(-y).2-2x-x(-y)\}\\
&=(x-y+2)(x^2+y^2+4+2y-2x+xy).
\end{aligned}$$

Example 3. Resolve into factors x^6+32x^3-64.

The given expression

$$\begin{aligned}
&=x^6+8x^3-64+24x^3\\
&=(x^2)^3+(2x)^3+(-4)^3-3.x^2.2x.(-4)\\
&=\{x^2+2x+(-4)\}\{(x^2)^2+(2x)^2+(-4)^2-2x(-4)\\
&\qquad\qquad-(-4)x^2-x^2.2x\}\\
&=(x^2+2x-4)(x^4+4x^2+16+8x+4x^2-2x^3)\\
&=(x^2+2x-4)(x^4-2x^3+8x^2+8x+16).
\end{aligned}$$

Example 4. Find the quotient of $a^3+b^3+1-3ab$ by $a+b+1$.

Since, $a^3+b^3+1-3ab=a^3+b^3+1^3-3ab.1$
$$=(a+b+1)(a^2+b^2+1^2-b.1-1.a-ab)$$
$$=(a+b+1)(a^2+b^2+1-b-a-ab);$$

$\therefore$ the reqd. quotient $=a^2+b^2+1-b-a-ab$.

EXERCISE 77

Factorise :

1. $x^3+y^3-z^3+3xyz$.
2. $p^3-8q^3-r^3-6pqr$.
3. $8x^3-27y^3-z^3-18xyz$.
4. $a^3+8b^3+1-6ab$.
5. $8a^3+27b^3-64+72ab$.
6. Find the quotient of $x^3-y^3+6xy+8$ by $x-y+2$.
7. Factorise x^6+5x^3+8.
8. Resolve into factors
$$(x-y)^3-(y-z)^3+(z-x)^3+3(y-z)(z-x)(x-y).$$
9. Factorise a^6-18a^3+125.

Find the quotient of :

10. $x^3+27-5y(25y^2-9x)$ by $x^2+25y^2+9+5xy-3x+15y$.
11. $a^3+b^3-c^3+3abc$ by $a+b-c$.
12. $x^3-y^3-1-3xy$ by $x-y-1$.
13. $x^3-8y^3+27z^3+18xyz$ by $x-2y+3z$.
14. $8a^3-27b^3-c^3-18abc$ by $4a^2+9b^2+c^2+6ab+2ac-3bc$.

135. To factorise expressions of the form $(a+b+c)(bc+ca+ab)-abc$.

The expression $=\{a+(b+c)\}\{a(b+c)+bc\}-abc$
$$=a^2(b+c)+a(b+c)^2+bc(b+c)$$
$$=(b+c)\{a^2+a(b+c)+bc\}$$
$$=(b+c)(a+b)(a+c)$$
$$=(b+c)(c+a)(a+b).$$

Cor. 1. $(a+b+c)(bc+ca+ab)-(b+c)(c+a)(a+b)=abc$.

Cor. 2. $(b+c)(c+a)(a+b)+abc=(a+b+c)(bc+ca+ab)$.

136. To factorise expressions of the form

(i) $P+2abc$

and (ii) $P+3abc$, where P stands for any of the equivalent forms

(1) $a^2(b+c)+b^2(c+a)+c^2(a+b)$,

(2) $bc(b+c)+ca(c+a)+ab(a+b)$,

(3) $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$.

(i) Taking the 1st value of P, we have

$$\begin{aligned}
P+2abc &= a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\\
&= a^2(b+c)+a(b^2+2bc+c^2)+b^2c+bc^2\\
&\qquad\qquad [\text{ arranging according to powers of } a\ ]\\
&= a^2(b+c)+a(b+c)^2+bc(b+c)\\
&= (b+c)\{a^2+a(b+c)+bc\}\\
&= (b+c)(a+b)(a+c)=(b+c)(c+a)(a+b).
\end{aligned}$$

(ii) Taking the 2nd value of P, we have

$$\begin{aligned}
P+3abc &= bc(b+c)+ca(c+a)+ab(a+b)+3abc\\
&= bc(b+c)+ca(c+a)+ab(a+b)+abc+abc+abc\\
&= \{bc(b+c)+abc\}+\{ca(c+a)+abc\}+\{ab(a+b)+abc\}\\
&= bc(a+b+c)+ca(c+a+b)+ab(a+b+c)\\
&= (a+b+c)(bc+ca+ab).
\end{aligned}$$

137. To factorise expressions of the type Q, where Q stands for any of the equivalent forms

(1) $a^2(b-c)+b^2(c-a)+c^2(a-b)$,

(2) $bc(b-c)+ca(c-a)+ab(a-b)$,

(3) $-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}$.

From the first form of Q, we have

$$\begin{aligned}
&a^2(b-c)+b^2(c-a)+c^2(a-b)\\
&\qquad= a^2(b-c)-a(b^2-c^2)+b^2c-bc^2\\
&\qquad\qquad\qquad [\text{ arranging according to powers of } a\ ]\\
&\qquad= a^2(b-c)-a(b^2-c^2)+bc(b-c)\\
&\qquad= (b-c)\{a^2-a(b+c)+bc\}\\
&\qquad= (b-c)(a-b)(a-c)=-(b-c)(c-a)(a-b).
\end{aligned}$$

Cor. Putting a^2, b^2 and c^2 for a, b and c respectively in the above, we have

$$\begin{aligned}
&a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)\\
&\qquad= -(b^2-c^2)(c^2-a^2)(a^2-b^2)\\
&\qquad= -(b-c)(c-a)(a-b)(b+c)(c+a)(a+b).
\end{aligned}$$

Example. Factorise $(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)$.

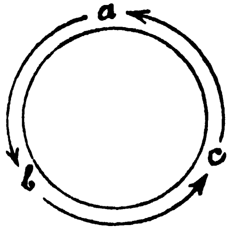
$$\begin{aligned}\text{The exp.}&=(x^2-2ax+a^2)(b-c)+(x^2-2bx+b^2)(c-a)\\&\qquad+(x^2-2cx+c^2)(a-b)\\&=x^2\{(b-c)+(c-a)+(a-b)\}-2x\{a(b-c)+b(c-a)+c(a-b)\}\\&\qquad+\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}\\&\qquad\qquad[\text{ arranged according to powers of } x\ ]\\&=x^2.0-2x.0-(b-c)(c-a)(a-b)=-(b-c)(c-a)(a-b).\end{aligned}$$

138. To factorise $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

$$\begin{aligned}&a^3(b-c)+b^3(c-a)+c^3(a-b)\\&\quad=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)\\&\qquad\qquad[\text{ arranging according to powers of } a\ ]\\&\quad=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}\\&\quad=(b-c)\{-b^2(a-c)-bc(a-c)+a(a^2-c^2)\}\\&\qquad\qquad[\text{ arranging according to powers of } b\ ]\\&\quad=(b-c)(a-c)\{-b^2-bc+a(a+c)\}\\&\quad=(b-c)(a-c)\{c(a-b)+a^2-b^2\}\quad[\text{ arranging the last factor in powers of } c\ ]\\&\quad=(b-c)(a-c)(a-b)(c+b+a)=-(b-c)(c-a)(a-b)(a+b+c).\end{aligned}$$

Note. *It must be observed that (i) as soon as the given expression is arranged according to powers of a, one of the factors, namely, b − c, becomes obvious ; (ii) when the expression within larger brackets is arranged according to powers of b, the next factor, a − c, becomes obvious ; (iii) when the expression now within larger brackets is arranged according to powers of c, the third factor, a − b, becomes obvious.*

139. Cyclic Order. There is a certain peculiarity in the arrangement of three letters a, b, c in the different expressions of Arts. 137 and 138. Thus, in any of the equivalent forms of Q in Art. 137, we get the second term by changing a, b, c of the first into b, c, a respectively; the third term by changing b, c, a of the second into c, a, b respectively; and the first term by changing c, a, b of the third into a, b, c respectively. The orders in which the letters a, b, c are to be changed successively will be best understood in the following way :



Let the letters a, b, c be arranged round the circumference of a circle as shown in the diagram, starting from the letter a and moving in the direction of the arrow-head we notice that the order of the letters is abc. Similarly starting from b and c successively and moving in the same direction, we notice that the orders of the letters are bca and cab respectively.

The letters a, b, c when arranged in this manner, are said to be in *cyclic order*.

Thus, a, b, c are arranged in cyclic order in the following :

(i) $b+c$, $c+a$ and $a+b$;

(ii) $b-c$, $c-a$ and $a-b$;

(iii) $b+c-a$, $c+a-b$ and $a+b-c$;

(iv) bc, ca and ab ;

(v) $a^2(b-c)$, $b^2(c-a)$ and $c^2(a-b)$;

and so on.

140. To factorise $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.

In this expression also the letters occur in *cyclic order* and we can at once proceed as in the last example.

$$
\begin{aligned}
&a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)\\
&\quad=a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)\\
&\qquad\qquad[\text{ arranged according to powers of } a\ ]\\
&\quad=(b-c)\{a^3(b+c)-a^2(b^2+bc+c^2)+b^2c^2\}\\
&\quad=(b-c)\{-b^2(a^2-c^2)+ba^2(a-c)+a^2c(a-c)\}\\
&\qquad\qquad[\text{ arranged according to powers of } b\ ]\\
&\quad=(b-c)(a-c)\{-b^2(a+c)+ba^2+a^2c\}\\
&\quad=(b-c)(a-c)\{c(a^2-b^2)+ab(a-b)\}\\
&\qquad\qquad[\text{ arranged according to powers of } c\ ]\\
&\quad=(b-c)(a-c)(a-b)\{c(a+b)+ab\}\\
&\quad=-(b-c)(c-a)(a-b)(bc+ca+ab).
\end{aligned}
$$

141. To factorise $(a+b+c)^3-a^3-b^3-c^3$. [See Art. 132, Cor.]

142. To factorise $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$.

The given expression

$$
\begin{aligned}
&=4b^2c^2-(a^4+b^4+c^4+2b^2c^2-2c^2a^2-2a^2b^2)\\
&=(2bc)^2-(a^2-b^2-c^2)^2\\
&=\{2bc+(a^2-b^2-c^2)\}\{2bc-(a^2-b^2-c^2)\}\\
&=\{a^2-(b^2-2bc+c^2)\}\{(b^2+2bc+c^2)-a^2\}\\
&=\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}\\
&=\{a+(b-c)\}\{a-(b-c)\}\{(b+c)+a\}\{(b+c)-a\}\\
&=(a+b-c)(a-b+c)(b+c+a)(b+c-a)\\
&=(a+b+c)(b+c-a)(c+a-b)(a+b-c).
\end{aligned}
$$

EXERCISE 78

Resolve into factors :

1. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
2. $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$.
3. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.
4. $bc(b^3-c^3)+ca(c^3-a^3)+ab(a^3-b^3)$.
5. $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$.
6. $x(y-z)^2+y(z-x)^2+z(x-y)^2+8xyz$.
7. $x^2(y-z)^3+y^2(z-x)^3+z^2(x-y)^3$. 8. $(y-z)^5+(z-x)^5+(x-y)^5$.
9. $(x^2+2x+4)(y-z)+(y^2+2y+4)(z-x)+(z^2+2z+4)(x-y)$.
10. $\{x^2-(b+c)x+bc\}(b-c)+\{x^2-(c+a)x+ca\}(c-a)$
$+\{x^2-(a+b)x+ab\}(a-b)$.
11. $(x+b)(x+c)(b-c)+(x+c)(x+a)(c-a)+(x+a)(x+b)(a-b)$.
12. $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc$.
13. $8x^3-(y-z)^3-(z+x)^3-(x-y)^3$.
14. $a^6(b^3-c^3)+b^6(c^3-a^3)+c^6(a^3-b^3)$.
15. $x^6(y^4-z^4)+y^6(z^4-x^4)+z^6(x^4-y^4)$.
16. $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$.
17. $yz(y+z)+zx(z+x)+xy(x+y)-x^3-y^3-z^3-2xyz$.
18. $(x+1)^2(y-z)+(y+1)^2(z-x)+(z+1)^2(x-y)$.
19. $(x+1)^3(y-z)+(y+1)^3(z-x)+(z+1)^3(x-y)$.
20. $x(y-z)^3+y(z-x)^3+z(x-y)^3$.
21. $2b^2c^2y^2z^2+2c^2a^2z^2x^2+2a^2b^2x^2y^2-a^4x^4-b^4y^4-c^4z^4$.
22. $72y^2z^2+18z^2x^2+8x^2y^2-x^4-16y^4-81z^4$.
23. Find the value of $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$,
when $b+c-a=7$, $c+a-b=10$ and $a+b-c=3$.
24. Evaluate $a^2(b+c)+b^2(c+a)+c^2(a+b)$,
when $a+b+c=20$, $bc+ca+ab=18$ and $abc=37$.
25. Evaluate $(a+b+c)^3-a^3-b^3-c^3+3abc$,
when $a+b+c=13$ and $a^2+b^2+c^2=69$.

143. Factors of Reciprocal Expressions.

Definition. An algebraical expression in which coefficients of the terms equidistant from the beginning and end are same, is called a **reciprocal or recurring** expression.

Thus, $x^4+4x^3+5x^2+4x+1$ is a reciprocal expression.

Example 1. Resolve into factors $x^4+2x^3+3x^2+2x+1$.

The expression $=(x^4+1)+(2x^3+2x)+3x^2$

[collecting terms with equal coefficients]

$$=\{(x^2+1)^2-2x^2\}+2x(x^2+1)+3x^2$$
$$=(x^2+1)^2+2x(x^2+1)+3x^2-2x^2$$
$$=(x^2+1)^2+2(x^2+1).x+x^2$$
$$=\{(x^2+1)+x\}^2=(x^2+x+1)^2.$$

Example 2. Factorise $a^4-5a^3-12a^2-5a+1$.

The expression $=(a^4+1)-(5a^3+5a)-12a^2$

[collecting terms with equal coefficients]

$$=\{(a^2+1)^2-2a^2\}-5a(a^2+1)-12a^2$$
$$=(a^2+1)^2-5(a^2+1).a-2a^2-12a^2$$
$$=x^2-5xa-14a^2 \qquad [\text{ putting } x \text{ for } a^2+1\ ]$$
$$=(x+2a)(x-7a)$$
$$=(a^2+1+2a)(a^2+1-7a) \qquad [\text{ restoring the value of } x\ ]$$
$$=(a+1)^2(a^2-7a+1).$$

144. Factors by trial.

Example 1. Resolve into factors x^3-2x^2-5x+6.

On inspection we find that the given expression can be split up into parts each of which is divisible by $x-1$.

Thus, the exp. $=x^3-x^2-x^2+x-6x+6$

$$=(x^3-x^2)-(x^2-x)-(6x-6)$$
$$=x^2(x-1)-x(x-1)-6(x-1)$$
$$=(x-1)(x^2-x-6)=(x-1)(x+2)(x-3).$$

Note. *It is important for the student to observe that the given expression vanishes when* 1, -2, *or* 3 *is substituted for* x. *Thus, it may be remembered as a general rule that if any expression involving* x *vanishes when* $x=a$, $x-a$ *is a factor of that expression.*

The above general rule leads to the following particular cases :

(1) If in any expression containing integral powers of x, the sum of the coefficients is zero, $x-1$ is a factor of that expression.

(2) If in any expression containing integral powers of x, the sum of the coefficients of odd powers of x is equal to the sum of the remaining coefficients, $x+1$ is a factor of that expression.

Thus, in example 1 above, the sum of the coefficients of the expression $=1+(-2)+(-5)+6=1-2-5+6=0$.

Hence, $x-1$ is a factor of the expression.

Again, in the expression x^3+3x^2+3x+1, the odd powers of x are x^3 and x.

The sum of their coefficients $=1+3=4$ and the sum of the remaining coefficients $=3+1=4$.

These two sums being equal, the expression x^3+3x^2+3x+1 must have $(x+1)$ as a factor.

Example 2. Resolve into factors $x^3+6x^2+11x+6$.

The sum of the coefficients of odd powers of $x=1+11=12$ and the sum of the remaining coefficients $=6+6=12$.

These two sums being equal, $x+1$ must be a factor of the given expression. Now, grouping the terms into parts each of which is divisible by $x+1$, we have

$$\begin{aligned}\text{the expression} &= x^3+x^2+5x^2+5x+6x+6\\ &=(x^3+x^2)+(5x^2+5x)+(6x+6)\\ &=x^2(x+1)+5x(x+1)+6(x+1)\\ &=(x+1)(x^2+5x+6)=(x+1)(x+2)(x+3).\end{aligned}$$

Example 3. Resolve into factors $8x^3+16x-9$.

Putting y for $2x$, the given expression

$$=(2x)^3+8.2x-9=y^3+8y-9.$$

Now, the sum of the coefficients of y^3+8y-9

$$=1+8-9=0.$$

Hence, $y-1$ is a factor of this expression. Next, arranging it into parts such that each part is divisible by $y-1$, we have

$$\begin{aligned}y^3+8y-9&=y^3-y+9y-9=y(y^2-1)+9(y-1)\\ &=(y-1)\{y(y+1)+9\}=(y-1)(y^2+y+9)\\ &=(2x-1)(4x^2+2x+9). \qquad [\text{ restoring the value of } y\ ]\end{aligned}$$

Example 4. Resolve into factors $x^5+4x^4-13x^3-13x^2+4x+1$.

We notice that the sum of the coefficients of odd powers of x

$$=1+(-13)+4=-8,$$

and the sum of the remaining coefficients

$$=4+(-13)+1=-8.$$

These two sums being equal, $x+1$ must be a factor.

Now, grouping the terms into parts each of which is divisible by $x+1$, we have the given expression

$$\begin{aligned}&=(x^5+x^4)+(3x^4+3x^3)-(16x^3+16x^2)+(3x^2+3x)+(x+1)\\ &=x^4(x+1)+3x^3(x+1)-16x^2(x+1)+3x(x+1)+(x+1)\\ &=(x+1)(x^4+3x^3-16x^2+3x+1).\end{aligned}$$

The factor $x^4+3x^3-16x^2+3x+1$ is a reciprocal expression. Hence proceeding as in Art. 143, we have

$$\begin{aligned}
&x^4+3x^3-16x^2+3x+1\\
&\quad=(x^4+1)+(3x^3+3x)-16x^2,\\
&\qquad\qquad\text{[grouping terms with equal coefficients]}\\
&\quad=\{(x^2+1)^2-2x^2\}+3x(x^2+1)-16x^2\\
&\quad=(x^2+1)^2+3(x^2+1).x-2x^2-16x^2\\
&\quad=y^2+3yx-18x^2 \qquad \text{[putting } y \text{ for } x^2+1 \text{]}\\
&\quad=(y-3x)(y+6x)\\
&\quad=(x^2+1-3x)(x^2+1+6x) \qquad \text{[restoring the value of } y \text{]}\\
&\quad=(x^2-3x+1)(x^2+6x+1).
\end{aligned}$$

Hence, the given expression $=(x+1)(x^2-3x+1)(x^2+6x+1)$.

Example 5. Resolve into factors $x^3+x^2-21x-38$.

By trial we find that the given expression vanishes when $x=-2$.

Hence, $x-(-2)=x+2$ is a factor. Thus, we have

$$\begin{aligned}
x^3+x^2-21x-38&=(x^3+2x^2)-(x^2+2x)-(19x+38)\\
&\qquad\text{[splitting into parts divisible by } x+2 \text{]}\\
&=x^2(x+2)-x(x+2)-19(x+2)\\
&=(x+2)(x^2-x-19).
\end{aligned}$$

145. Factors of Homogeneous expressions of two dimensions.

The following examples will illustrate the process :

Example 1. Resolve into factors $6a^2+7ab+2b^2+11ac+7bc+3c^2$.

If $a=0$, the expression becomes $2b^2+7bc+3c^2$,

which $=(2b+c)(b+3c)$. ··· (1)

If $b=0$, the expression reduces to $6a^2+11ac+3c^2$,

which $=(3a+c)(2a+3c)$. ··· (2)

If $c=0$, the expression reduces to $6a^2+7ab+2b^2$,

which $=(3a+2b)(2a+b)$. ··· (3)

Now, comparing the results (1), (2) and (3), we notice that the given expression must be $=(3a+2b+c)(2a+b+3c)$, [since it is these factors which reduce to the form (1) when $a=0$, to the form (2) when $b=0$, and to the form (3) when $c=0$].

Alternative Method : Arranging the terms in descending powers of any one of the letters, say, a, we have

$$\begin{aligned}
\text{the given expression}&=6a^2+(7b+11c)a+(2b^2+7bc+3c^2)\\
&=6a^2+(7b+11c)a+(2b+c)(b+3c).
\end{aligned}$$

Now, split the product of (the coefficient of a^2) and (the term independent of a) into two factors whose sum = the coefficient of a.

Thus, split $6 \times (2b+c)(b+3c)$ into factors whose sum $=7b+11c$.

By trial, the factors are $2(2b+c)$ and $3(b+3c)$.

Hence, the given expression

$$\begin{aligned}&=6a^2+2(2b+c)a+3(b+3c)a+(2b+c)(b+3c)\\&=2a\{3a+(2b+c)\}+(b+3c)\{3a+(2b+c)\}\\&=(3a+2b+c)(2a+b+3c).\end{aligned}$$

Example 2. Factorise $x^2-3xy+2y^2-2yz-4z^2$.

The given expression is homogeneous in x, y and z.

If $x=0$, the given expression reduces to $2y^2-2yz-4z^2$,

$$\begin{aligned}\text{which}&=2(y^2-yz-2z^2)=2(y+z)(y-2z)\\&=(2y+2z)(y-2z). \quad \cdots \quad \cdots \quad (1)\end{aligned}$$

If $y=0$, the given expression reduces to x^2-4z^2,

$$\text{which}=(-x+2z)(-x-2z). \quad \cdots \quad \cdots \quad (2)$$

If $z=0$, the given expression reduces to $x^2-3xy+2y^2$,

$$\text{which}=(-x+2y)(-x+y). \quad \cdots \quad \cdots \quad (3)$$

Now, comparing the results (1), (2) and (3), the given expression is evidently equal to $(-x+2y+2z)(-x+y-2z)=(x-2y-2z)(x-y+2z)$.

Alternative Method : Arranging the expression in descending powers of any one of the letters, say, x, we have

the expression $=x^2-3yx+(2y^2-2yz-4z^2)=x^2-3yx+2(y+z)(y-2z)$.

Next, splitting the product of (the coefficient of x^2) $\times$ (the term independent of x), *i.e.*, $2(y+z)(y-2z)$ into two factors whose sum

= the coefficient of x, *i.e.*, $-3y$,

we notice by trial that these factors are $-2(y+z)$ and $-(y-2z)$.

Hence, the given expression

$$\begin{aligned}&=x^2-2(y+z)x-(y-2z)x+2(y+z)(y-2z)\\&=x\{x-2(y+z)\}-(y-2z)\{x-2(y+z)\}\\&=(x-2y-2z)(x-y+2z).\end{aligned}$$

146. Factors of general expressions of the second degree in two or more letters.

Example. Factorise $6a^2+7ab+2b^2+11a+7b+3$.

Arranging the expression in descending powers of any one of the letters, say, a,

$$\begin{aligned}\text{the given expression}&=6a^2+(7b+11)a+(2b^2+7b+3)\\&=6a^2+(7b+11)a+(2b+1)(b+3).\end{aligned}$$

Now, split the product of (the coefficient of a^2) and (the term independent of a), *i.e.*, $6\times(2b+1)(b+3)$ into two factors whose sum $=$ the coefficient of a, *i.e.*, $7b+11$.

The factors are evidently $2(2b+1)$ and $3(b+3)$.

Hence, the given expression

$$=6a^2+2(2b+1)a+3(b+3)a+(2b+1)(b+3)$$
$$=2a\{3a+(2b+1)\}+(b+3)\{3a+(2b+1)\}$$
$$=(3a+2b+1)(2a+b+3).$$

147. Factors found by suitable arrangement and grouping of terms.

There are some expressions of which the factors become obvious after re-arrangement of the terms in a certain way, but there are others again which do not exactly come under this category. Hence, no definite method can be specified as applicable to all cases that may be practically included in this article. We must, therefore, content ourselves only with directing the student's attention to a few important cases, more or less isolated, which will fairly introduce him to the subject under consideration.

Example 1. Resolve into factors $(3x^2-4b^2)a+(3a^2-4x^2)b$.

The given expression $=3x^2a-4b^2a+3a^2b-4x^2b$

$$=(3x^2a+3a^2b)-(4b^2a+4x^2b)$$

[taking the 3rd term with the 1st, and the 4th with the 2nd]

$$=3a(x^2+ab)-4b(ab+x^2)$$
$$=(x^2+ab)(3a-4b).$$

Example 2. Resolve into factors $x^4+x^2y^2-y^2z^2-z^4$.

Combining the 4th term with the 1st, and the 2nd with the 3rd, we have

$$x^4+x^2y^2-y^2z^2-z^4=(x^4-z^4)+(x^2y^2-y^2z^2)$$
$$=(x^2+z^2)(x^2-z^2)+y^2(x^2-z^2)$$
$$=(x^2-z^2)\{(x^2+z^2)+y^2\}$$
$$=(x+z)(x-z)(x^2+y^2+z^2).$$

Example 3. Resolve into factors $x^3+7x^2-21x-27$.

The given expression $=(x^3-27)+(7x^2-21x)$

$$=(x-3)(x^2+3x+9)+7x(x-3)$$
$$=(x-3)\{(x^2+3x+9)+7x\}$$
$$=(x-3)(x^2+10x+9)=(x-3)(x+9)(x+1).$$

Example 4. Resolve into factors $4a^2+12ab+9b^2-8a-12b$.

$$\begin{aligned}\text{The given exp.}&=(4a^2+12ab+9b^2)-(8a+12b)\\&=(2a+3b)^2-4(2a+3b)\\&=(2a+3b)\{(2a+3b)-4\}\\&=(2a+3b)(2a+3b-4).\end{aligned}$$

Example 5. Resolve into factors $2a^2-2bc+6b^2+ac-7ab$.

We observe that the 1st, 3rd and 5th terms are of the second degree in a and b, whilst the 2nd and 4th terms are of the first degree in those letters.

Putting the former set of terms in one group and the latter in another, we have

$$\begin{aligned}\text{the given exp.}&=(2a^2-7ab+6b^2)+c(a-2b)\\&=(a-2b)(2a-3b)+c(a-2b)\\&=(a-2b)(2a-3b+c).\end{aligned}$$

Example 6. Resolve into factors $x^2-y^2-z^2+2yz+x+y-z$.

$$\begin{aligned}\text{The given exp.}&=(x^2-y^2-z^2+2yz)+(x+y-z)\\&=\{x^2-(y-z)^2\}+(x+y-z)\\&=(x+y-z)(x-y+z)+(x+y-z)\\&=(x+y-z)\{(x-y+z)+1\}\\&=(x+y-z)(x-y+z+1).\end{aligned}$$

Example 7. Resolve into factors

$$a^2x^3+a^5-2abx^3+b^2x^3+a^3b^2-2a^4b.$$

We observe that the 1st, 3rd and 4th terms have got x^3 for a common factor, whilst the others have got a^3.

Hence, putting the 1st, 3rd and 4th terms in one group and the remaining terms in another, we have

$$\begin{aligned}\text{the given exp.}&=(a^2x^3-2abx^3+b^2x^3)+(a^5+a^3b^2-2a^4b)\\&=x^3(a^2-2ab+b^2)+a^3(a^2+b^2-2ab)\\&=(a^2-2ab+b^2)(x^3+a^3)\\&=(a-b)^2(x+a)(x^2-xa+a^2).\end{aligned}$$

EXERCISE 79

Resolve into factors :

1. x^3+x^2+x+1.
2. x^3+x^2-x-1.
3. x^3-x^2-x+1.
4. $bc(a^2+1)+a(b^2+c^2)$.
5. $x^4-ab^3+xb^3-x^3a$.
6. $ab(x^2+y^2)+xy(a^2+b^2)$.
7. $x^2+xy-yz-z^2$.
8. $xb-ac-xc+ab$.

9. $(2x^2+3b^2)a-(2a^2+3x^2)b.$ 10. $a(a+c)-b(b+c).$

11. $4a^2+8ac-12bc-9b^2.$ 12. $a^2x^2+acxz-b^2y^2-bcyz.$

13. $x^4-y^3z+y^2x^2-y^2z^2.$ 14. $16x^2-15ab+12bx-25a^2.$

15. $a^2(a+2b)+b^2(2a+b).$ 16. $m^3-2m^2n+2mn^2-n^3.$

17. $a^4+2a^3b-2ab^3-b^4.$ 18. $x^3(x-2y)+y^3(2x-y).$

19. $a^3+5a^2+10a+8.$ 20. $x^3-17x^2+85x-125.$

21. $8a^3+18a^2b-27ab^2-27b^3.$ 22. $x^2-2xy+y^2-x+y.$

23. $4a^2-4ab+b^2-6a+3b.$ 24. $x^4-2ax^3+2a^2x^2-2a^3x+a^4.$

25. $a^4-3a^3b+4a^2b^2-6ab^3+4b^4.$

26. $a^2+3ab+2b^2+ac+2bc.$ 27. $x^2-4xy+3y^2+xz-3yz.$

28. $m^2+2pm-5mn-4pn+6n^2.$ 29. $a^2-10ab-15bc+21b^2+5ac.$

30. $2x^2+4a(4b-3a)+x(4b+5a).$ 31. $a^2-3a(2b-1)+4b(2b-3).$

32. $3x(x+2)-2y(4x-1)-3y^2.$ 33. $a^2-b^2-c^2-2bc+a-b-c.$

34. $x^2-4y^2-9z^2+12yz+4x-8y+12z.$

35. $9x^2-4z^2-24xy+16y^2+20y-15x+10z.$

36. $2a^2x^4-5a^4x^2+3a^6-2b^2x^4+5a^2b^2x^2-3a^4b^2.$

37. $2x^3+(2a-3b)x^2-(2b+3ab)x+3b^2.$

38. $(a^2+b^2)x^2-a^2b(2a+b)+a(2bx^2-a^3).$

39. $2a^4-5a^3+6a^2-5a+2.$ 40. $a^5-4a^4-13a^3+13a^2+4a-1.$

41. $2x^2+6xy+4y^2+5xz+6yz+2z^2.$

42. $2x^2+xy-3y^2-xz-4yz-z^2.$ 43. $a^8-5a^6-12a^4-5a^2+1.$

44. $4x^2-4xy-3y^2+12yz-9z^2.$

45. $x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5.$ 46. $x^3+7x^2+14x+8.$

148. Miscellaneous Examples.

Example 1. Resolve into factors $a^3+7ab^2-22b^3.$

We find that the expression can be split up into parts each of which is divisible by $a-2b$ in either of the two following ways :

(i) $(a^3-8b^3)+7b^2(a-2b)$;
(ii) $a(a^2-4b^2)+11b^2(a-2b).$

Hence, choosing the former way, we have

$$\begin{aligned}a^3+7ab^2-22b^3&=(a^3-8b^3)+7b^2(a-2b)\\&=(a-2b)\{(a^2+2ab+4b^2)+7b^2\}\\&=(a-2b)(a^2+2ab+11b^2).\end{aligned}$$

Example 2. Resolve into factors $x^2+2(a^2+b^2)+3ax-b(3x+5a)$.

Arranging the expression according to descending powers of x, we have it

$$\begin{aligned}
&=x^2+3(a-b)x+(2a^2-5ab+2b^2)\\
&=x^2+3(a-b)x+(2a-b)(a-2b)\\
&=x^2+\{(2a-b)+(a-2b)\}x+(2a-b)(a-2b)\\
&=x\{x+(2a-b)\}+(a-2b)\{x+(2a-b)\}\\
&=\{x+(2a-b)\}\{x+(a-2b)\}\\
&=(x+2a-b)(x+a-2b).
\end{aligned}$$

Example 3. Resolve into factors $x^2-6xy+8y^2-z^2+2yz$.

$$\begin{aligned}
\text{The given expression}&=(x^2-6xy+9y^2)-(y^2+z^2-2yz)\\
&=(x-3y)^2-(y-z)^2\\
&=\{(x-3y)+(y-z)\}\{(x-3y)-(y-z)\}\\
&=(x-2y-z)(x-4y+z).
\end{aligned}$$

Example 4. Resolve into factors $(a^2-b^2)(x^2-y^2)+4abxy$.

$$\begin{aligned}
\text{The given expression}&=a^2x^2-a^2y^2-b^2x^2+b^2y^2+4abxy\\
&=(a^2x^2+b^2y^2+2abxy)-(a^2y^2+b^2x^2-2abxy)\\
&=(ax+by)^2-(ay-bx)^2\\
&=\{(ax+by)+(ay-bx)\}\{(ax+by)-(ay-bx)\}\\
&=\{(a-b)x+(a+b)y\}\{(a+b)x-(a-b)y\}.
\end{aligned}$$

Example 5. Resolve into factors $x^4+6x^3+4x^2-15x+6$.

$$\begin{aligned}
\text{The given expression}&=(x^4+6x^3+9x^2)-(5x^2+15x)+6\\
&=(x^2+3x)^2-5(x^2+3x)+6\\
&=\{(x^2+3x)-2\}\{(x^2+3x)-3\}\\
&=(x^2+3x-2)(x^2+3x-3).
\end{aligned}$$

Example 6. Resolve into factors $x^4+2x^3y+3x^2y^2+2xy^3+y^4$.

$$\begin{aligned}
\text{The given expression}&=(x^4+2x^2y^2+y^4)+x^2y^2+(2x^3y+2xy^3)\\
&=(x^2+y^2)^2+(xy)^2+2(xy)(x^2+y^2)\\
&=\{(x^2+y^2)+xy\}^2=(x^2+xy+y^2)^2.
\end{aligned}$$

Example 7. Resolve into factors $(x-1)(x-2)(x+3)(x+4)+4$.

$$\begin{aligned}
(x-1)(x-2)(x+3)(x+4)&=\{(x-1)(x+3)\}\{(x-2)(x+4)\}\\
&=(x^2+2x-3)(x^2+2x-8).
\end{aligned}$$

Hence, putting z for x^2+2x,

$$\begin{aligned}
\text{the given expression}&=(z-3)(z-8)+4=z^2-11z+28=(z-4)(z-7)\\
&=(x^2+2x-4)(x^2+2x-7).
\end{aligned}$$

Note. *The student must carefully notice why in multiplying together the four binomials,* $x-1$, $x-2$, $x+3$, $x+4$, *we combine* $x+3$ *with* $x-1$, *and* $x+4$ *with* $x-2$.

Example 8. $x+y=a$ and $xy=b^2$, find the value of (i) x^4+y^4 and (ii) $x^3-x^2y-xy^2+y^3$ in terms of a and b.

(i) $x^4+y^4=(x^2+y^2)^2-2x^2y^2=\{(x+y)^2-2xy\}^2-2x^2y^2$,

and $\therefore$ the required value $=(a^2-2b^2)^2-2b^4=a^4-4a^2b^2+2b^4$.

(ii)
$$\begin{aligned}x^3-x^2y-xy^2+y^3&=x^2(x-y)-y^2(x-y)\\&=(x-y)(x^2-y^2)=(x-y)^2(x+y)\\&=\{(x+y)^2-4xy\}(x+y)=(a^2-4b^2)a.\end{aligned}$$

Example 9. Find the value of $x^4-x^3+x^2+2$, when $x^2+2=2x$.

$$\begin{aligned}x^4-x^3+x^2+2&=(x^4+x^3+x^2)-2(x^3-1)\\&=x^2(x^2+x+1)-2(x-1)(x^2+x+1)\\&=(x^2+x+1)\{x^2-2(x-1)\}\\&=(x^2+x+1)(x^2-2x+2),\end{aligned}$$

and $\therefore$ the required value $=(x^2+x+1)\times 0=0$.

Example 10. Find the value of

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2, \text{ when } a+b=c$$

The given expression

$$\begin{aligned}&=-(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\\&=-(a+b-c)(a-b+c)(a+b+c)(b+c-a), \qquad [\text{Art. } 142]\end{aligned}$$

and $\therefore$ $=0$, when $a+b=c$.

EXERCISE 80

Resolve into factors :

1. $x^3+8x^2+19x+12$.
2. $x^3+9x^2+26x+24$.
3. $x^3-6x^2+11x-6$.
4. $x^3+5x^2-2x-24$.
5. x^3-4x^2+x+2.
6. x^3+5x^2-2x-6.
7. $x^3-6x^2+13x-10$.
8. $x^4-3x^3-9x^2+12x+20$.
9. $x^4-3x^3-x^2+13x-10$.
10. $x^4-5x^3+x^2+13x+6$.
11. $x^4+5x^3-8x^2-30x+36$.
12. $x^4-7x^3+9x^2+26x-56$.
13. $x^3-7x^2+13x-15$.
14. $x^3-5x+12$.
15. x^3-6x^2+32.
16. $2x^3-3x^2-4$.
17. $x^3-9xy^2-10y^3$.
18. $a^3+4a^2b-9b^3$.
19. $5a^3-3a^2b-28b^3$.
20. $8x^3+4x-3$.
21. $2x^3+5x^2-4x-3$.
22. x^3-3x-2.
23. $2a^3-a^2b-b^3$.
24. $3x^3+8x^2-8x-3$.
25. $x^3-6xy^2+9y^3$.
26. $x^2+bx-(a^2-3ab+2b^2)$.

27. $x^4+4abx^2y^2-(a^2-b^2)^2y^4$.

28. $a^4+2(x^2+y^2)a^2b^2+(x^2-y^2)^2b^4$.

29. $a^2+(x+y)a-2x^2+5xy-2y^2$.

30. $x(x+a)-2a^2+3b(a+x)+2b^2$.

31. $x^2+4xy+3y^2+2yz-z^2$.

32. $x^2+2xy-8y^2-4z^2+12yz$.

33. $a^2+2ab-c^2+2bc$.

34. x^2+2x-y^2+2y.

35. x^2+6x-y^2+4y+5.

36. $a^2+4ab-5b^2-c^2+6bc$.

37. $x^2-6xy+5y^2-z^2+4yz$.

38. $x^2-10xy+16y^2-4z^2+12yz$.

39. $a^2-12ab-13b^2-9c^2+42bc$.

40. $x^2+12xy-9z^2+36yz$.

41. $x^2-14xy-15y^2-25z^2+80yz$.

42. $4a^2-4ab-3b^2+12bc-9c^2$.

43. $x^4+6x^3+8x^2+6x-9$.

44. $a^4-4a^3b-5a^2b^2+6ab^3-b^4$.

45. $4x^4-20x^3+24x^2+6x-9$.

46. $x^4-2x^3+2x^2-2x+1$.

47. $a^4-9a^2+30a-25$.

48. $a^2-2abx-(ac-b^2)x^2+bcx^3$.

49. $x^4y^4+x^2y^2-z^2+2xyz+1$.

50. $x^2(y^2-z^2)+4xyz-y^2+z^2$.

51. $(a^2-b^2)(x^2+y^2)+2(a^2+b^2)xy$.

52. $x^4-4x^3-x^2+10x+4$.

53. $a^4-6a^3+15a^2-18a+5$.

54. $4x^4+12x^3-5x^2-21x+12$.

55. $x^4-5x^3y+6x^2y^2-5xy^3+y^4$.

56. $x^4-5x^3+14x^2-20x+16$.

57. $a^4-7a^3b+14a^2b^2-14ab^3+4b^4$.

58. $x^4+4x^3-11x^2+20x+25$.

59. $a^4+4a^3b-10a^2b^2+4ab^3+b^4$.

60. $x^4+8x^3+24x^2+32x-20$.

61. $(x+1)(x+3)(x-4)(x-6)+13$.

62. $(x+2)(x+3)(x+4)(x+5)-360$.

63. $x(2x+1)(x-2)(2x-3)-63$.

64. Find the value of $xy(x+y)+yz(y+z)+zx(z+x)+3xyz$, when $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$.

65. Find the value of $(y-z)(y^2-z^2)-x\{(y-z)^2+x(y+z)\}+x^3$, when $x=a^2-b^2$, $y=b^2-c^2$, $z=a^2-c^2$.

66. Find the value of $x^2-2(y-2)x-3y^2+20y-32$, when $x+y=4$.

67. Find the value of $x^2-y^2+4x+14y-45$, when $x+y=25$ and $x-y=6$.

68. Find the value of $8xy(x^2+y^2)$, $\left.\begin{array}{l}\text{when } x+y=\sqrt{3}\\ \text{when } x-y=\sqrt{2}\end{array}\right\}$

II. Identities

149. We shall in this section consider some important identities of a somewhat harder type than those treated of in Chapter XIII and establish their truth with the aid of the foregoing formulæ and principles.

The following general method should be remembered :

Reduce the more difficult side of the identity to the form of the other with the help of the preceding formulæ.

If both the sides of the identity are complex, reduce each to its simplest form and establish their equality.

Sometimes an identity follows easily by transposition of terms or addition of some terms to both its sides.

Sometimes an identity may be proved very easily by substituting a new letter for a group of letters occurring in the identity. Make such substitutions wherever necessary.

The following examples will illustrate the process :

Example 1. Prove that

$$(x-a)(x-b)(a-b)+(x-b)(x-c)(b-c)+(x-c)(x-a)(c-a)$$
$$=-(b-c)(c-a)(a-b).$$

Substituting p for $x-a$, q for $x-b$ and r for $x-c$, we have

$$q-p=a-b,\ r-q=b-c,\ p-r=c-a.$$

$\therefore$ the left side $=pq(q-p)+qr(r-q)+rp(p-r)$

$=-(q-p)(r-q)(p-r)$

$=-(a-b)(b-c)(c-a)$. [restoring values of $q-p$, $r-q$, $p-r$]

Example 2. Prove that

$$(y+z)^2(2x+y+z)+(z+x)^2(x+2y+z)+(x+y)^2(x+y+2z)$$
$$+2(y+z)(z+x)(x+y)=(2x+y+z)(x+2y+z)(x+y+2z).$$

Putting a for $y+z$, b for $z+x$, c for $x+y$, we have

$$b+c=2x+y+z,\ c+a=x+2y+z,\ a+b=x+y+2z.$$

$\therefore$ the left side $=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$

$=(b+c)(c+a)(a+b)=(2x+y+z)(x+2y+z)(x+y+2z).$

Example 3. Prove that $x^3+6(y+z)x^2+12(y+z)^2x+8(y+z)^3$

$=4(3x+2y+6z)y^2+(x+6y+2z)(x+2z)^2$. [M. M. 1881]

The left side $=x^3+3x^2.\{2(y+z)\}+3.x.\{2(y+z)\}^2+\{2(y+z)\}^3$

$=\{x+2(y+z)\}^3=(x+2y+2z)^3=\{2y+(x+2z)\}^3$

$=(2y)^3+3(2y)^2(x+2z)+3(2y)(x+2z)^2+(x+2z)^3$

$=8y^3+12y^2(x+2z)+6y(x+2z)^2+(x+2z)^3$

$=4y^2\{2y+3(x+2z)\}+\{6y+(x+2z)\}(x+2z)^2$

$=4(3x+2y+6z)y^2+(x+6y+2z)(x+2z)^2.$

Example 4. Prove that

$$x^3+y^3+z^3+24xyz=(x+y+z)^3-3\{x(y-z)^2+y(z-x)^2+z(x-y)^2\}.$$

By transposition of terms, this identity is equivalent to the form

$$3\{x(y-z)^2+y(z-x)^2+z(x-y)^2\}+24xyz$$
$$=(x+y+z)^3-x^3-y^3-z^3. \quad \cdots \quad (1)$$

If the latter identity can be established, the former can be deduced by transposing terms.

Now,
$$3\{x(y-z)^2+y(z-x)^2+z(x-y)^2\}+24xyz$$
$$=3\{x(y^2-2yz+z^2)+y(z^2-2zx+x^2)+z(x^2-2xy+y^2)\}+24xyz$$
$$=3\{x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)-2xyz$$
$$-2yzx-2zxy+8xyz\}$$
$$=3\{x(y^2+z^2)+y(z^2+x^2)+z(x^2+y^2)+2xyz\}$$
$$=3(y+z)(z+x)(x+y)=(x+y+z)^3-x^3-y^3-z^3.$$

∴ by transposition,
$$x^3+y^3+z^3+24xyz=(x+y+z)^3-3\{x(y-z)^2+y(z-x)^2+z(x-y)^2\}.$$

Example 5. Prove that
$$-x(b-c)(c-a)(a-b)=a(b-c)(x-b)(x-c)+b(c-a)(x-c)(x-a)$$
$$+c(a-b)(x-a)(x-b).$$

The 1st expression of the right side
$$=a(b-c)\{x^2-x(b+c)+bc\}$$
$$=x^2a(b-c)-xa(b^2-c^2)+abc(b-c).$$

The 2nd expression of the right side
$$=b(c-a)\{x^2-x(c+a)+ca\}$$
$$=x^2b(c-a)-xb(c^2-a^2)+abc(c-a).$$

The 3rd expression of the right side
$$=c(a-b)\{x^2-x(a+b)+ab\}$$
$$=x^2c(a-b)-xc(a^2-b^2)+abc(a-b).$$

∴ the right side (adding the columns vertically)
$$=x^2\{a(b-c)+b(c-a)+c(a-b)\}-x\{a(b^2-c^2)$$
$$+b(c^2-a^2)+c(a^2-b^2)\}+abc\{(b-c)+(c-a)+(a-b)\}$$
$$=x^2.0-x\{(b-c)(c-a)(a-b)\}+abc.0.$$ [Formulæ XXIII XIV and XXII, Art. 133]
$$=-x(b-c)(c-a)(a-b).$$

Example 6. Prove that
$$(1-x^2)(1-y^2)(1-z^2)-(x+yz)(y+zx)(z+xy)$$
$$=(1+xyz)(1-x^2-y^2-z^2-2xyz).$$

The left side
$$=(1-x^2)(1-y^2)(1-z^2)-\frac{(xyz+x^2)}{x}.\frac{(xyz+y^2)}{y}.\frac{(xyz+z^2)}{z}$$
$$=\{1-(x^2+y^2+z^2)+y^2z^2+z^2x^2+x^2y^2-x^2y^2z^2\}-\frac{1}{xyz}\{(xyz)^3$$
$$+(xyz)^2(x^2+y^2+z^2)+(xyz)(y^2z^2+z^2x^2+x^2y^2)+x^2y^2z^2$$

$$=(1-x^2-y^2-z^2)+(y^2z^2+z^2x^2+x^2y^2)-x^2y^2z^2-x^2y^2z^2$$
$$-xyz(x^2+y^2+z^2)-(y^2z^2+z^2x^2+x^2y^2)-xyz$$
$$=(1-x^2-y^2-z^2)-xyz-xyz(x^2+y^2+z^2)-2x^2y^2z^2$$
$$=1-x^2-y^2-z^2-2xyz+xyz-xyz(x^2+y^2+z^2)-2x^2y^2z^2$$
$$=(1-x^2-y^2-z^2-2xyz)+xyz(1-x^2-y^2-z^2-2xyz)$$
$$=(1+xyz)(1-x^2-y^2-z^2-2xyz).$$

150. Conditional Identities. We shall now establish certain important *Conditional* identities and deduce the truth of other identities from them.

151. If $a+b+c=0$, prove that

(1) $a^2+b^2+c^2=-2(bc+ca+ab)$.

We have $(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.$

$$\therefore \quad 0^2=a^2+b^2+c^2+2(bc+ca+ab).$$

Transposing, $a^2+b^2+c^2=-2(bc+ca+ab).$

(2) $a^3+b^3+c^3=3abc$.

We have $a^3+b^3+c^3-3abc$

$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
$$=0\times(a^2+b^2+c^2-bc-ca-ab)=0.$$

$\therefore$ transposing, $a^3+b^3+c^3=3abc.$ [See Art. 99, Ex. 10]

(3) $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2=\frac{1}{4}(a^2+b^2+c^2)^2$.

We have $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)$

$$=b^2c^2+c^2a^2+a^2b^2+2abc\times 0$$
$$=b^2c^2+c^2a^2+a^2b^2.$$

Also, from (1) above, $bc+ca+ab=-\frac{1}{2}(a^2+b^2+c^2).$

$$\therefore \quad (bc+ca+ab)^2=\tfrac{1}{4}(a^2+b^2+c^2)^2.$$

Hence, $(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2=\frac{1}{4}(a^2+b^2+c^2)^2.$

(4) $a^4+b^4+c^4=2(b^2c^2+c^2a^2+a^2b^2)=\frac{1}{2}(a^2+b^2+c^2)^2$.

We have $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$
$$=0\times(b+c-a)(c+a-b)(a+b-c) \qquad \text{[Art. 142]}$$
$$=0.$$

Hence, transposing,

$$a^4+b^4+c^4=2b^2c^2+2c^2a^2+2a^2b^2=2(b^2c^2+c^2a^2+a^2b^2)$$
$$=\tfrac{1}{2}(a^2+b^2+c^2)^2. \qquad \text{[from (3)]}$$

(5) $$\begin{aligned} a^5+b^5+c^5 &= -5abc(bc+ca+ab) \\ &= \tfrac{5}{2}abc(a^2+b^2+c^2) \\ &= \tfrac{5}{6}(a^2+b^2+c^2)(a^3+b^3+c^3). \end{aligned}$$

Since, $a+b+c=0$, we have, by transposition, $a+b=-c$;

$$\therefore \quad (a+b)^5=(-c)^5,$$

or, $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=-c^5$. [Art. 127]

By transposition,

$$\begin{aligned} a^5+b^5+c^5 &= -5a^4b-10a^3b^2-10a^2b^3-5ab^4 \\ &= -5ab(a^3+2a^2b+2ab^2+b^3) \\ &= -5ab(a+b)(a^2+ab+b^2) \qquad \text{[factorising]} \\ &= -5ab(-c)\{(a+b)^2-ab\} \qquad \text{[since, } a+b=-c \text{]} \\ &= 5abc\{(a+b)(-c)-ab\} \\ &= 5abc(-ac-bc-ab) \\ &= -5abc(bc+ca+ab) \\ &= \frac{5abc}{2}(a^2+b^2+c^2) \qquad \text{[by (1)]} \\ &= \tfrac{5}{6}(a^2+b^2+c^2).3abc \\ &= \tfrac{5}{6}(a^2+b^2+c^2)(a^3+b^3+c^3). \qquad \text{[since, } a^3+b^3+c^3 = 3abc \text{]} \end{aligned}$$

(6) $$\begin{aligned} a^7+b^7+c^7 &= 7abc(bc+ca+ab)^2 \\ &= \tfrac{7}{12}(a^2+b^2+c^2)^2(a^3+b^3+c^3). \end{aligned}$$

Since, $a+b+c=0$, we have, by transposition, $a+b=-c$;

$$\therefore \quad (a+b)^7=(-c)^7,$$

or, $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$
$=-c^7$. [Art. 127]

Transposing, $a^7+b^7+c^7$

$$\begin{aligned} &= -7a^6b-21a^5b^2-35a^4b^3-35a^3b^4-21a^2b^5-7ab^6 \\ &= -7ab(a^5+3a^4b+5a^3b^2+5a^2b^3+3ab^4+b^5) \\ &= -7ab(a+b)(a^4+2a^3b+3a^2b^2+2ab^3+b^4) \qquad \text{[factorising]} \\ &= -7ab(-c)(a^2+ab+b^2)^2 \\ &= 7abc(a^2+ab+b^2)^2 \\ &= 7abc(bc+ca+ab)^2 \qquad \text{[as in (5)]} \\ &= \tfrac{7}{3}(bc+ca+ab)^2.3abc \\ &= \frac{7}{3}\cdot\left(\frac{a^2+b^2+c^2}{2}\right)^2\cdot(a^3+b^3+c^3) \qquad \text{[from (2) \& (3)]} \\ &= \tfrac{7}{12}(a^2+b^2+c^2)^2(a^3+b^3+c^3). \end{aligned}$$

Example 1. Prove that

$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y)$$

Putting a for $(y-z)$, b for $(z-x)$ and c for $(x-y)$, we have

$$a+b+c=y-z+z-x+x-y=0.$$

$$\therefore \quad a^3+b^3+c^3=3abc. \quad [\text{ by } (3)\ ]$$

Restoring values of a, b and c,

$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y).$$

Example 2. Prove that $\dfrac{(y-z)^5+(z-x)^5+(x-y)^5}{5}$

$$=\frac{(y-z)^2+(z-x)^2+(x-y)^2}{2}\cdot\frac{(y-z)^3+(z-x)^3+(x-y)^3}{3}.$$

Putting a for $(y-z)$, b for $(z-x)$ and c for $(x-y)$, we have

$$a+b+c=y-z+z-x+x-y=0.$$

$$\therefore \quad a^5+b^5+c^5=\tfrac{5}{6}(a^2+b^2+c^2)(a^3+b^3+c^3), \qquad [\text{ from } (5)\ ]$$

$$\text{or,} \quad \frac{a^5+b^5+c^5}{5}=\frac{a^2+b^2+c^2}{2}\cdot\frac{a^3+b^3+c^3}{3}.$$

$\therefore$ restoring values of a, b, c, we obtain

$$\frac{(y-z)^5+(z-x)^5+(x-y)^5}{5}=\frac{(y-z)^2+(z-x)^2+(x-y)^2}{2}$$
$$\times\frac{(y-z)^3+(z-x)^3+(x-y)^3}{3}.$$

Example 3. Prove that $(y+z-x)^3+(z+x-y)^3+(x+y-z)^3$
$=3(y+z-x)(z+x-y)(x+y-z)=-24xyz$, if $x+y+z=0$.

Putting a for $y+z-x$, b for $z+x-y$ and c for $x+y-z$,
we have $a+b+c=(y+z-x)+(z+x-y)+(x+y-z)$
$=x+y+z=0.$

Hence, $a^3+b^3+c^3=3abc$.

$\therefore$ restoring values of a, b, c, we obtain

$$(y+z-x)^3+(z+x-y)^3+(x+y-z)^3$$
$$=3(y+z-x)(z+x-y)(x+y-z).$$

Also, since $a+b+c=0$, we have, by transposition,

$$a=-(b+c)=-\{(z+x-y)+(x+y-z)\}=-2x,$$
$$b=-(c+a)=-\{(x+y-z)+(y+z-x)\}=-2y,$$
$$c=-(a+b)=-\{(y+z-x)+(z+x-y)\}=-2z\,;$$

$$\therefore \quad 3abc=3(-2x)(-2y)(-2z)=-24xyz,$$

$$\text{or,} \quad 3(y+z-x)(z+x-y)(x+y-z)=-24xyz.$$

Hence, $(y+z-x)^3+(z+x-y)^3+(x+y-z)^3$

$$=3(y+z-x)(z+x-y)(x+y-z)=-24xyz$$

Example 4. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, show that

$$x^3+y^3+z^3-3xyz=(a^3+b^3+c^3-3abc)^2.$$

We have
$$\begin{aligned}x+y+z&=a^2-bc+b^2-ca+c^2-ab\\&=a^2+b^2+c^2-bc-ca-ab\,;\\ y-z&=b^2-ca-(c^2-ab)=b^2-c^2+ab-ca\\&=(b-c)(b+c)+a(b-c)=(b-c)\{(b+c)+a\}\\&=(b-c)(a+b+c).\end{aligned}$$

Similarly,
$$\begin{aligned}z-x&=(c-a)(a+b+c),\\ x-y&=(a-b)(a+b+c).\end{aligned}$$

Now, $x^3+y^3+z^3-3xyz$
$$\begin{aligned}&=\tfrac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\}\\&=\tfrac{1}{2}(a^2+b^2+c^2-bc-ca-ab)\{(b-c)^2(a+b+c)^2\\&\qquad+(c-a)^2(a+b+c)^2+(a-b)^2(a+b+c)^2\}\\&=(a^2+b^2+c^2-bc-ca-ab)\\&\qquad\times\tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}(a+b+c)^2\\&=(a+b+c)^2(a^2+b^2+c^2-bc-ca-ab)^2\\&=\{(a+b+c)(a^2+b^2+c^2-bc-ca-ab)\}^2\\&=(a^3+b^3+c^3-3abc)^2.\end{aligned}$$

Example 5. If $s=a+b+c$, prove that

$$\begin{aligned}s(s-2b)(s-2c)+s(s-2c)(s-2a)+s(s-2a)(s-2b)\\=(s-2a)(s-2b)(s-2c)+8abc.\end{aligned}$$

The sum of the first two terms of the left side
$$\begin{aligned}&=s(s-2c)\{(s-2b)+(s-2a)\}\\&=s(s-2c)\{2s-2(a+b)\}=s(s-2c)\times 2c\,;\end{aligned}$$

and the third term $=(\overline{s-2c}+2c)(s-2a)(s-2b)$
$$=(s-2c)(s-2a)(s-2b)+2c(s-2a)(s-2b).$$

Hence, the left side
$$\begin{aligned}&=s(s-2c)2c+\{(s-2c)(s-2a)(s-2b)+2c(s-2a)(s-2b)\}\\&=(s-2a)(s-2b)(s-2c)+2c\{s(s-2c)+(s-2a)(s-2b)\}.\end{aligned}$$

But $s(s-2c)+(s-2a)(s-2b)$
$$\begin{aligned}&=(s^2-2cs)+\{s^2-2s(a+b)+4ab\}\\&=2s^2-2s(a+b+c)+4ab\\&=2s^2-2s.s+4ab=4ab.\end{aligned}$$

$\therefore$ the left side $=(s-2a)(s-2b)(s-2c)+8abc.$

Example 6. If $s=a+b+c$, show that

$$(s-a)(s-b)(s-c)=(a+b+c)(bc+ca+ab)-abc.$$

The left side $=s^3-(a+b+c)s^2+(bc+ca+ab)s-abc$

$$=s^3-s.s^2+(bc+ca+ab)(a+b+c)-abc$$

$$=(bc+ca+ab)(a+b+c)-abc.$$

Example 7. If $a+b+c+d=0$, prove that

$$\begin{aligned}(a+b)(a+c)(a+d)&=(b+a)(b+d)(b+c)\\&=(c+d)(c+a)(c+b)\\&=(d+c)(d+b)(d+a).\end{aligned}$$

Since, $a+b+c+d=0$, we have, by transposition,

$$\begin{aligned}a+b&=-(c+d),\\a+c&=-(b+d),\\a+d&=-(b+c);\end{aligned}$$

$$\begin{aligned}\therefore (a+b)(a+c)(a+d)&=(a+b)\{-(b+d)\}\{-(b+c)\}\\&=(a+b)(b+d)(b+c)\\&=(b+a)(b+d)(b+c).\end{aligned}$$

Similarly,

$$\begin{aligned}(a+b)(a+c)(a+d)&=\{-(c+d)\}(a+c)\{-(b+c)\}\\&=(c+d)(a+c)(b+c)\\&=(c+d)(c+a)(c+b);\end{aligned}$$

$$\begin{aligned}(a+b)(a+c)(a+d)&=-(c+d)\{-(b+d)\}(a+d)\\&=(c+d)(b+d)(a+d)\\&=(d+c)(d+b)(d+a).\end{aligned}$$

Example 8. Prove that

$$(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+24xyz$$
$$=(2x+y-z)^3+(y+z)^3-(x+y-z)^3-6x(x-2z)(x+y).$$

Putting a for $2x+y-z$, b for $y+z$ and c for $-(x+y-z)$,

we have
$$\begin{aligned}a+b+c&=x+y+z,\\b+c&=2z-x,\\c+a&=x,\\a+b&=2(x+y);\end{aligned}$$

$\therefore$ the right side

$$=(2x+y-z)^3+(y+z)^3+\{-(x+y-z)\}^3+3.x(2z-x).2(x+y)$$
$$=a^3+b^3+c^3+3(c+a)(b+c)(a+b)$$
$$=(a+b+c)^3$$
$$=(x+y+z)^3 \qquad \text{[restoring values of } a, b, c \text{]}$$

$=\{(y+z-x)+(z+x-y)+(x+y-z)\}^3$ [since, $(y+z-x)+(z+x-y)+(x+y-z)=x+y+z$]

$=(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+3\{(z+x-y)+(x+y-z)\}\{(x+y-z)+(y+z-x)\}\{(y+z-x)+(z+x-y)\}$

[Formula XVII, Art. 133]

$=(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+3.2x.2y.2z$

$=(y+z-x)^3+(z+x-y)^3+(x+y-z)^3+24xyz.$

EXERCISE 81

Prove that :

1. $a^2x+b^2y+c^2z=(x+y+z)(a^2+b^2+c^2)$, if $x^2-yz=a^2,\ y^2-zx=b^2,\ z^2-xy=c^2$.
2. $ax+by+cz=(a+b+c)(x+y+z)$, if $x=a^2-bc,\ y=b^2-ca,\ z=c^2-ab$.
3. $(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)+(b-c)(c-a)(a-b)=0$.
4. $27(a+b+c)^3-(a+2b)^3-(b+2c)^3-(c+2a)^3$ $=3(a+3b+2c)(b+3c+2a)(c+3a+2b)$.
5. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)$ $+c(s-a)(s-b)=abc$, if $2s=a+b+c$.
6. $s(s-a)(s-b)+s(s-a)(s-c)+s(s+a)(s-c)+c(s+a)(s+b)$ $=(s+a)(s+b)(s+c)$, if $s=a+b+c$.
7. $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$ $=\frac{1}{2}(a^3+b^3+c^3-3abc)$, if $2s=a+b+c$.
8. $(3x+2y+5z)^3-(3x+2y-5z)^3-30z\{(3x+2y)^2-25z^2\}$ $=(20x-y+8z)^3+(y+2z-20x)^3+30z(20x-y+8z)(y+2z-20x)$.
9. $(x+y+2z)(x+2y+z)(2x+y+z)-(y+z)(z+x)(x+y)$ $=2(x+y+z)^3+2xyz$.
10. $(a+b+c)(x+y+z)+(a+b-c)(x+y-z)+(b+c-a)(y+z-x)$ $+(c+a-b)(z+x-y)=4(ax+by+cz)$.
11. $(y-z)(1+xy)(1+xz)+(z-x)(1+yz)(1+yx)$ $+(x-y)(1+zx)(1+yz)=(y-z)(z-x)(x-y)$.
12. $(x-1)(x^2+x+4)(y-z)+(y-1)(y^2+y+4)(z-x)$ $+(z-1)(z^2+z+4)(x-y)=-(y-z)(z-x)(x-y)(x+y+z)$.
13. $x^3+y^3+z^3+w^3+3(y+z)(z+x)(x+y)=0$, if $x+y+z+w=0$.
14. $\frac{(b-c)^5+(c-a)^5+(a-b)^5}{5}\cdot\frac{(b-c)^2+(c-a)^2+(a-b)^2}{2}$ $=\frac{(b-c)^7+(c-a)^7+(a-b)^7}{7}$.

15. $(x+y)(x+z)(x^2-yz)=(x+y+z)(x-z)(x^2+y^2)$,
if $x=a^3+a^2$, $y=a^2+a$ and $z=a+1$. [M. U. 1909]

16. $(y+z-2x)(z+x-2y)+(z+x-2y)(x+y-2z)+(x+y-2z)(y+z-2x)$
$=3\{(y-z)(z-x)+(z-x)(x-y)+(x-y)(y-z)\}$.

17. $(y-z)^4+(z-x)^4+(x-y)^4=2(x^2+y^2+z^2-yz-zx-xy)^2$.

18. $(b-c)(b+c-2a)^3+(c-a)(c+a-2b)^3+(a-b)(a+b-2c)^3=0$.

19. $x^3(y-z)^3+y^3(z-x)^3+z^3(x-y)^3=3xyz(y-z)(z-x)(x-y)$.

20. $a^6(b^2-c^2)^3+b^6(c^2-a^2)^3+c^6(a^2-b^2)^3$
$=3a^2b^2c^2(b+c)(c+a)(a+b)(b-c)(c-a)(a-b)$.

21. $(b+c)(b-c)^3+(c+a)(c-a)^3+(a+b)(a-b)^3$
$=2(b-c)(c-a)(a-b)(a+b+c)$.

22. $x(y-z)^3+y(z-x)^3+z(x-y)^3=(y-z)(z-x)(x-y)(x+y+z)$.

23. $4(a^2+ab+b^2)^3-(a-b)^2(a+2b)^2(2a+b)^2=27a^2b^2(a+b)^2$.

24. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
$=16s(s-a)(s-b)(s-c)$, if $2s=a+b+c$.

25. $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$, if $2s=a+b+c$.

26. $\dfrac{(b-c)^3+(c-a)^3+(a-b)^3}{3}\cdot\dfrac{(b-c)^7+(c-a)^7+(a-b)^7}{7}$
$=\left\{\dfrac{(b-c)^5+(c-a)^5+(a-b)^5}{5}\right\}^2$.

27. $(ax+by+cz)^2+(bx+cy+az)^2+(cx+ay+bz)^2$
$-\{(bx+cy+az)(cx+ay+bz)+(cx+ay+bz)(ax+by+cz)$
$+(ax+by+cz)(bx+cy+az)\}$
$=(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$.

28. $(ax+by+cz)^3+(bx+cy+az)^3+(cx+ay+bz)^3$
$-3(ax+by+cz)(bx+cy+az)(cx+ay+bz)$
$=(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$.

If $a+b+c=0$, prove that

29. $ca-b^2=ab-c^2=bc-a^2=bc+ca+ab=-\frac{1}{2}(a^2+b^2+c^2)$.

30. $b^2+bc+c^2=c^2+ca+a^2=a^2+ab+b^2=-(bc+ca+ab)$.

31. $a(c+a)(a+b)=b(a+b)(b+c)=c(a+c)(b+c)=abc$.

32. $a(b+c)^2+b(c+a)^2+c(a+b)^2=3abc$.

33. $a(b-c)^2+b(c-a)^2+c(a-b)^2=0$.

34. $X^3+Y^3+Z^3=3XYZ$, where $X=ax+by+cz$,
$Y=bx+cy+az$ and $Z=cx+ay+bz$.

35. $(2a+b+c)^3+(a+2b+c)^3+(a+b+2c)^3$
$=3(2a+b+c)(a+2b+c)(a+b+2c)$.

36. Prove that $(3x+y+z)^3+(x+3y+z)^3+(x+y+3z)^3$

$$-3(3x+y+z)(x+3y+z)(x+y+3z)=20(x^3+y^3+z^3-3xyz).$$

37. If $a+b+c=1$, prove that

$$(a+bc)(b+c)=(b+ca)(c+a)=(c+ab)(a+b)=(1-a)(1-b)(1-c).$$

Prove that:

38. $(x+y)^2(y+z-x)(z+x-y)+(x-y)^2(x+y+z)(x+y-z)$

$$=4xyz^2+(y^2-z^2)(y^4+y^2z^2+z^4)+(z^2-x^2)(z^4+z^2x^2+x^4)$$
$$+(x^2-y^2)(x^4+x^2y^2+y^4).$$

39. $2x(y+z-x)+(z+x-y)(x+y-z)$

$$=2y(z+x-y)+(x+y-z)(y+z-x)$$
$$=2z(x+y-z)+(y+z-x)(z+x-y)$$
$$=(y+z-x)(z+x-y)+(z+x-y)(x+y-z)+(x+y-z)(y+z-x).$$

40. $x^3+y^3+z^3=a^3-3ab+3c$,

when $x+y+z=a$, $yz+zx+xy=b$, $xyz=c$.

41. $yz(y+z)+zx(z+x)+xy(x+y)+3xyz=\frac{1}{2}(p^3-pq^2)$,

when $x+y+z=p$ and $x^2+y^2+z^2=q^2$.

42. $x^7+y^7+z^7=7qr^2$, when $x+y=-z$, $xyz=q$, $yz+zx+xy=r$.

43. $x^4+y^4+z^4=\frac{1}{2}q^4$, when $x^2+y^2+z^2=q^2$, $x+y=13$, $z=-13$.

44. $(x+y+z)(yz+zx+xy)-(y+z)(z+x)(x+y)=120$,

when $yz=45$, $zx=64$, $xy=5$.

CHAPTER XXIII

THE REMAINDER THEOREM AND DIVISIBILITY

152. Important Theorems in Division.

Theorem I. *If px^2+qx+r is divided by $x-a$ until the remainder does not contain x, the remainder will be pa^2+qa+r.*

By actual division, we have

$$\begin{array}{r|l|l} x-a & px^2+qx+r & px+(ap+q) \\ & \underline{px^2-apx} & \\ & (ap+q)x+r & \\ & \underline{(ap+q)x-a(ap+q)} & \\ & a(ap+q)+r & \end{array}$$

$\therefore$ the remainder $=a(ap+q)+r=pa^2+qa+r$.

Note. *Observe that the remainder is of the same form as the dividend with a in the place of x.*

Theorem II. *If* px^3+qx^2+rx+s *is divided by* $x-a$ *until the remainder does not involve* x, *the remainder will be* pa^3+qa^2+ra+s.

By actual division,

$$\begin{array}{l|l}
x-a\,\big) & px^3+qx^2+rx+s\,\big(\,px^2+(ap+q)x+(pa^2+qa+r) \\
 & \underline{px^3-apx^2} \\
 & (ap+q)x^2+rx+s \\
 & \underline{(ap+q)x^2-a(ap+q)x} \quad \therefore \text{ the remainder required} \\
 & (pa^2+qa+r)x+s \qquad\qquad = pa^3+qa^2+ra+s. \\
 & \underline{(pa^2+qa+r)x-a(pa^2+qa+r)} \\
 & \qquad\qquad pa^3+qa^2+ra+s
\end{array}$$

Note. *Here also, notice that the remainder is of the same form as the dividend with* a *in the place of* x.

Example 1. Find the remainder independent of x when x^3-5x^2+6x+9 is divided by $x-2$.

By Theorem II, the remainder required

$$\begin{aligned}&=\text{the value of } x^3-5x^2+6x+9, \text{ when } x=2\\ &=2^3-5.2^2+6.2+9=8-20+12+9=9.\end{aligned}$$

Example 2. Find the remainder independent of x when x^3-216 is divided by $x-6$.

The remainder required $=$ the value of x^3-216, when $x=6$

$$=6^3-216=216-216=0.$$

Note. *The student is recommended to verify these results by actual division.*

153. Rational and Integral Expressions. We shall now establish a more general theorem known as the **Remainder Theorem** by dividing an expression of the type $px^n+qx^{n-1}+rx^{n-2}+\cdots\cdots+lx+m$ by $x-a$, n being a *positive integer* and $p, q, r, \ldots\ldots l, m$ being constants.

An algebraic expression of this kind in which every power of x is a *positive integer* is called a **rational and integral expression** in x.

Thus, $x^2-3x+13$, x^3+px+r, etc. are each a rational and integral expression in x.

154. The Remainder Theorem. *If any rational and integral expression in* x *is divided by* $x-a$, *the remainder independent of* x, *is obtained by putting* a *for* x *in the given expression.*

Let $px^n+qx^{n-1}+rx^{n-2}+\cdots\cdots+lx+m$ be the rational and integral expression. Let Q be the quotient and R, the remainder *independent of* x when the above expression is divided by $x-a$.

Then, since, (Dividend) = (Divisor) × (Quotient) + (**Remainder**), **we have**

$$px^n+qx^{n-1}+rx^{n-2}+\cdots\cdots+lx+m=(x-a)Q+R \ (\textit{ identically }).$$

Since, R does not contain x, it remains the same whatever value be given to x. If a is put for x in the above relation, we have

$$pa^n + qa^{n-1} + ra^{n-2} + \cdots\cdots + la + m = (a-a)Q' + R,$$

(where Q' is the value of Q when a is put for x)

$$= 0 \times Q' + R = 0 + R = R.$$

$\therefore$ the remainder $R = pa^n + qa^{n-1} + ra^{n-2} + \cdots\cdots + la + m$.

Thus, the remainder is of the same form as the dividend with a in the place of x.

Hence, the theorem is established.

Cor. *If any rational and integral expression in x be divided by $x+a$, the remainder independent of x is obtained by putting $-a$ for x in the given expression.*

Since, $x+a = x-(-a)$, the above corollary follows at once from the Remainder Theorem.

Note. *If any rational and integral expression in x is divided by $ax+b$, the remainder independent of x is obtained by putting $-\frac{b}{a}$ for x in the expression.*

Example 1. If $x^2 - 5x + 9$ be divided by $x+2$, find the remainder independent of x.

From the corollary, the remainder required = the value of the expression $x^2 - 5x + 9$, when -2 is put for x

$$= (-2)^2 - 5(-2) + 9 = 4 + 10 + 9 = 23.$$

Example 2. If $2x^3 - 3x^2 + 7x + 5$ be divided by $2x+3$, find the remainder independent of x.

From the note above, the required remainder = the value of the expression $2x^3 - 3x^2 + 7x + 5$ when $-\frac{3}{2}$ is put for x

$$= 2\left(-\frac{27}{8}\right) - 3\cdot\frac{9}{4} + 7\cdot\left(-\frac{3}{2}\right) + 5$$

$$= -\frac{27}{4} - \frac{27}{4} - \frac{21}{2} + 5 = -19.$$

Example 3. If $x^2 + px + q$ be divided by $x+a$, find the remainder independent of x.

From the corollary above, the remainder required

$$= \text{the value of } x^2 + px + q, \text{ when } x = -a$$

$$= (-a)^2 + p(-a) + q = a^2 - pa + q.$$

Note. *The student is advised to verify these results by actual division.*

Example 4. Find, without actual substitution, the value of $x^6 - 19x^5 + 69x^4 - 151x^3 + 229x^2 + 166x + 26$, when $x = 15$.

By the Ramainder Theorem, the value of the expression, when 15 is put for x, = the remainder on division by $x - 15$.

But the given expression

$$= x^6 - 15x^5 - (4x^5 - 60x^4) + (9x^4 - 135x^3)$$
$$- (16x^3 - 240x^2) - (11x^2 - 165x) + (x - 15) + 41$$
$$= x^5(x - 15) - 4x^4(x - 15) + 9x^3(x - 15) - 16x^2(x - 15)$$
$$- 11x(x - 15) + (x - 15) + 41.$$

Evidently, the remainder on division by $x - 15 = 41$.

Hence, the value required = 41.

155. Divisibility and Factor Theorem. *If any rational and integral expression in x vanishes identically when a is substituted for x, the expression is exactly divisible by $x - a$ and contains $x - a$ as a factor.*

Let the given expression be $px^n + qx^{n-1} + rx^{n-2} + \cdots\cdots + lx + m$.

The remainder on division by $x - a$

= the value of the dividend when a is put for x

[by the Remainder Theorem]

$$= pa^n + qa^{n-1} + ra^{n-2} + \cdots\cdots + la + m.$$

The given expression is exactly divisible by $x - a$ if the remainder is zero, *i.e.*, if $pa^n + qa^{n-1} + ra^{n-2} + \cdots\cdots + la + m = 0$.

Also, $\because$ (Dividend) = (Divisor) × (Quotient) + (Remainder), we have the given expression

$$= (x - a) \times Q + (pa^n + qa^{n-1} + ra^{n-2} + \cdots\cdots + la + m)$$

[Q being the quotient]

$$= (x - a)Q, \text{ if } pa^n + qa^{n-1} + \cdots\cdots + la + m = 0 ;$$

$\therefore$ $x - a$ is a factor of $px^n + qx^{n-1} + \cdots\cdots + lx + m$,

if $pa^n + qa^{n-1} + \cdots\cdots + la + m = 0$.

Thus, the theorem is established.

Cor. $x + a$ is a factor of $px^n + qx^{n-1} + rx^{n-2} + \cdots\cdots + lx + m$, if $p(-a)^n + q(-a)^{n-1} + r(-a)^{n-2} + \cdots + l(-a) + m = 0$

Since, $x + a = x - (-a)$, the corollary follows at once.

Note *If any rational and integral expression in x vanishes when $-\frac{b}{a}$ is substituted for x, the expression is exactly divisible by $ax + b$ and contains $ax + b$ as a factor*

Example 1. Show that $3x^3-2x^2+x-18$ is exactly divisible by $x-2$ and contains $x-2$ as a factor.

The value of $3x^3-2x^2+x-18$, when 2 is put for x

$$=3.2^3-2.2^2+2-18=24-8+2-18=0.$$

Hence, by the above theorem, $3x^3-2x^2+x-18$ is exactly divisible by $x-2$ and contains $x-2$ as a factor.

Example 2. Show that $3x^3-2x^2y-13xy^2+10y^3$ is exactly divisible by $x-2y$ and contains $x-2y$ as a factor.

Putting $2y$ for x in the given expression, we have

$$3.(2y)^3-2.(2y)^2.y-13.(2y).y^2+10y^3$$
$$=24y^3-8y^3-26y^3+10y^3=0.$$

$\therefore$ by the above theorem, the given expression is exactly divisible by $x-2y$ and contains $x-2y$ as a factor.

Example 3. Find the condition that the rational and integral expression $ax^n+bx^{n-1}+cx^{n-2}+\cdots\cdots+lx+m$ may be divisible by $x-1$.

The value of the given expression, when 1 is put for x

$$=a.1^n+b.1^{n-1}+c.1^{n-2}+\cdots\cdots\cdots+l.1+m$$
$$=a+b+c+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+l+m$$

[since, $1^n=1\times1\times1\cdots$to n factors$=1$
and similarly $1^{n-1}=1^{n-2}=\cdots=1$]

$\therefore$ the given expression is exactly divisible by $x-1$,

$$\text{if } a+b+c+\cdots\cdots+l+m=0,$$

i.e., if the algebraic sum of the coefficients of the expression be **zero.**

Example 4. Prove that $x+3$ is a factor of x^3+4x^2+5x+6.

$$x+3=x-(-3).$$

The value of x^3+4x^2+5x+6, when $x=-3$

$$=(-3)^3+4.(-3)^2+5.(-3)+6=-27+36-15+6=0.$$

Hence, by the corollary of the factor theorem the expression is exactly divisible by $x+3$ and contains $x+3$ as a factor.

Example 5. If the expression x^3+3x^2+4x+p contains $x+6$ as a factor, find p.

$$x+6=x-(-6).$$

The value of x^3+3x^2+4x+p for $x=-6$

$$=(-6)^3+3.(-6)^2+4.(-6)+p=-216+108-24+p=p-132.$$

By the above theorem, $(x+6)$ is a factor, if $p-132=0$;

$\therefore$ the required value of $p=132$.

Example 6. Find the condition that $x^3+(p+q)x+a$ is divisible by $x+p+q$. [P. U. 1896]

The divisor $=x+p+q$. If $x+p+q=0$, $x=-(p+q)$.

Hence, if the expression (*viz.*, dividend) vanishes, when $-(p+q)$ is put for x, it is exactly divisible by $x+p+q$.

Therefore, $\{-(p+q)\}^3+(p+q)\{-(p+q)\}+a=0$,

$$\text{or,}\quad (p+q)^3+(p+q)^2=a\,; \qquad \text{or,}\quad (p+q)^2(p+q+1)=a.$$

$\therefore$ the required condition is $(p+q)^2(p+q+1)=a$.

Example 7. Find the condition that x^2+3x+p and x^2+4x+q may have a common factor.

Let $x-a$ be the common factor.

Putting a for x, the value of

$$x^2+3x+p=a^2+3a+p=0. \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

[since, $x-a$ is a factor of x^2+3x+p]

Similarly, $a^2+4a+q=0$. $\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (2)$

[since, $x-a$ is a factor of x^2+4x+q]

From (1) and (2), by subtraction, we have

$$(a^2+4a+q)-(a^2+3a+p)=0,$$

$$\text{or,}\quad a+q-p=0, \qquad \text{or,}\quad a=p-q. \qquad [\text{ transposing }]$$

Substituting this value of a in (1), we have

$$0=a^2+3a+p=(p-q)^2+3(p-q)+p$$

$$=p^2-2pq+q^2+4p-3q\,;$$

$\therefore$ the required condition is $p^2-2pq+q^2+4p-3q=0$.

156. Important Theorems on Divisibility. In Chapter X we have already considered the divisibility of expressions a^n+b^n and a^n-b^n by $a+b$ and $a-b$ in particular cases. We propose now to establish the propositions generally.

Theorem I. *The expression a^n-b^n is always divisible by $a-b$, if n is **any** positive integer, odd or even.*

Divide a^n-b^n by $a-b$ until the remainder is independent of a Let Q be the quotient and R, the remainder.

Since, (Dividend) $=$ (Quotient) × (Divisor) + (Remainder),

we have $\quad a^n-b^n=Q\times(a-b)+R \quad$ (*identically*).

Now, since R is independent of a, it remains the same whatever value be given to a.

Let $a=b$ in the above relation. Then, we have

$$b^n-b^n=Q'\times(b-b)+R,$$

[Q' being the value of
when b is put for a

$$\text{or,}\quad 0=Q'\times 0+R=0+R\,; \qquad \therefore\ R=0.$$

Hence, the remainder being zero, $a^n - b^n$ is exactly divisible by $a-b$.

Thus, if the division be actually performed, we have

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots\cdots + ab^{n-2} + b^{n-1}).$$

Example. Each of the expressions a^2-b^2, a^3-b^3, a^4-b^4, a^5-b^5, etc. is exactly divisible by $a-b$.

Theorem II. *The expression $a^n - b^n$ is exactly divisible by $a+b$, when n is any* **even** *positive integer, but not if n is* **odd.**

Divide $a^n - b^n$ by $a+b$ till the remainder does not contain a. Then, if Q be the quotient and R, the remainder, we have

$$a^n - b^n = Q \times (a+b) + R \quad (\textit{identically}).$$

Since, R does not contain a, it remains the same whatever value be given to a. Putting $-b$ for a in the above identity, we have

$$(-b)^n - b^n = Q' \times (-b+b) + R. \quad [Q' \text{ being the value of } Q \text{ when } -b \text{ is put for } a]$$

$$= Q' \times 0 + R = R;$$

when n is even, $(-b)^n - b^n = b^n - b^n = 0$;

when n is odd, $(-b)^n - b^n = -b^n - b^n = -2b^n$;

$$\therefore \quad R = 0, \quad \text{when } n \text{ is even};$$

but, since $R = -2b^n$, when n is odd, the remainder is not zero, when n is an odd integer.

Hence, $a^n - b^n$ is exactly divisible by $a+b$, when n is even, but not, if n is odd.

Thus, by actual division, we have

$$a^n - b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + ab^{n-2} - b^{n-1}), \text{ when } n \text{ is even.}$$

Example. Each of the expressions a^2-b^2, a^4-b^4, a^6-b^6, etc. is exactly divisible by $a+b$; but a^3-b^3, a^5-b^5, a^7-b^7, etc. are *not* exactly divisible by $a+b$.

Theorem III. *The expression $a^n + b^n$ is exactly divisible by $a+b$, if n is* **odd,** *but not if n be* **even.**

Divide $a^n + b^n$ by $a+b$ till the remainder does not contain a. Let Q be the quotient and R, the remainder. Then,

$$a^n + b^n = Q \times (a+b) + R \quad (\textit{identically}).$$

Since, R does not contain a, it remains the same whatever value be given to a. Let $a = -b$ in the above identity. Then, we have

$$(-b)^n + b^n = Q' \times (-b+b) + R = Q' \times 0 + R = R,$$

when n is odd, $\qquad (-b)^n + b^n = -b^n + b^n = 0.$

But, when n is even, $(-b)^n+b^n=b^n+b^n=2b^n$, which is, therefore, not zero.

Hence, $R=0$, if n is odd, but not if n is even.

$\therefore$ a^n+b^n is exactly divisible by $a+b$, when n is odd, but not when n is even.

Thus, by actual division, we have

$$a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots-ab^{n-2}+b^{n-1}),$$

when n is odd.

Example. a^3+b^3, a^5+b^5, a^7+b^7 are all exactly divisible by $a+b$, while a^2+b^2, a^4+b^4, a^6+b^6 are not so.

Theorem IV. *The expression* a^n+b^n *is never divisible by* $a-b$, *whether* n *is even or odd.*

Divide a^n+b^n by $a-b$ till the remainder does not contain a. Let Q be the quotient and R, the remainder. Then,

$$a^n+b^n=Q\times(a-b)+R \quad (\textit{identically}).$$

Since, R does not contain a, it remains the same whatever value be given to a. Put b for a in the above identity and we have

$$b^n+b^n=Q'\times(b-b)+R=Q'\times 0+R=R, \quad \text{or,} \quad R=2b^n.$$

Since, R does not vanish for any value of n, a^n+b^n is never divisible by $a-b$.

Example. Thus, a^2+b^2, a^3+b^3, a^4+b^4, etc. are never divisible by $a-b$.

Example 1. Show that $3^{4n}-4^{3n}$ is divisible by 17, if n is any positive integer.

The expression $3^{4n}-4^{3n}=(3^4)^n-(4^3)^n=(81)^n-(64)^n$;

$\therefore$ by Theorem I, Art. 156, the given expression is divisible by $81-64$, *i.e.*, 17.

Example 2. Show that the last two digits of $2^{6n}-6^{2n}$ are 0's, n being any even positive integer.

The given expression $=(2^6)^n-(6^2)^n=(64)^n-(36)^n$.

Since n is even, the given expression is exactly divisible by $64+36$, *i.e.*, by 100. [Theorem II, Art. 156]

Hence, 100 being a factor of the given expression, the last two digits must be 0's.

Example 3. Show that

$$(x^3+3ax^2+3a^2x+a^3)^{2m+1}+(x^3-3ax^2+3a^2x-a^3)^{2m+1}$$

contains $2x$ as a factor, m being a positive integer.

The given expression $=\{(x+a)^3\}^{2m+1}+\{(x-a)^3\}^{2m+1}$

$$=(x+a)^{3(2m+1)}+(x-a)^{3(2m+1)}.$$

Since, $3(2m+1)$ is an odd positive integer, the given expression is exactly divisible by $(x+a)+(x-a)$, *i.e.*, $2x$. [Theorem III]

Example 4. Show that $(b-c)^{2n+1}+(c-a)^{2n+1}+(a-b)^{2n+1}$ is divisible by $(b-c)(c-a)(a-b)$, n being any positive integer.

The given expression is a rational and integral expression in a, b and c.

If we substitute c for b in the expression, we have the expression

$$\begin{aligned}&=(c-c)^{2n+1}+(c-a)^{2n+1}+(a-c)^{2n+1}\\&=(0)^{2n+1}\quad+(c-a)^{2n+1}+\{-(c-a)\}^{2n+1}\\&=0\quad+(c-a)^{2n+1}-(c-a)^{2n+1}.\end{aligned}$$

Since, $\{-(c-a)\}^{2n+1}$

$$\begin{aligned}&=\{-1\times(c-a)\}^{2n+1}\\&=\{-1\times(c-a)\}\times\{-1\times(c-a)\}\times\cdots\text{ to }(2n+1)\text{ factors}\\&=(-1)\times(-1)\times(-1)\cdots\cdots\text{ to }(2n+1)\text{ factors}\\&\qquad\times(c-a)\times(c-a)\times(c-a)\cdots\text{ to }(2n+1)\text{ factors}\\&=-1\times(c-a)^{2n+1}=-(c-a)^{2n+1};\end{aligned}$$

$\therefore$ the expression $=0$;

$\therefore$ by Art. 155, the given expression is divisible by $b-c$.

Similarly, putting a for c, in the given expression, it may be shown that the expression is divisible by $c-a$, and putting b for a, it may be shown that the given expression is divisible by $a-b$.

Hence, the given expression is divisible by the product $(b-c)(c-a)(a-b)$.

Example 5. If n be any positive integer, show that

$(ab)^n-(bc)^n+(cd)^n-(da)^n$ is divisible by $ab-bc+cd-da$.

[M. M. 1873]

Evidently, $ab-bc+cd-da=b(a-c)+d(c-a)=(c-a)(d-b)$.

Now, if we put a for c in the given expression, we have the expression $=(ab)^n-(ba)^n+(ad)^n-(da)^n$

$$=(ab)^n-(ab)^n+(ad)^n-(ad)^n=0;$$

$\therefore$ by Art. 155, the given expression is exactly divisible by $c-a$.

Similarly, putting b for d in the expression, it may be shown that the expression is divisible by $d-b$.

$\therefore$ the expression is divisible by the product of $c-a$ and $d-b$, *i.e.*, by $(c-a)(d-b)$, *i.e.*, by $ab-bc+cd-da$.

Example 6. Show that $x^{n+1}-x^n-x+1$ is exactly divisible by $(x-1)^2$, when n is any positive integer.

The given expression $=x^{n+1}-x^n-x+1=x^n(x-1)-(x-1)$

$$=(x-1)(x^n-1).$$

Thus, $x-1$ is a factor of the given expression.

Since, by Theorem I, Art. 156, x^n-1 is exactly divisible by $x-1$,

$\therefore$ the given expression is divisible by $(x-1)\times(x-1)$, *i.e.*, $(x-1)^2$.

Example 7. Find the continued product of

$$(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8).$$

Let A denote the continued product required.

Then $A=(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)$.

Multiplying both the sides by $x-a$, we have

$$\begin{aligned}(x-a)A&=\{(x-a)(x+a)\}(x^2+a^2)(x^4+a^4)(x^8+a^8)\\&=\{(x^2-a^2)(x^2+a^2)\}(x^4+a^4)(x^8+a^8)\\&=\{(x^4-a^4)(x^4+a^4)\}(x^8+a^8)\\&=(x^8-a^8)(x^8+a^8)=x^{16}-a^{16};\end{aligned}$$

$$\therefore\quad A=\frac{x^{16}-a^{16}}{x-a}=x^{15}+x^{14}a+x^{13}a^2+\cdots\cdots+xa^{14}+a^{15}.$$

Example 8. If $x+a$ be the H.C.F. of x^2+px+q and $x^2+p'x+q'$, show that $a=\dfrac{q-q'}{p-p'}$.

Since, $x+a$ is the H.C.F. of the two expressions x^2+px+q and $x^2+p'x+q'$, these expressions must be exactly divisible by $x+a$.

$\therefore$ by the Divisibility Theorem, we have

$$(-a)^2+p(-a)+q=0,\quad i.e.,\quad a^2-pa+q=0,$$

and $$(-a)^2+p'(-a)+q'=0,\quad i.e.,\quad a^2-p'a+q'=0.$$

$\therefore$ by subtraction, $\quad (a^2-p'a+q')-(a^2-pa+q)=0,$

or, $\quad a(p-p')+q'-q=0.$

Transposing, $\quad a(p-p')=q-q'$; $\quad\therefore\quad a=\dfrac{q-q'}{p-p'}$.

EXERCISE 82

Find the remainder, without actual division, when

1. $x^4+2x^3+3x^2+4x+5$ is divided by $x-3$.
2. $3x^9+5x^7+11$ is divided by $x+1$.
3. $3x^3+7x^2+11x+2$ is divided by $3x-1$.
4. $4x^3+5x^2+9x+7$ is divided by $2x+3$.
5. ax^3+bx^2+cx+d is divided by $ax+b$.

In the following examples, show that the first expression is divisible by the second :

6. $6x^3 + 13x^2 + 17x + 6$ by $2x + 1$.

7. $apx^3 + (2p + aq)x^2 + (2q + ar)x + 2r$ by $ax + 2$.

8. $6x^4 + 13x^3y + 18x^2y^2 + 23xy^3 + 10y^4$ by $3x + 2y$.

9. $a^{57} + b^{57}$ by $a + b$. 10. $64x^6 - 729y^6$ by $2x + 3y$.

11. $x^{2n} - y^{2n}$ by $x + y$ (n being a positive integer).

12. $x^{12}y^3 - x^3y^{12}$ by $x^2y^2(x - y)$.

13. $(3a + 2b)^{2n+1} + b^{2n+1}$ by $a + b$ (n being any positive integer).

14. $x^{2n+1} - ax^{2n} - xa^{2n} + a^{2n+1}$ by $(x - a)^2$.

15. $64 + 32x + 2x^5 + x^6$ by $x^2 + 4x + 4$.

16. Find the condition that (i) $x^7 + 9x^4 - 7x^3 + 11ax + 5a^2$ may contain $x + 1$ as a factor.

(ii) $ax^3 + (b + c)x + d$ may contain $x + b + c$ as a factor.

17. For what values of a will $3x^5 + 9x^4 - 7x^3 - 5x^2 - 4ax + 3a^2$ contain $x - 1$ as a factor ?

18. What must be the form of m in order that $a^m - x^m$ may have both $a^n + x^n$ and $a^n - x^n$ for divisors, n being any positive integer ? [M. M. 1875]

19. If n be any positive integer, show that

$$(x^2 + 7x + 6)^n - (2 + x)^{2n}, \text{ is divisible by } 3x + 2.$$

20. Show that the quotient of $3x^3 + x^2 - 11x + 7$ when divided by $x - 1$ is exactly divisible by $x - 1$.

Show that each of the following expressions is exactly divisible by $(a - b)(b - c)(c - a)$:

21. $a^2b^2(a - b) + b^2c^2(b - c) + c^2a^2(c - a)$.

22. $a^3b^3(a - b) + b^3c^3(b - c) + c^3a^3(c - a)$.

23. $a^2(b - c)^3 + b^2(c - a)^3 + c^2(a - b)^3$. 24. $(a - b)^9 + (b - c)^9 + (c - a)^9$.

25. $a^7b^7(a - b)^{69} + b^7c^7(b - c)^{69} + c^7a^7(c - a)^{69}$.

26. Show, by the principle of divisibility that $(a + b + c)(ab + bc + ca) - abc$ contains the factors $b + c$, $c + a$ and $a + b$.

27. Shew that $(ax + by)(bx + cy)(cx + ay) - (ay + bx)(by + cx)(cy + ax)$ contains the factors $x - y$, $a - b$, $b - c$ and $c - a$. [M. U. 1874]

28. Shew that $a^n(b - c) + b^n(c - a) + c^n(a - b)$ contains each of the factors $a - b$, $b - c$ and $c - a$. [P. U. 1916]

29. Resolve $a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2)$ into factors by the principle of divisibility.

30. Shew that $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$ is divisible by $(b+c)(c+a)(a+b)(a-b)(b-c)(c-a)$.

31. Shew that the last digit in $(41)^n-1$, where n is any positive integer, is zero.

32. Shew that $7^{2m}-1$, where m is any positive integer, is divisible by each of the factors 2, 3, 4, 6, 8, 12, 16, 24 and 48.

33. Shew that $17^6+13^7-5^8+2^7$ is divisible by 3.

34. Shew that x^3-x-6 and $x^3-11x+14$ contain a common factor of the *form* $x-m$.

35. Shew that the expression $(81)^m.(121)^m-1$, where m is *any* positive integer, is divisible by 100.

36. Find the continued product of $(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$.

37. Shew that $13^n=12(13^{n-1}+13^{n-2}+\cdots\cdots+1)+1$, n being any positive integer.

38. Find the continued product of $11\times 101\times 10001$.

39. Shew that $x^n-nx+n-1$ is exactly divisible by $(x-1)^2$.

40. Shew that $a^m(a-1)+b^m(b-1)$ is not divisible by $a+b$, m being any positive integer.

Write down the quotients in the following divisions :

41. x^5+y^5 by $x+y$. 42. x^6-y^6 by $x+y$. 43. x^7-y^7 by $x-y$.

44. $x^{16}-y^{16}$ by x^2+y^2. 45. $x^{16}-y^{16}$ by $x-y$.

46. If $x+3$ be the H.C.F. of $ax^2+5x+2p$ and $ax^3+3x+p+6$, find p and a.

47. If $x-5$ be the H.C.F. of bx^2-px+5 and $bx^2-2x-2p$, prove that $p=5$ and $b=\frac{4}{5}$.

48. If $x-a$ be the H.C.F. of qx^2+2x+p and qx^2+x+r prove that $a=r-p$ and $q(r-p)^2+2r-p=0$.

49. Find, without actual substitution, the value of $x^9-3x^8+5x^7-15x^6+13x^5-39x^4+7x^3-21x^2+17x-51$, when $x=3$.

50. What is the value of $32x^5-48x^4+40x^3-60x^2+26x-38$, when $x=1{\cdot}5$?

CHAPTER XXIV

HARDER H.C.F. AND L.C.M.

157. In Chapters XIV and XV we have explained the methods of finding the H.C.F. and the L.C.M. of compound expressions, whose factors can be determined easily. We shall now proceed to consider more difficult cases.

I. Harder Highest Common Factors

158. If the H.C.F. of two or more compound expressions be a compound expression, it cannot generally be found by inspection. In such cases the following methods should be adopted.

159. The ordinary method of finding the H.C.F. of two multinomial expressions which have no monomial factors.

Rule. *Arrange the two expressions according to descending powers of some common letter ; divide the expression which is of higher degree, in that letter by the other, or if they be of the same degree, either of them by the other ; if there be any remainder, take it for a new divisor and the preceding divisor for the dividend, and continue the process till there is no remainder. The last divisor will be the H.C.F. required. Of any divisor and the corresponding dividend either may be multiplied or divided by any number which is not a factor of the other.*

This method is similar to 'Division Method' of finding H.C.F. in Arithmetic.

It depends on the principles :

(*a*) *When an expression contains a particular factor, any multiple of the expression is exactly divisible by that factor* ; and

(*b*) *if two expressions have a common factor, that common factor will exactly divide their sum and difference, and also the sum and difference of any multiple of the expressions.*

This rule may be proved as follows :

Let A and B stand for two such expressions both arranged according to descending powers of some common letter* and let the index of the highest power of that letter in A be not less than the index of the highest power of that letter in B.

Divide A by B, and let Q be the quotient and C, the remainder.

Then, we must have $C = A - BQ$, ... (1)

or, $A = BQ + C$. ... (2)

---

* The letter is called the *letter of reference.*

From (1), it is clear that every common factor of A and B is a factor of C [for if $A=pa$ and $B=pb$, we have $C=p(a-bQ)$]. Hence, if H denote the H.C.F. of A and B, H also is a factor of C, and is, therefore, a *common factor* of B and C.

It is clear, therefore, that the H.C.F. of B and C is either H or an expression of higher dimensions than H. ... (a)

Now, from (2), it is evident that *every* common factor of B and C is a factor of A, and is, therefore, a common factor of B and A. Hence, the H.C.F. of B and C also is a common factor of B and A, and therefore, cannot be of higher dimensions than H.

Hence, from (a), the H.C.F. of B and C is H.

Thus, the H.C.F. of B and C is the H.C.F. required.

Similarly, if B be divided by C, and D be the new remainder, the H.C.F. of C and D is the same as the H.C.F. of B and C, and is, therefore, the H.C.F. required.

Now, divide C by D, and let there be no remainder. Then D is the H.C.F. of C and D and is, therefore, the H.C.F. required.

Cor. 1. As the H.C.F. of any divisor and the corresponding dividend is the H.C.F. required, it is clear that, for the sake of convenience, either of them may be multiplied or divided by any monomial expression *which is not a factor of the other.* [See Note 8, Art. 100]

Cor. 2. In dividing A by B if we stop before the complete* quotient is obtained so that q is the partial quotient and C' the corresponding remainder, then the H.C.F. of B and C' just as the H.C.F. of B and C is the H.C.F. required. Hence, by Cor. 1, in dividing C' by B (or if convenient, B by C' when C' is *not* of higher degree than B) we can multiply or divide either of them, if necessary, by any monomial expression *which is not a factor of the other.*

The following examples will illustrate the process :

Example 1. Find the H.C.F. of $3x^3-7x^2-18x-8$ and $2x^3-3x^2-17x-12$.

The H.C.F. required is evidently the H.C.F. of $3x^3-7x^2-18x-8$ and $3(2x^3-3x^2-17x-12)$ [Cor. 1]. Let us, therefore, multiply the 2nd expression by 3 and divide the product by the 1st :

$$\begin{array}{r|l}
 & 2x^3-3x^2-17x-12 \\
 & 3 \\ \hline
3x^3-7x^2-18x-8 & 6x^3-9x^2-51x-36 \;(\,2 \\
 & 6x^3-14x^2-36x-16 \\ \hline
 & \qquad\;\; 5x^2-15x-20
\end{array}$$

Hence, the H.C.F. required is the H.C.F. of $3x^3-7x^2-18x-8$ and $5x^2-15x-20=5(x^2-3x-4)$, and is, therefore, the H.C.F. of $3x^3-7x^2-18x-8$ and x^2-3x-4. [Cor. 1]

---

* The quotient obtained is said to be complete when the remainder is of lower degree in the letter of reference than the divisor.

We must proceed then as follows:

$$\begin{array}{r@{\,}l@{\,}l}
 & 5)\ \underline{5x^2-15x-20} & \\
x^2-3x-4 & \big)\ 3x^3-7x^2-18x-8 & \big(\,3x+2 \\
 & \ \underline{3x^3-9x^2-12x} & \\
 & \phantom{\big)\ 3x^3-{}}2x^2-6x-8 & \\
 & \phantom{\big)\ 3x^3-{}}\underline{2x^2-6x-8} &
\end{array}$$

Hence, the H.C.F. required $= x^2-3x-4$.

The work of finding the H.C.F. in compact form:

$$\begin{array}{r|r|r|l}
3x & 3x^3-7x^2-18x-8 & 2x^3-3x^2-17x-12 & \\
 & \underline{3x^3-9x^2-12x} & \underline{3} & \\
2 & 2x^2-6x-8 & 6x^3-9x^2-51x-36 & 2 \\
 & \underline{2x^2-6x-8} & \underline{6x^3-14x^2-36x-16} & \\
 & & 5\ )\ \underline{5x^2-15x-20} & \\
 & & x^2-3x-4 &
\end{array}$$

The given expressions are arranged according to descending powers of x. As the coefficients of the highest powers of x are 3 and 2 and therefore, one is not exactly divisible by the other, $2x^3$ is multiplied by 3. The product is divided by $3x^3-7x^2-18x-8$ and the quotient 2 is written to the right. Since 5 is a factor of the remainder and not of the first divisor, it is removed and x^2-3x-4 is taken as the new divisor and $3x^3-7x^2-18x-8$ is divided by it. The quotients are written to the left. As there is no remainder, x^2-3x-4 is the required H.C.F.

With detached coefficients the work would stand as follows:

$$\begin{array}{r|r|r|l}
3 & 3-7-18-8 & 2-3-17-12 & \\
 & \underline{3-9-12} & \underline{3} & \\
2 & 2-6-8 & 6-9-51-36 & 2 \\
 & \underline{2-6-8} & \underline{6-14-36-16} & \\
 & & 5\ )\ \underline{5-15-20} & \\
 & & 1-3-4 &
\end{array}$$

Example 2. Find the H.C.F. of

$$22x^6-78x^5-16x^2 \text{ and } 2x^5-78x^2-44x.$$

The 1st expression $=2x^2(11x^4-39x^3-8)$.

The 2nd expression $=2x(x^4-39x-22)$.

Hence, by Note 7, Art. 100, H.C.F. required

$= ($ the H.C.F. of $2x^2$ and $2x$ $) \times ($ the H.C.F. of $11x^4-39x^3-8$ and $x^4-39x-22$ $)$

$= 2x \times X$, putting X for the H.C.F. of the multinomials.

Now, let us find X, as in the last example :

$$\begin{array}{rl}
x^4-39x-22\,\big) & 11x^4-39x^3-8\,\big(\,11 \\
 & \underline{11x^4-429x-242} \\
-3\,\big) & \underline{-39x^3+429x+234} \\
13\,\big) & \underline{13x^3-143x-78} \\
 & x^3-11x-6
\end{array}$$

$$\begin{array}{rl}
x^3-11x-6\,\big) & x^4-39x-22\,\big(\,x \\
 & \underline{x^4-11x^2-6x} \\
11\,\big) & \underline{11x^2-33x-22} \\
 & x^2-3x-2
\end{array}$$

$$\begin{array}{rl}
x^2-3x-2\,\big) & x^3-11x-6\,\big(\,x+3 \\
 & \underline{x^3-3x^2-2x} \\
 & 3x^2-9x-6 \\
 & \underline{3x^2-9x-6}
\end{array}$$

Thus, $X=x^2-3x-2$.

Hence, the H.C.F. required $=2x(x^2-3x-2)$.

Example 3. Find the H.C.F. of $12x^4a^2+54x^3a^3+6x^2a^4-72xa^5$ and $8x^6a+60x^5a^2+160x^4a^3+180x^3a^4+72x^2a^5$.

The 1st expression $=6xa^2(2x^3+9x^2a+xa^2-12a^3)$.

The 2nd expression $=4x^2a(2x^4+15x^3a+40x^2a^2+45xa^3+18a^4)$.

Hence, if X denote the H.C.F. of the multinomial factors of the given expressions, we must have the required H.C.F. $=$ (the H.C.F. of $6xa^2$ and $4x^2a$) $\times X=2xa\times X$.

Now, to find X.

$$\begin{array}{rl}
2x^3+9x^2a+xa^2-12a^3\,\big) & 2x^4+15x^3a+40x^2a^2+45xa^3+18a^4\,\big(\,x \\
 & \underline{2x^4+9x^3a+x^2a^2-12xa^3} \\
3a\,\big) & \underline{6x^3a+39x^2a^2+57xa^3+18a^4} \\
 & 2x^3+13x^2a+19xa^2+6a^3
\end{array}$$

Hence, X is the H.C.F. of $2x^3+9x^2a+xa^2-12a^3$ and $2x^3+13x^2a+19xa^2+6a^3$, and as they are both of the same degree we can divide either of them by the other.

$$\begin{array}{rl}
2x^3+9x^2a+xa^2-12a^3\,\big) & 2x^3+13x^2a+19xa^2+6a^3\,\big(\,1 \\
 & \underline{2x^3+9x^2a+xa^2-12a^3} \\
2a\,\big) & \underline{4x^2a+18xa^2+18a^3} \\
 & 2x^2+9xa+9a^2
\end{array}$$

$$\begin{array}{r|l|l}
2x^2+9xa+9a^2 & 2x^3+9x^2a+\ xa^2-12a^3 & x \\
 & \underline{2x^3+9x^2a+9xa^2\qquad\qquad} & \\
 & \quad -4a^2\,)\,\underline{-8xa^2-12a^3} & \\
 & \qquad\qquad\ \ 2x\ \ +\ 3a &
\end{array}$$

$$\begin{array}{r|l|l}
2x+3a & 2x^2+9xa+9a^2 & x+3a \\
 & \underline{2x^2+3xa\qquad\ \ } & \\
 & \qquad\quad 6xa+9a^2 & \\
 & \qquad\quad \underline{6xa+9a^2} &
\end{array}$$

Thus, $X=2x+3a$

Hence, the H.C.F. required $=2xa(2x+3a)$.

Example 4. Find the H.C.F. of

$$4x^4+11x^3+27x^2+17x+5 \text{ and } 6x^4+14x^3+36x^2+14x+10.$$

The second expression $=2(3x^4+7x^3+18x^2+7x+5)$, but 2 is not a factor of the 1st expression. Hence, the H.C.F. required is the H.C.F. of the 1st expression and $3x^4+7x^3+18x^2+7x+5$.

$$\begin{array}{r|l|l}
 & 4x^4+11x^3+27x^2+17x+5 & \\
 & \underline{3\qquad\qquad\qquad\qquad\qquad\qquad} & \\
3x^4+7x^3+18x^2+7x+5 & 12x^4+33x^3+81x^2+51x+15 & 4 \\
 & \underline{12x^4+28x^3+72x^2+28x+20} & \\
 & \qquad 5x^3+\ \ 9x^2+23x-\ \ 5 &
\end{array}$$

$$\begin{array}{r|l|l}
 & 3x^4+7x^3+18x^2+7x+5 & \\
 & \underline{5\qquad\qquad\qquad\qquad\qquad} & \\
5x^3+9x^2+23x-5 & 15x^4+35x^3+90x^2+35x+25 & 3x \\
 & \underline{15x^4+27x^3+69x^2-15x\qquad\ } & \\
 & \qquad 8x^3+21x^2+50x+25 & \\
 & \qquad \underline{5\qquad\qquad\qquad\qquad\ \ } & \\
 & \qquad 40x^3+105x^2+250x+125 & 8 \\
 & \qquad \underline{40x^3+\ \ 72x^2+184x-\ \ 40} & \\
 & \qquad\quad 33\,)\,\underline{33x^2+\ \ 66x+165} & \\
 & \qquad\qquad\quad\ \ x^2+\ \ \ 2x+5 &
\end{array}$$

$$\begin{array}{r|l|l}
x^2+2x+5 & 5x^3+\ \ 9x^2+23x-5 & 5x-1 \\
 & \underline{5x^3+10x^2+25x\qquad} & \\
 & \qquad\quad -x^2-\ \ 2x-5 & \\
 & \qquad\quad \underline{-x^2-\ \ 2x-5} &
\end{array}$$

Thus, the required H.C.F. $=x^2+2x+5$.

Example 5. Find the H.C.F. of

$$4x^4-16x^3+108 \text{ and } 6x^5-14x^3-40x^2+36$$

The first expression $=4(x^4-4x^3+27)$.

The second expression $=2(3x^5-7x^3-20x^2+18)$.

Hence, if X denote the H.C.F. of the multinomial factors of the given expressions, the H.C.F. required $=2X$.

Let us then find X.

$$\begin{array}{r|l|l}
x^4-4x^3+27 & 3x^5-7x^3-20x^2+18 & 3x+12 \\
 & 3x^5-12x^4+81x & \\
\hline
 & 12x^4-7x^3-20x^2-81x+18 & \\
 & 12x^4-48x^3+324 & \\
\hline
 & 41x^3-20x^2-81x-306 &
\end{array}$$

$$\begin{array}{r|l|l}
 & x^4-4x^3+27 & \\
 & 41 & \\
\hline
41x^3-20x^2-81x-306 & 41x^4-164x^3+1107 & x \\
 & 41x^4-20x^3-81x^2-306x & \\
\hline
-9\,) & -144x^3+81x^2+306x+1107 & \\
 & 16x^3-9x^2-34x-123 & \\
 & 41 & \\
\hline
 & 656x^3-369x^2-1394x-5043 & 16 \\
 & 656x^3-320x^2-1296x-4896 & \\
\hline
-49\,) & -49x^2-98x-147 & \\
 & x^2+2x+3 &
\end{array}$$

$$\begin{array}{r|l|l}
x^2+2x+3 & 41x^3-20x^2-81x-306 & 41x-102 \\
 & 41x^3+82x^2+123x & \\
\hline
 & -102x^2-204x-306 & \\
 & -102x^2-204x-306 & \\
\hline
\end{array}$$

Hence, the H.C.F. required $=2(x^2+2x+3)$.

EXERCISE 83

Find the H.C.F. of :

1. $2x^2+5x-3$ and $2x^3+3x^2-32x+15$.
2. $3x^2+16x-12$ and $3x^3+4x^2-28x+16$.
3. $2x^2-3ax-20a^2$ and $2x^3+3ax^2-45a^2x-100a^3$.
4. $3x^4+7x^3-14x^2-24x$ and $6x^4-10x^3-24x^2$.
5. $6a^3-11a^2-3a+2$ and $3a^3+20a^2+23a-10$.
6. $6a^3-25a^2b+32ab^2-12b^3$ and $4a^3+12a^2b-7ab^2-30b^3$.
7. $3x^3+5x^2+5x+2$ and $2x^3+5x^2+5x+3$.
8. $4x^3-7x^2y+7xy^2-3y^3$ and $3x^3-7x^2y+7xy^2-4y^3$.
9. $6x^4+7x^3+5x^2+2x$ and $4x^5-18x^4-8x^3-10x^2$.
10. $3x^4+10x^3+7x^2+4x+1$ and $2x^3+3x^2-7x-3$.
11. $4x^3+13x^2-8x-3$ and $3x^4+13x^3+9x^2+9x+2$.

12. $12a^3+11a^2x+6ax^2+x^3$ and $21a^3+17a^2x+9ax^2+x^3$.

13. $35a^3+31a^2x+13ax^2+2x^3$ and $65a^3+54a^2x+22ax^2+3x^3$.

14. $70x^3-9ax^2+11a^2x+6a^3$ and $91x^3-25ax^2+20a^2x+4a^3$.

15. $75x^3-35x^2+24x+4$ and $85x^3-36x^2+25x+6$.

16. $35x^3-34x^2+3x+2$ and $49x^3-49x^2+5x+3$.

17. $4x^6+2ax^5+14a^2x^4+10a^3x^3+24a^4x^2$ and $6x^6+21ax^5+30a^2x^4+24a^3x^3$.

18. $4a^4+32a^3+72a^2+44a+8$ and $6a^4+54a^3+138a^2+78a+12$.

19. $2x^4-19x^2+21x-6$ and $6x^4+21x^3+3x-6$.

20. $12x^4-30x^2+126x+90$ and $15x^4-25x^3+145x-75$.

21. $18x^4+117x^3+162x^2+72x+9$ and $12x^4+68x^3+72x^2+108x+20$.

22. $x^5-5x^2+6x+12$ and $x^4-8x^2-24x-32$.

23. $x^4+5x^3+3x^2-14x-40$ and $x^5-4x^3+45x+75$.

24. $4x^5-8x^3a^2+28x^2a^3-24xa^4+24a^5$ and $6x^4+24x^3a-12x^2a^2-24xa^3+96a^4$.

25. $9x^4-18x^3y-13x^2y^2-38xy^3-12y^4$ and $6x^5+4x^4y+5x^3y^2+4x^2y^3+8y^5$.

26. $2x^5-11x^2-9$ and $4x^5+11x^4+81$.

27. $32a^4+104a^3-20a^2-122a+30$ and $60a^5+10a^4-45a^3+45a^2-50a$.

28. $x^5+2x^4-5x^2-7x+3$ and $3x^6-3x^4-18x^3+x^2+2x+3$.

160. In some cases the H.C.F. may be found more easily by the application of the following principles:

If A and B denote two expressions having no monomial factors, and if m, n, p, q be any four numerical quantities such that $mq-np$ is not equal to zero, then the H.C.F. of A and B is the same as the H.C.F. of $mA+nB$ and $pA+qB$, numerical common factors, if any, being left out. This may be proved as follows:

Let H denote the H.C.F. of A and B, and H' the H.C.F. of $mA+nB$ and $pA+qB$, after removal from them of any numerical common factors that may occur.

Now, since *every* common factor of A and B is a factor of $mA+nB$ and also of $pA+qB$, therefore, H is a common factor of $mA+nB$ and $pA+qB$. Hence, H' is either equal to H or is an expression of higher dimensions than H. ... ... (a)

Again, since $q(mA+nB)-n(pA+qB)=(mq-np)A$,

and $m(pA+qB)-p(mA+nB)=(mq-np)B$,

it is clear that *every* common factor of $mA+nB$ and $pA+qB$ is a factor of $(mq-np)A$, and also of $(mq-np)B$. Hence, as $mq-np$ is only a numerical quantity, *every* common factor of those expressions *other than numerical* must be a factor of A as well as of B. Hence, H' is a common factor of A and B, and therefore, cannot be of higher dimensions than H.

Hence, by (*a*), $H'=H$, which proves the proposition.

Cor. 1. The H.C.F. of A and B is the same as the H.C.F. of $A+B$ and $A-B$. Here $m=1$, $n=1$, $p=1$ and $q=-1$.

Cor. 2. The H.C.F. of A and B is the same as the H.C.F. of $A\pm B$ and B; here $m=1$, $n=\pm 1$, $p=0$ and $q=1$. Similarly, it is the same as the H.C.F. of $A\pm B$ and A.

Example 1. Find the H.C.F. of

$$x^4+x^3-5x^2-3x+2 \text{ and } x^4-3x^3+x^2+3x-2.$$

Let $A=x^4+x^3-5x^2-3x+2$,

and $B=x^4-3x^3+x^2+3x-2$.

Then, $A+B=2x^4-2x^3-4x^2=2x^2(x^2-x-2)$,

and $A-B=4x^3-6x^2-6x+4=2(2x^3-3x^2-3x+2)$.

Hence, by Cor. 1, the required H.C.F. is the H.C.F. of $x^2(x^2-x-2)$ and $2x^3-3x^2-3x+2$, and therefore, of x^2-x-2 and $2x^3-3x^2-3x+2$.

Let $A'=x^2-x-2$, and $B'=2x^3-3x^2-3x+2$.

Then, $A'+B'=2x^3-2x^2-4x=2x(x^2-x-2)$.

Hence, the required H.C.F.

$$=\text{the H.C.F. of } A' \text{ and } A'+B' \qquad [\text{Cor. } 2]$$
$$=x^2-x-2.$$

Example 2. Find the H.C.F. of

$$4x^4+11x^3+27x^2+17x+5 \text{ and } 3x^4+7x^3+18x^2+7x+5.$$

Let $A=4x^4+11x^3+27x^2+17x+5$,

and $B=3x^4+7x^3+18x^2+7x+5$.

Then, $A-B=x^4+4x^3+9x^2+10x=x(x^3+4x^2+9x+10)$,

and $3A-4B=5x^3+9x^2+23x-5$.

Hence, the H.C.F. of $x^3+4x^2+9x+10$ and

$5x^3+9x^2+23x-5$ is the H.C.F. required.

Let $A'=x^3+4x^2+9x+10$, and $B'=5x^3+9x^2+23x-5$.

Then, $A'+2B'=11x^3+22x^2+55x=11x(x^2+2x+5)$,

and $5A'-B'=11x^2+22x+55=11(x^2+2x+5)$.

Hence, the H.C.F. required is the H.C.F. of $x(x^2+2x+5)$ and x^2+2x+5, and is, therefore $=x^2+2x+5$.

Example 3. Find the H.C.F. of

$2x^5-11x^2-9$ and $4x^5+11x^4+81.$ [C. U. 1865]

Let $A=4x^5+11x^4+81,$

and $B=2x^5-11x^2-9.$

Then, $A-2B=11x^4+22x^2+99=11(x^4+2x^2+9),$

and $A+9B=22x^5+11x^4-99x^2=11x^2(2x^3+x^2-9).$

Hence, the required H.C.F. is the same as the H.C.F. of x^4+2x^2+9 and $x^2(2x^3+x^2-9)$, and, therefore, of x^4+2x^2+9 and $2x^3+x^2-9.$

Let $A'=x^4+2x^2+9$ and $B'=2x^3+x^2-9.$

Then, $A'+B'=x^4+2x^3+3x^2=x^2(x^2+2x+3).$

Hence, the H.C.F. of

$$\left.\begin{array}{l} 2x^3+x^2-9\ (=B') \\ \text{and}\quad x^2+2x+3\ (=C' \text{ say}) \end{array}\right\} \text{is the H.C.F. required.}$$

Now, since $B'+3C'=2x^3+4x^2+6x$
$=2x(x^2+2x+3);$

$\therefore$ the H.C.F. reqd. = the H.C.F. of C' and $B'+3C'$
$=x^2+2x+3.$

EXERCISE 84

Find the H.C.F. of:

1. $x^3-3x^2-4x+12$ and $x^3-7x^2+16x-12.$
2. $2x^3-17x+12$ and $4x^4-2x^3-34x^2+41x-12.$
3. $4x^3+13x^2+19x+4$ and $2x^3+5x^2+5x-4.$
4. $3x^3-5x^2+7$ and $6x^4-7x^3-5x^2+14x+7.$
5. $6x^4-11x^3+16x^2-22x+8$ and $6x^4-11x^3-8x^2+22x-8.$
6. $2x^4+19x^3+20x^2-31x+8$ and $2x^4+7x^3-64x^2+62x-16.$
7. $3x^4-7x^3-27x^2-6x+2$ and $3x^4-13x^3-40x^2-9x+3.$
8. $5x^4-18x^3-7x^2+12x+3$ and $5x^4-23x^3-9x^2+16x+4.$
9. $2x^4-5x^3-17x^2-2x+2$ and $6x^5+23x^4+34x^3+21x^2-2x-2.$
10. $6x^5+9x^4-13x^3-4x^2+9x-3$ and $9x^5+12x^4-18x^3-5x^2+12x-4.$
11. x^5-x^3+8 and $x^5-x^2+4.$
12. $3x^5+139x^2-44$ and $39x^5+139x^4-16.$

161. The H.C.F. of three or more expressions whose factors cannot be easily found.

Let A, B, C stand for any three expressions of which the H.C.F. is to be found.

Let G denote the H.C.F. of A and B, and H that of G and C.

Then G being the product of *all* the elementary common factors of A and B, every factor of G is a common factor of A and B, and therefore, every common factor of G and C is a common factor of A, B and C.

Hence, H also is a common factor of A, B and C. Therefore, the H.C.F. required is either H or an expression of higher dimensions than H. ... ... (β)

But, since every common factor of A and B is a factor of G, *every* common factor of A, B and C is a common factor of G and C. Hence, the H.C.F. required is a common factor of G and C, and therefore, cannot be of higher dimensions than H.

Hence, by (β), the H.C.F. required $=H$.

By a similar reasoning it follows that if D be a fourth expression, then the H.C.F. of H and D is the H.C.F. of A, B, C and D.

Thus, we have the following rule:

To find the H.C.F. of any number of expressions A, B, C, D, &c., first find the H.C.F. of A and B, then the H.C.F. of this result and C, and so on; the result obtained last of all is the H.C.F. required.

Example 1. Find the H.C.F. of $2x^4-7x^3-17x^2+58x-24$,

$3x^4+14x^3-11x^2-70x+24$ and $5x^4+9x^3-47x^2-81x+18$.

Let us first find the H.C.F. of the first two expressions.

Put $A=2x^4-7x^3-17x^2+58x-24$,

and $B=3x^4+14x^3-11x^2-70x+24$.

Then, $A+B=5x^4+7x^3-28x^2-12x$

$=x(5x^3+7x^2-28x-12)$,

and $-3A+2B=49x^3+29x^2-314x+120$.

Hence, the H.C.F. of A and B is the H.C.F of $5x^3+7x^2-28x-12$ and $49x^3+29x^2-314x+120$.

Let $A'=5x^3+7x^2-28x-12$,

and $B'=49x^3+29x^2-314x+120$.

Then, $10A'+B'=99x^3+99x^2-594x$

$=99x(x^2+x-6)$.

Hence, the H.C.F. of A and B is the same as the H.C.F. of

$$\left.\begin{array}{ll} & 5x^3+7x^2-28x-12\ (=A') \\ \text{and} & x^2+x-6\ (=C', \text{say}). \end{array}\right\}$$

Now, $A' - 2C' = 5x^3 + 5x^2 - 30x = 5x(x^2 + x - 6)$;

$\therefore$ the H.C.F. of A and B = the H.C.F. of C' and $A' - 2C'$

$= x^2 + x - 6.$

Hence, the H.C.F. required is the H.C.F. of $x^2 + x - 6$ and $5x^4 + 9x^3 - 47x^2 - 81x + 18$, which can be found as follows :

$$\begin{array}{r|l|l}
x^2+x-6 & 5x^4+9x^3-47x^2+81x+18 & 5x^2+4x \\
 & \underline{5x^2+5x^3-30x^2} & \\
 & \quad 4x^3-17x^2-81x+18 & \\
 & \quad \underline{4x^3+\ \ 4x^2-24x} & \\
-3 & \quad -21x^2-57x+18 & \\
 & \qquad 7x^2+19x-\ \ 6 & 7 \\
 & \qquad \underline{7x^2+\ \ 7x-42} & \\
12 & \qquad \underline{12x+36} & \\
 & \qquad\quad x+\ \ 3 &
\end{array}
\qquad
\begin{array}{r|l|l}
x+3 & x^2+\ \ x-6 & x-2 \\
 & \underline{x^2+3x} & \\
 & \quad -2x-6 & \\
 & \quad \underline{-2x-6} &
\end{array}$$

Thus, the required H.C.F. $= x + 3$.

Example 2. If $(x+m)$ is the H.C.F. of $x^2 + ax + b$ and $x^2 - a_1x + b_1$, prove that $m = \dfrac{b - b_1}{a + a_1}$.

$(x + m)$ is the H.C.F. of the given expressions ;

$\therefore$ $(x + m)$ is a factor of each of the expressions.

$\therefore$ by Remainder Theorem,

$x = (-m)$ makes each of the expressions $= 0$.

$\therefore$ $m^2 - am + b = 0$, and $m^2 + a_1m + b_1 = 0$.

By subtracting,

$(a_1 + a)\,m + (b_1 - b) = 0,$

or, $(a_1 + a)\,m = b - b_1$; $\quad \therefore\ m = \dfrac{b - b_1}{a + a_1}.$

EXERCISE 85

Find the H. C. F. of :

1. $2x^3 + 7x^2 - 5x - 4$, $x^3 + 8x^2 + 11x - 20$ and $2x^3 + 19x^2 + 49x + 20$.
2. $2x^4 + 3x^3 + 8x^2 + 15x - 10$, $2x^4 - 5x^3 + 12x^2 - 25x + 10$ and $2x^4 - 5x^3 + 10x^2 - 20x + 8$.
3. $2x^4 + 7x^3 - 19x^2 - 14x + 30$, $2x^4 + 5x^3 - 16x^2 - 10x + 24$ and $2x^4 + 5x^3 - 10x^2 + 5x - 12$.
4. $2x^4 - 4x^3 - 60x^2 - 2x - 35$, $2x^4 - 6x^3 - 55x^2 - 3x - 28$ and $2x^4 + 18x^3 + 41x^2 + 9x + 20$.

5. $3a^3+28a^2b+52ab^2-48b^3$, $3a^3+4a^2b-28ab^2+16b^3$ and $3a^3+10a^2b-44ab^2+24b^3$.

6. $6a^3+5a^2b-34ab^2+15b^3$, $6a^3-37a^2b+57ab^2-20b^3$ and $3a^3-8a^2b-31ab^2+60b^3$.

7. $3x^4+11x^3-32x^2-44x+80$, $3x^4-x^3-52x^2+124x-80$, $3x^4+2x^3-20x^2-8x+32$ and $3x^4+2x^3-83x^2-50x+200$.

8. $6x^5+14x^4-53x^3-37x^2+66x+24$, $6x^5-28x^4+17x^3+54x^2-39x-18$, $6x^5+8x^4-79x^3-36x^2+105x+36$ and $2x^5-2x^4-31x^3+51x^2+42x-72$.

9. If the H.C.F. of x^2+px+q and $x^2+p'x+q'$ is $x+a$, show that $(p-p')\,a=q-q'$. [C. U. 1941]

II. Harder L.C.M.

162. L.C.M. of two expressions whose factors are not obvious by inspection.

Let A and B stand for two such expressions, and suppose their H.C.F. is found to be H.

Divide A and B by H and let the respective quotients be a and b Then, we have

$$\left.\begin{array}{l} A=aH \\ B=bH \end{array}\right\}$$

Hence, since a and b have no common factors, *every* common multiple of A and B must *necessarily* contain $a\times H\times b$ as a factor.

Hence, the L.C.M. required $=aHb$.

$$\left.\begin{array}{ll} \text{But,} & aHb=a(Hb)=\dfrac{A}{H}\times B \\ & \text{or, } =(aH)b=A\times\dfrac{B}{H} \end{array}\right\}$$

Hence, the required L.C.M. $=\dfrac{A}{H}\times B$, or $=A\times\dfrac{B}{H}$.

Thus, *to find the L.C.M. of any two expressions we have to divide one of them by their H.C.F. and multiply the quotient by the other.*

Cor. If L denote the L.C.M. of A and B, we have $L\times H=A\times B$; that is, *the product of the L.C.M. and H.C.F. of any two expressions is equal to the product of these expressions.*

Note. *If any two expressions have no common factor, their L.C.M. is evidently equal to their product.*

Example. Find the L.C.M. of

$$6x^3+25x^2+16x+7 \text{ and } 6x^3-11x^2-8x-5.$$

$$
\begin{array}{rl}
6x^3-11x^2-8x-5\,\big)\, & 6x^3+25x^2+16x+7\,\big(\,1 \\
& \underline{6x^3-11x^2-8x-5} \\
12\,\big)\, & \underline{36x^2+24x+12} \\
& 3x^2+2x+1
\end{array}
$$

$$
\begin{array}{rl}
3x^2+2x+1\,\big)\, & 6x^3-11x^2-8x-5\,\big(\,2x-5 \\
& \underline{6x^3+4x^2+2x} \\
& -15x^2-10x-5 \\
& \underline{-15x^2-10x-5}
\end{array}
$$

Thus the H.C.F. of the given expressions $=3x^2+2x+1$

Hence, the L.C.M. required

$$=\frac{6x^3-11x^2-8x-5}{3x^2+2x+1}(6x^3+25x^2+16x+7)$$
$$=(2x-5)(6x^3+25x^2+16x+7)$$
$$=12x^4+20x^3-93x^2-66x-35$$

EXERCISE 86

Find the L.C.M. of :

1. $3x^3+2x^2-11x+4$ and $3x^3+14x^2+13x-8$
2. $6x^3+17x^2+9x-4$ and $6x^3-7x^2-27x+8$.
3. $12x^3-4x^2-25x+12$ and $12x^3-28x^2+7x+12$.
4. $9x^3-12x^2-15x+20$ and $15x^3+12x^2-25x-20$.
5. $4x^3-10x^2-18x+45$ and $6x^3+8x^2-27x-36$.
6. $4x^4+4x^3+7x^2+11x+4$ and $6x^4+7x^3+4x^2+5x+2$.
7. $8x^4-6x^3-8x^2+9x-6$ and $16x^4-12x^3+20x^2-9x+6$.
8. $4x^4+8x^3+21x^2+18x+27$ and $3x^4+6x^3+17x^2+16x+24$.

9. The H.C.F. and the L.C.M. of two expressions are $x-7$ and $x^3-10x^2+11x+70$ respectively; if one of them is $x^2-5x-14$, what is the other? [B. U. 1925]

10. The H.C.F. and the L.C.M. of two expressions of second degree are $x-1$ and x^3-7x+6 respectively. Find the expressions. [D. B. 1927]

11. If h be the Highest Common Divisor and l, the Lowest Common Multiple of two quantities x and y, and if $h+l=x+y$, prove that $h^3+l^3=x^3+y^3$. [P. U. 1891]

168. L.C.M. of three or more expressions whose factors are not obvious by inspection.

Let A, B, C stand for three such expressions; to find their L.C.M.

Let L denote the L.C.M. of A and B, and M that of L and C.

Then evidently *every* common multiple of L and C is a common multiple of A, B, C; ... ... (1)

also *every* common multiple of A, B, C is a multiple of M. ... (2)

From (1), M is a common multiple of A, B, C. Hence, either M or an expression of a lower degree than M is the L.C.M. of A, B, C.

But an expression of a lower degree than M cannot be the L.C.M. of A, B, C; because from (2), the L.C.M. of A, B, C is a common multiple of L and C.

Hence, the required L.C.M. $= M$.

Thus, *to find the L.C.M. of any number of expressions A, B, C, D, &c., we have first to find the L.C.M. of A and B, then the L.C.M. of the result and C, and so on; the last result thus obtained is the L.C.M. required.*

Example 1. Find the L.C.M. of x^3-2x^2-x+2, $3x^3-7x^2+4$ and $2x^3-3x^2+1$.

$$\begin{array}{r|r|r|l}
x & x^3-2x^2-x+2 & 3x^3-7x^2+4 & 3\\
 & x^3-3x^2+2x & 3x^3-6x^2-3x+6 & \\
\hline
1 & x^2-3x+2 & -1\ \ \ -x^2+3x-2 & \\
 & x^2-3x+2 & x^2-3x+2 & \\
\hline
\end{array}$$

Thus, the H.C.F. of the expressions considered $= x^2-3x+2$.

Hence, the L.C.M. of the first two expressions

$$= \frac{x^3-2x^2-x+2}{x^2-3x+2} \times 3x^3-7x^2+4$$

$$= (x+1)(3x^3-7x^2+4) = 3x^4-4x^3-7x^2+4x+4.$$

Now, to find the L.C.M. of the result and $2x^3-3x^2+1$.

$$\begin{array}{r|r|r|l}
 & 2x^3-3x^2+1 & 3x^4-4x^3-7x^2+4x+4 & \\
 & 5 & 2 & \\
\hline
2x & 10x^3-15x^2+5 & 6x^4-8x^3-14x^2+8x+8 & 3x\\
 & 10x^3-4x^2-6x & 6x^4-9x^3+3x & \\
\hline
 & -11x^2+6x+5 & x^3-14x^2+5x+8 & \\
 & -5 & 2 & \\
\hline
11 & 55x^2-30x-25 & 2x^3-28x^2+10x+16 & 1\\
 & 55x^2-22x-33 & 2x^3-3x^2+1 & \\
\hline
 & -8\ \ \ -8x+8 & -5\ \ \ -25x^2+10x+15 & \\
 & x-1 & 5x^2-2x-3 & 5x\\
 & & 5x^2-5x & \\
\hline
 & & 3x-3 & 3\\
 & & 3x-3 & \\
\hline
\end{array}$$

Thus, the H.C.F. of the expressions considered $= x - 1$.

Hence, their L.C.M. $= \dfrac{(2x^3 - 3x^2 + 1)}{(x - 1)} \times (3x^4 - 4x^3 - 7x^2 + 4x + 4)$

$= (2x^2 - x - 1)(3x^4 - 4x^3 - 7x^2 + 4x + 4)$

$= 6x^6 - 11x^5 - 13x^4 + 19x^3 + 11x^2 - 8x - 4.$

EXERCISE 87

Find the L.C.M. of :

1. $6x^3 + 11x^2 + 6x + 1$, $4x^3 - 7x - 3$ and $6x^3 - x^2 - 10x - 3$.
2. $6x^3 + 25x^2 + 16x + 7$, $6x^3 - 11x^2 - 8x - 5$ and $2x^3 + 5x^2 - 5x + 7$.
3. $x^3 - 2x^2 - 19x + 20$, $x^3 + 2x^2 - 23x - 60$ and $x^4 + 7x^3 - 4x^2 - 52x + 48$. [B. U. 1891]
4. $2x^4 + 4x^3 + x^2 + 6x - 3$, $4x^4 + 8x^3 - 7x^2 - 6x + 3$ and $8x^4 + 4x^3 - 2x^2 - 3x - 3$.

CHAPTER XXV

HARDER FRACTIONS

164. In this Chapter we shall consider fractions of a harder type than those treated of in Chapter XVI.

I. Reduction of Fractions

165. A fraction is said to be reduced to its lowest terms, when its numerator and denominator have no common factor. In all cases where the numerator and the denominator can be factorised by inspection, the reduction is effected by simply removing the common factors. Otherwise, divide both the numerator and the denominator by their highest common factor.

Example 1. Reduce to its lowest terms

$$\frac{a^3 + b^3 + c^3 - 3abc}{a^2(b+c) + b^2(c+a) + c^2(a+b) + 3abc}$$

The fraction $= \dfrac{(a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab)}{(a+b+c)(bc + ca + ab)}$

$= \dfrac{a^2 + b^2 + c^2 - bc - ca - ab}{bc + ca + ab}.$

Example 2. Simplify $\dfrac{8(x+y+z)^3 - (y+z)^3 - (z+x)^3 - (x+y)^3}{3(2x+y+z)(x+2y+z)(x+y+2z)}.$

The fraction $= \dfrac{3(2x+y+z)(x+2y+z)(x+y+2z)}{3(2x+y+z)(x+2y+z)(x+y+2z)}$. [Ex. 1, Art. 132]

$= 1.$

Example 3. Reduce to its lowest terms

$$\frac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}. \qquad \text{[C. U. 1889]}$$

The numerator $= 3(x^3-9ax^2+26a^2x-24a^3)$.

The denominator $= 2(x^3+5ax^2-2a^2x-24a^3)$.

Now, to find their H.C.F.

$$\begin{array}{rl}
 & x^3+5ax^2-2a^2x-24a^3 \\
 & \underline{x^3-9ax^2+26a^2x-24a^3} \\
14ax) & \underline{14ax^2-28a^2x} \\
 & x-2a
\end{array}$$

$$x-2a\,\Big)\begin{array}{l}
x^3-9ax^2+26a^2x-24a^3 \\
\underline{x^3-2ax^2} \\
-7ax^2+26a^2x-24a^3 \\
\underline{-7ax^2+14a^2x} \\
12a^2x-24a^3 \\
\underline{12a^2x-24a^3}
\end{array}\Big(\,x^2-7ax+12a^2$$

Thus, the H.C.F. required $= x-2a$.

Hence, the required result

$$= \frac{3(x^3-9ax^2+26a^2x-24a^3)\div(x-2a)}{2(x^3+5ax^2-2a^2x-24a^3)\div(x-2a)} = \frac{3(x^2-7ax+12a^2)}{2(x^2+7ax+12a^2)}$$

Example 4. Reduce $\dfrac{2x^4-x^3-9x^2+13x-5}{7x^3-19x^2+17x-5}$ to its lowest terms.

[C. U. 1870]

The H.C.F. of the numerator and the denominator of the given fraction can be found as follows: [See Cor. 2, Art. 160]

$$\begin{array}{rl}
 & 2x^4-x^3-9x^2+13x-5 \\
 & \underline{7x^3-19x^2+17x-5} \\
2x) & \underline{2x^4-8x^3+10x^2-4x} \\
 & x^3-4x^2+5x-2
\end{array}$$

$$x^3-4x^2+5x-2\,\Big)\begin{array}{rl}
 & 7x^3-19x^2+17x-5 \\
 & \underline{7x^3-28x^2+35x-14} \\
9) & \underline{9x^2-18x+9} \\
 & x^2-2x+1
\end{array}\Big(\,7$$

$$x^2-2x+1\,\Big)\begin{array}{l}
x^3-4x^2+5x-2 \\
\underline{x^3-2x^2+x} \\
-2x^2+4x-2 \\
\underline{-2x^2+4x-2}
\end{array}\Big(\,x-2$$

Thus, the H.C.F. required $= x^2 - 2x + 1$.

Hence, the required result

$$= \frac{(2x^4 - x^3 - 9x^2 + 13x - 5) \div (x^2 - 2x + 1)}{(7x^3 - 19x^2 + 17x - 5) \div (x^2 - 2x + 1)} = \frac{2x^2 + 3x - 5}{7x - 5}.$$

EXERCISE 88

Reduce to the lowest terms :

1. $\dfrac{x^3 + 4x^2 + x - 6}{x^2 + x - 2}$.

2. $\dfrac{x^3 - 7x + 6}{x^3 + 2x^2 - 13x + 10}$.

3. $\dfrac{a^3 + 2a^2b - 2ab^2 + 3b^3}{a^3 - 5a^2b + 5ab^2 - 4b^3}$.

4. $\dfrac{x^4 + (2b^2 - a^2)x^2 + b^4}{x^4 + 2ax^3 + a^2x^2 - b^4}$.

5. $\dfrac{3x^3 + 4x^2y - 7xy^2 + 2y^3}{2x^3 + 9x^2y + 8xy^2 - 5y^3}$.

6. $\dfrac{1 + 3x - x^3 - 3x^4}{1 - x + 2x^2 + x^3 + 3x^4}$.

7. $\dfrac{x^4 - x^3 - x + 1}{x^4 - 2x^3 - x^2 - 2x + 1}$.

8. $\dfrac{x^4 + x^2 + 25}{x^4 - 9x^2 + 30x - 25}$.

9. $\dfrac{2x^3 + 3ax^2 + 5a^2x - 21a^3}{4x^3 - 12ax^2 + 19a^2x - 15a^3}$.

10. $\dfrac{2x^4 + x^3 - 3x^2 + 2x + 3}{3x^4 + x^3 - 4x^2 + 3x + 4}$.

11. $\dfrac{9x^3 - 7a^2x - 2a^3}{9x^3 + 6ax^2 - 5a^2x - 2a^3}$.

12. $\dfrac{2a^3 - 16a^2b + 44ab^2 - 42b^3}{3a^3 + 6a^2b - 24ab^2 - 63b^3}$.

13. $\dfrac{9x^4 + 30x^3 + 12x^2 - 6x - 45}{8x^4 + 28x^3 + 16x^2 - 4x - 48}$.

14. $\dfrac{6a^6 - 9a^5b + a^4b^2 + 3a^3b^3 - a^2b^4}{4a^5 - 6a^4b + 3a^2b^3 - ab^4}$.

15. $\dfrac{24x^5 + 16x^4y - 28x^3y^2 - 24x^2y^3 - 12xy^4}{45x^4y + 30x^3y^2 - 15x^2y^3 - 20xy^4 - 10y^5}$.

16. $\dfrac{(b+c)^3(b-c) + (c+a)^3(c-a) + (a+b)^3(a-b)}{(b+c)^2(b-c) + (c+a)^2(c-a) + (a+b)^2(a-b)}$.

17. $\dfrac{(1-x^2)(1-y^2)(1-z^2) - (yz+x)(zx+y)(xy+z)}{1 - x^2 - y^2 - z^2 - 2xyz}$.

18. $\dfrac{(x+y-2z)^3 + (y+z-2x)^3 + (z+x-2y)^3}{12(x+y-2z)(y+z)(y+z-2x)}$.

19. $\dfrac{(y-z)^2 + (z-x)^2 + (x-y)^2}{(x-y)(x-z) + (y-z)(y-x) + (z-x)(z-y)}$.

20. $\dfrac{7x^3 - 2x^2y - 63xy^2 + 18y^3}{5x^4 - 3x^3y - 43x^2y^2 + 27xy^3 - 18y^4}$. [P. U. 1912]

21. $\dfrac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{a^3(b-c)^3 + b^3(c-a)^3 + c^3(a-b)^3}$. [B. U. 1895]

II. Addition and Subtraction of Fractions

136. We know $\frac{p}{a}+\frac{q}{a}+\frac{r}{a}+\ldots\ldots=\frac{p+q+r+\cdots}{a}$, so that the sum of any number of fractions which have a common denominator is a fraction whose denominator is the same and whose numerator is the sum of the numerators of the given fractions.

Hence, to obtain the sum of any number of fractions which have not the same denominator we must first of all reduce them to equivalent fractions having a common denominator by the method of Art. 138, and then proceed as above.

Example 1. Simplify $\frac{a^2}{(x-a)^n}+\frac{2a}{(x-a)^{n-1}}+\frac{1}{(x-a)^{n-2}}$. [M. M. 1865]

$$\text{The expression}=\frac{a^2}{(x-a)^n}+\frac{2a(x-a)}{(x-a)^n}+\frac{(x-a)^2}{(x-a)^n}=\frac{a^2+2a(x-a)+(x-a)^2}{(x-a)^n}$$
$$=\frac{\{a+(x-a)\}^2}{(x-a)^n}=\frac{x^2}{(x-a)^n}.$$

Example 2. Simplify $\frac{1}{x-2}-\frac{1}{x+2}-\frac{4}{x^2+4}+\frac{32}{x^4+16}$.

To simplify examples like this, we combine two suitable terms; then the result thus obtained with a third, and so on.

$$\text{Thus, we have } \frac{1}{x-2}-\frac{1}{x+2}=\frac{(x+2)-(x-2)}{x^2-4}=\frac{4}{x^2-4};$$
$$\frac{4}{x^2-4}-\frac{4}{x^2+4}=\frac{4(x^2+4)-4(x^2-4)}{x^4-16}=\frac{32}{x^4-16}.$$
$$\text{Lastly,}\quad \frac{32}{x^4-16}+\frac{32}{x^4+16}=\frac{32(x^4+16)+32(x^4-16)}{x^8-256}$$
$$=\frac{64x^4}{x^8-256}, \text{ which is the required result.}$$

Example 3. Simplify $\frac{1}{a+b}-\frac{1}{a+2b}-\frac{1}{a+3b}+\frac{1}{a+4b}$.

Instead of simplifying all the terms together, it is convenient to combine them by groups.

$$\text{Thus, the given expression}=\left\{\frac{1}{a+b}-\frac{1}{a+2b}\right\}-\left\{\frac{1}{a+3b}-\frac{1}{a+4b}\right\}.$$
$$\text{Now, we have } \frac{1}{a+b}-\frac{1}{a+2b}=\frac{(a+2b)-(a+b)}{(a+b)(a+2b)}=\frac{b}{(a+b)(a+2b)};$$
$$\text{and}\quad \frac{1}{a+3b}-\frac{1}{a+4b}=\frac{(a+4b)-(a+3b)}{(a+3b)(a+4b)}=\frac{b}{(a+3b)(a+4b)}.$$

Lastly, $\dfrac{b}{(a+b)(a+2b)}-\dfrac{b}{(a+3b)(a+4b)}$

$$=\frac{b(a+3b)(a+4b)-b(a+b)(a+2b)}{(a+b)(a+2b)(a+3b)(a+4b)},$$

of which the numerator $=b(a^2+7ab+12b^2)-b(a^2+3ab+2b^2)$

$$=b(4ab+10b^2)=2b^2(2a+5b).$$

Hence, the reqd. result $=\dfrac{2b^2(2a+5b)}{(a+b)(a+2b)(a+3b)(a+4b)}$

Example 4. Simplify $\dfrac{x+3}{x^2-3x+2}+\dfrac{x+2}{x^2-4x+3}+\dfrac{x+1}{x^2-5x+6}$.

[P. U. 1904]

The first denominator $=x^2-3x+2=(x-1)(x-2)$.

The second denominator $=x^2-4x+3=(x-3)(x-1)$.

The third denominator $=x^2-5x+6=(x-2)(x-3)$.

$\therefore$ the L.C.M. of the denominators $=(x-1)(x-2)(x-3)$.

Hence, the given expression

$$=\frac{x+3}{(x-1)(x-2)}+\frac{x+2}{(x-3)(x-1)}+\frac{x+1}{(x-2)(x-3)}$$

$$=\frac{(x+3)(x-3)+(x+2)(x-2)+(x+1)(x-1)}{(x-1)(x-2)(x-3)}$$

$$=\frac{x^2-9+x^2-4+x^2-1}{(x-1)(x-2)(x-3)}=\frac{3x^2-14}{(x-1)(x-2)(x-3)}.$$

Example 5. Simplify

$$\frac{x-y}{(a+x)(a+y)}+\frac{y-z}{(a+y)(a+z)}+\frac{z-x}{(a+z)(a+x)}.$$

[A. U. 1916]

The L.C.M. of the denominators $=(a+x)(a+y)(a+z)$.

$\therefore$ the given expression

$$=\frac{(a+z)(x-y)+(a+x)(y-z)+(a+y)(z-x)}{(a+x)(a+y)(a+z)}.$$

The numerator $=a\{(x-y)+(y-z)+(z-x)\}$

$$+z(x-y)+x(y-z)+y(z-x)$$

$=0.$ [simplifying]

$\therefore$ the given expression

$$=\frac{0}{(a+x)(a+y)(a+z)}=0.$$

Otherwise : Since,

$$\frac{1}{a+y}-\frac{1}{a+x}=\frac{(a+x)-(a+y)}{(a+x)(a+y)}=\frac{x-y}{(a+x)(a+y)};$$

$$\frac{1}{a+z}-\frac{1}{a+y}=\frac{(a+y)-(a+z)}{(a+y)(a+z)}=\frac{y-z}{(a+y)(a+z)},$$

and $$\frac{1}{a+x}-\frac{1}{a+z}=\frac{(a+z)-(a+x)}{(a+z)(a+x)}=\frac{z-x}{(a+z)(a+x)}.$$

We have the given expression

$$=\frac{1}{a+y}-\frac{1}{a+x}+\frac{1}{a+z}-\frac{1}{a+y}+\frac{1}{a+x}-\frac{1}{a+z}=0.$$

Example 6. Simplify $\frac{a}{a+b}+\frac{2a^2}{a^2+b^2}+\frac{4a^2b^2}{a^4-b^4}$.

Such expressions are easily simplified by adding and subtracting a suitable fraction. Thus, adding and subtracting $\frac{a}{a-b}$, the given expression $=\frac{a}{a-b}+\frac{a}{a+b}+\frac{2a^2}{a^2+b^2}+\frac{4a^2b^2}{a^4-b^4}-\frac{a}{a-b}$.

Now, $$\frac{a}{a-b}+\frac{a}{a+b}=\frac{a(a+b)+(a-b)a}{a^2-b^2}=\frac{2a^2}{a^2-b^2}.$$

Again, $$\frac{2a^2}{a^2-b^2}+\frac{2a^2}{a^2+b^2}=\frac{2a^2(a^2+b^2)+2a^2(a^2-b^2)}{a^4-b^4}=\frac{4a^4}{a^4-b^4};$$

and $$\frac{4a^4}{a^4-b^4}+\frac{4a^2b^2}{a^4-b^4}=\frac{4a^4+4a^2b^2}{a^4-b^4}=\frac{4a^2(a^2+b^2)}{a^4-b^4}=\frac{4a^2}{a^2-b^2}.$$

∴ the given expression

$$=\frac{4a^2}{a^2-b^2}-\frac{a}{a-b}=\frac{4a^2-a(a+b)}{a^2-b^2}=\frac{3a^2-ab}{a^2-b^2}=\frac{a(3a-b)}{a^2-b^2}.$$

EXERCISE 89

Simplify :

1. $\frac{1}{2x^2-6ax+9a^2}-\frac{1}{2x^2+6ax+9a^2}+\frac{12ax}{4x^4-81a^4}$.

2. $\frac{1}{(x+a)(x+2a)}+\frac{1}{(x+2a)(x+3a)}+\frac{1}{(x+3a)(x+4a)}$.

3. $\frac{a-b}{(x+a)(x+b)}+\frac{b-c}{(x+b)(x+c)}+\frac{c-d}{(x+c)(x+d)}$.

4. $\frac{1}{a^2-3a+2}+\frac{2}{a^2-5a+6}+\frac{3}{a^2-4a+3}$.

5. $\dfrac{1}{(x+1)^2(x+2)^2}-\dfrac{1}{(x+1)^2}+\dfrac{2}{x+1}-\dfrac{2}{x+2}$. [A. U. 1912]

6. $\dfrac{2(x-3)}{(x-4)(x-5)}-\dfrac{x-1}{(x-3)(x-4)}-\dfrac{x-2}{(x-5)(x-3)}$. [A. U. 1911]

7. $\dfrac{a^2+ac}{a^2c-c^3}-\dfrac{a-c}{a(a+c)}-\dfrac{2c}{a^2-c^2}$. [C. U. 1869]

8. $\dfrac{1}{1+a}+\dfrac{2}{1+a^2}+\dfrac{4}{1+a^4}+\dfrac{8}{1+a^8}-\dfrac{16}{1-a^{16}}$.

9. $\left(\sqrt{\dfrac{a+x}{x}}-\sqrt{\dfrac{x}{a+x}}\right)^2-\left(\sqrt{\dfrac{x}{a}}-\sqrt{\dfrac{a}{x}}\right)^2+\dfrac{x^2}{a(a+x)}$ [B. U. 1876]

10. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}$

11. $\dfrac{1}{x^2-5x+6}-\dfrac{2}{x^2-4x+3}+\dfrac{1}{x^2-3x+2}$.

12. $1+\dfrac{a}{x-a}+\dfrac{bx}{(x-a)(x-b)}+\dfrac{cx^2}{(x-a)(x-b)(x-c)}$
$$+\dfrac{dx^3}{(x-a)(x-b)(x-c)(x-d)}$$

13. $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$. [C. U. 1937]

14. $\dfrac{a^2+a+1}{(a-b)(a-c)}+\dfrac{b^2+b+1}{(b-c)(b-a)}+\dfrac{c^2+c+1}{(c-a)(c-b)}$. [A. U. 1892]

15. $\dfrac{1}{2}\cdot\dfrac{1}{x-1}-\dfrac{x-5}{x^2-7x+10}+\dfrac{1}{2}\cdot\dfrac{x-6}{x^2-9x+18}$. [C. U. 1864]

16. $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2x^3}{a^4+a^2x^2+x^4}$. [B. U. 1892]

III. Complex and Continued Fractions

167. Complex Fractions. A fraction which contains a fraction in its numerator or in its denominator or in both, is called a **complex fraction.**

Then, $\dfrac{\frac{x}{y}}{z}$, $\dfrac{x}{\frac{y}{z}}$, $\dfrac{\frac{x}{y}}{\frac{a}{b}}$ are complex fractions, which are, therefore, merely divisions of fractions.

We have already considered simplifications of such fractions in Art. 111.

168. Continued Fractions.

Fractions of the type

$$x+\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\text{etc.}}}}$$ are called **continued fractions.**

To simplify such fractions, begin from the bottom and proceed upwards step by step as in Arithmetic.

Example 1. Simplify $-1+\cfrac{a}{2(a+b)-\cfrac{a+b}{1-\cfrac{b}{a+b}}}$.

Since, $1-\frac{b}{a+b}=\frac{a+b-b}{a+b}=\frac{a}{a+b}$, we have, by simplifying from the bottom, the given expression

$$=-1+\cfrac{a}{2(a+b)-\cfrac{(a+b)}{\cfrac{a}{a+b}}}$$

$$=-1+\frac{a}{2(a+b)-\frac{(a+b)^2}{a}}=-1+\frac{a}{\frac{2a^2+2a^2-(a^2+2ab+b^2)}{a}}$$

$$=-1+\frac{a^2}{a^2-b^2}=\frac{-a^2+b^2+a^2}{a^2-b^2}=\frac{b^2}{a^2-b^2}.$$

Example 2. Simplify $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$. [A. U. 1912]

Proceeding from the bottom, the given expression

$$=\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{1+x}{x}}}}=\cfrac{1}{1+\cfrac{1}{1+\cfrac{x}{1+x}}}=\cfrac{1}{1+\cfrac{1}{\cfrac{1+x+x}{1+x}}}$$

$$=\cfrac{1}{1+\cfrac{1+x}{1+2x}}=\cfrac{1}{\cfrac{1+2x+1+x}{1+2x}}=\frac{1+2x}{2+3x}.$$

Example 3. Solve $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}}=\dfrac{3}{4}$.

By example 2 above, we have

the left-hand side $=\dfrac{1+2x}{2+3x}=\dfrac{3}{4}$;

or, $3(2+3x)=4(1+2x)$, *i.e.*, $6+9x=4+8x$,

or, $9x-8x=4-6$, [transposing]

or, $x=-2$.

EXERCISE 90

Simplify :

1. $\dfrac{\left(\dfrac{y}{z}-\dfrac{z}{y}\right)\left(\dfrac{z}{x}-\dfrac{x}{z}\right)\left(\dfrac{x}{y}-\dfrac{y}{x}\right)}{\left(\dfrac{1}{y^2}-\dfrac{1}{z^2}\right)\left(\dfrac{1}{z^2}-\dfrac{1}{x^2}\right)\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)}$. [B. U. 1926]

2. $\dfrac{\dfrac{a}{a-b}+\dfrac{b}{b-c}+\dfrac{c}{c-a}}{\dfrac{a+b}{a-b}+\dfrac{b+c}{b-c}+\dfrac{c+a}{c-a}+3}$.

3. $\dfrac{\dfrac{a^2}{x-a}+\dfrac{b^2}{x-b}+\dfrac{c^2}{x-c}}{\dfrac{ax}{x-a}+\dfrac{bx}{x-b}+\dfrac{cx}{x-c}-(a+b+c)}$.

4. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\times\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$. [C. U. 1921]

5. $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}}$.

6. $\dfrac{1}{1+\dfrac{1}{a+x}}+\dfrac{1}{1-\dfrac{1}{a+x}}+\dfrac{2}{1+\dfrac{1}{a^2+x^2}}$. [C. U. 1870]

7. $\dfrac{\dfrac{a}{a-x}+\dfrac{b}{b-x}+\dfrac{c}{c-x}}{\dfrac{3}{x}-\dfrac{1}{x-a}-\dfrac{1}{x-b}-\dfrac{1}{x-c}}$.

8. $\dfrac{\dfrac{a^3}{b^3}-\dfrac{b^3}{a^3}}{\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}-1\right)}\times\dfrac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{ab}}$.

9. $\dfrac{\dfrac{1}{2(x-1)}-\dfrac{1}{x-2}+\dfrac{1}{2(x-3)}}{\dfrac{1}{(x-2)(x-1)}+\dfrac{1}{(x-1)(x-3)}+\dfrac{1}{(x-2)(x-3)}}$.

10. $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x-y}{x+y}-\dfrac{x^3-y^3}{x^3+y^3}}$.

11. $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$.

12. $\dfrac{x}{x-\dfrac{x-1}{1-\dfrac{1}{x+1}}}$.

13. $a^2+\dfrac{b^4}{a^2-\dfrac{a^3+b^3}{a+\dfrac{b^2}{a-b}}}$

14. $\dfrac{m}{m^2-\dfrac{m^3-1}{m+\dfrac{1}{m+1}}}$

15. $\dfrac{x^2-2xy+y^2}{\dfrac{x+y}{x+y-\dfrac{(x-y)^2}{x+y}}-1}$.

16. $\dfrac{x^2(x+2)}{4x+8+\dfrac{2x^4-32}{x+2+\dfrac{(x-2)(x-2)}{x+2}}}$.

17. $\dfrac{\dfrac{a^2}{x-a}+\dfrac{b^2}{x-b}+\dfrac{c^2}{x-c}+a+b+c}{\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}}$ [C. U. 1925]

18. $\dfrac{1}{\left(1-\dfrac{b}{a}\right)\left(1-\dfrac{c}{a}\right)}+\dfrac{1}{\left(1-\dfrac{c}{b}\right)\left(1-\dfrac{a}{b}\right)}+\dfrac{1}{\left(1-\dfrac{a}{c}\right)\left(1-\dfrac{b}{c}\right)}$. [B. U. 1897]

19. $\dfrac{a^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b^2\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c^2\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}{a\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+b\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+c\left(\dfrac{1}{a}-\dfrac{1}{b}\right)}$. [C. U. 1880]

20. $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}}$. [C. U. 1946]

Solve :

21. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{2-x}}}=\dfrac{4}{3}$.

22. $\dfrac{2x}{1+\dfrac{1}{1+\dfrac{x}{1-x}}}=1$.

23. $1+\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{x}}}=\dfrac{13}{9}$.

24. $\dfrac{a}{a+\dfrac{a^2}{a+\dfrac{a^2}{x}}}=\dfrac{2}{3}$.

169. Fractions involving Cyclic Order. Certain fractions are easily simplified when the cyclic order of letters is maintained.

Example. Simplify $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}$.

Considering the denominator, we see that the factor $a-c$ is not in cyclic order.

Since, $a-c=-(c-a)$, $(a-b)(a-c)=-(a-b)(c-a)$.

Hence, the first fraction $=-\dfrac{bc}{(a-b)(c-a)}$,

Similarly, the second fraction $=-\dfrac{ca}{(b-c)(a-b)}$,

and the third fraction $=-\dfrac{ab}{(c-a)(b-c)}$.

The L.C.M. of the denominators $=(b-c)(c-a)(a-b)$.

$\therefore$ the given expression

$$=-\left[\frac{bc}{(a-b)(c-a)}+\frac{ca}{(a-b)(b-c)}+\frac{ab}{(c-a)(b-c)}\right]$$

$$=-\frac{bc(b-c)+ca(c-a)+ab(a-b)}{(b-c)(c-a)(a-b)}=\frac{(b-c)(c-a)(a-b)}{(b-c)(c-a)(a-b)}=1$$

170. Important Results in Cyclic Order. The following results can be easily verified and are very useful in simplifying many harder examples in fractions involving cyclic order.

If $\dfrac{1}{(a-b)(a-c)}=X$, $\dfrac{1}{(b-c)(b-a)}=Y$, and $\dfrac{1}{(c-a)(c-b)}=Z$,

then (i) $X+Y+Z=0$; (ii) $aX+bY+cZ=0$;

(iii) $a^2X+b^2Y+c^2Z=1$; (iv) $bcX+caY+abZ=1$;

(v) $a^3X+b^3Y+c^3Z=a+b+c$;

(vi) $a^4X+b^4Y+c^4Z=a^2+b^2+c^2+bc+ca+ab$.

Example 1. Simplify $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2-ca}{(b-c)(b-a)}+\dfrac{c^2-ab}{(c-a)(c-b)}$.

The given expression $=(a^2-bc)X+(b^2-ca)Y+(c^2-ab)Z$

[adopting above notations]

$=a^2X+b^2Y+c^2Z-(bcX+caY+abZ)$

$=1-1$ [Results (iii) and (iv)]

$=0$.

Example 2. Simplify

$$\frac{pa^3+qa^2bc+ra}{(a-b)(a-c)}+\frac{pb^3+qab^2c+rb}{(b-c)(b-a)}+\frac{pc^3+qabc^2+rc}{(c-a)(c-b)}.$$

The given expression

$=(pa^3+qa^2bc+ra)X+(pb^3+qab^2c+rb)Y+(pc^3+qabc^2+rc)Z$

[adopting above notations]

$=p(a^3X+b^3Y+c^3Z)+qabc(aX+bY+cZ)+r(aX+bY+cZ)$

$=p(a+b+c)+qabc.0+r.0=p(a+b+c)$.

Example 3. Show that

$$\frac{1}{(l-m)(l-n)(x+l)}+\frac{1}{(m-n)(m-l)(x+m)}+\frac{1}{(n-l)(n-m)(x+n)}$$
$$=\frac{1}{(x+l)(x+m)(x+n)}.$$

Putting a for $x+l$, b for $x+m$ and c for $x+n$, we have

$$a-b=l-m,\ a-c=l-n,\ b-c=m-n, \text{ etc.}$$

The given expression

$$=\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}$$
$$=\frac{1}{abc}\left[\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-a)(b-c)}+\frac{ab}{(c-a)(c-b)}\right]$$
$$=\frac{1}{abc}(bcX+caY+abZ)$$
$$=\frac{1}{abc}=\frac{1}{(x+l)(x+m)(x+n)}.\quad [\text{ restoring values of } a, b, c\ ]$$

171. Fractional Identities : Miscellaneous Examples.

Example 1. Show that

$$\frac{x}{x^2+a^2}=\frac{1}{x}-\frac{a^2}{x^3}+\frac{a^4}{x^5}-\frac{a^6}{x^7}+\frac{a^8}{x^7(x^2+a^2)}.$$

Let us divide x by x^2+a^2 :

$$\begin{array}{r|l}
x^2+a^2 & x\ \left(\dfrac{1}{x}-\dfrac{a^2}{x^3}+\dfrac{a^4}{x^5}-\dfrac{a^6}{x^7}\right. \\
 & \underline{x+\dfrac{a^2}{x}} \\
 & -\dfrac{a^2}{x} \\
 & \underline{-\dfrac{a^2}{x}-\dfrac{a^4}{x^3}} \\
 & \dfrac{a^4}{x^3} \\
 & \underline{\dfrac{a^4}{x^3}+\dfrac{a^6}{x^5}} \\
 & -\dfrac{a^6}{x^5} \\
 & \underline{-\dfrac{a^6}{x^5}-\dfrac{a^8}{x^7}} \\
 & \dfrac{a^8}{x^7}
\end{array}$$

Hence, proceeding no further with the division, we have

$$\frac{x}{x^2+a^2}=\frac{1}{x}-\frac{a^2}{x^3}+\frac{a^4}{x^5}-\frac{a^6}{x^7}+\frac{\left(\frac{a^8}{x^7}\right)}{x^2+a^2}$$

$$=\frac{1}{x}-\frac{a^2}{x^3}+\frac{a^4}{x^5}-\frac{a^6}{x^7}+\frac{a^8}{x^7(x^2+a^2)}.$$

Example 2. Find the value of

$$\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}, \text{ when } x=\frac{4ab}{a+b}. \qquad [\text{ O. U. 1886 }]$$

The given expression

$$=\left(\frac{x+2a}{x-2a}-1\right)+\left(\frac{x+2b}{x-2b}-1\right)+2=\frac{4a}{x-2a}+\frac{4b}{x-2b}+2$$

$$=\frac{4}{(x-2a)(x-2b)}\{a(x-2b)+b(x-2a)\}+2$$

$$=\frac{4}{(x-2a)(x-2b)}\{(a+b)x-4ab\}+2$$

$$=0+2 \qquad [\because (a+b)x=4ab]$$

$$=2.$$

Example 3. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, show that

$$\frac{1}{a^7}+\frac{1}{b^7}+\frac{1}{c^7}=\frac{1}{(a+b+c)^7}=\frac{1}{a^7+b^7+c^7}.$$

Since, $\frac{1}{a+b+c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{bc+ca+ab}{abc}$;

$\therefore$ $(a+b+c)(bc+ca+ab)=abc$,

or $(a+b+c)(bc+ca+ab)-abc=0$,

or, $(b+c)(c+a)(a+b)=0$;

$\therefore$ any one of these factors, say, $b+c=0$.

Hence, $b=-c$. $\therefore b^7=(-c)^7=-c^7$, or, $b^7+c^7=0$.

Also, since, $b=-c$, $\frac{1}{b}=-\frac{1}{c}$; $\therefore$ $\frac{1}{b^7}=\left(-\frac{1}{c}\right)^7=-\frac{1}{c^7}$.

Hence, $\frac{1}{a^7}+\frac{1}{b^7}+\frac{1}{c^7}=\frac{1}{a^7}-\frac{1}{c^7}+\frac{1}{c^7}=\frac{1}{a^7}=\frac{1}{(a+b+c)^7}$. $[\because b+c=0]$

Similarly, $\frac{1}{a^7}+\frac{1}{b^7}+\frac{1}{c^7}=\frac{1}{a^7}=\frac{1}{a^7+b^7+c^7}$. $[\because b^7+c^7=0]$

Hence, the identity is established.

Example 4. Reduce to its simplest form

$$\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(x-z)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}. \qquad \text{[C. U. 1866]}$$

We have, the 1st fraction $=\dfrac{\{x+(y-z)\}\{x-(y-z)\}}{\{(x+z)+y\}\{(x+z)-y\}}$

$$=\frac{(x+y-z)(x-y+z)}{(x+z+y)(x+z-y)}=\frac{x+y-z}{x+y+z}.$$

Similarly, the 2nd fraction $=\dfrac{(y+x-z)(y-x+z)}{(x+y+z)(x+y-z)}=\dfrac{y-x+z}{x+y+z},$

and the 3rd fraction $=\dfrac{(z+x-y)(z-x+y)}{(y+z+x)(y+z-x)}=\dfrac{z+x-y}{x+y+z}.$

Hence, the given exp. $=\dfrac{(x+y-z)+(y-x+z)+(z+x-y)}{x+y+z}=\dfrac{x+y+z}{x+y+z}=1$

Example 5. If $x+y+z=xyz$, prove that

$$\frac{y+z}{1-yz}+\frac{z+x}{1-zx}+\frac{x+y}{1-xy}=\frac{y+z}{1-yz}\cdot\frac{z+x}{1-zx}\cdot\frac{x+y}{1-xy}.$$

Since, $x+y+z=xyz$; we have $y+z=xyz-x=x(yz-1)$.

Hence, $\dfrac{y+z}{1-yz}=\dfrac{x(yz-1)}{1-yz}=-x.$

Similarly, $\dfrac{z+x}{1-zx}=-y$ and $\dfrac{x+y}{1-xy}=-z.$

$\therefore$ the left side $=\dfrac{y+z}{1-yz}+\dfrac{z+x}{1-zx}+\dfrac{x+y}{1-xy}$

$$=-x-y-z=-(x+y+z)=-xyz$$

$$=(-x).(-y).(-z)$$

$$=\frac{y+z}{1-yz}\cdot\frac{z+x}{1-zx}\cdot\frac{x+y}{1-xy}.$$

Example 6. Show that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

[C. U. 1867]

We have

$$\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=\left(\frac{c^2}{a^2}+2+\frac{a^2}{c^2}\right)+\left(\frac{a^2}{b^2}+2+\frac{b^2}{a^2}\right)$$

$$=4+a^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{1}{a^2}\left(b^2+c^2\right)$$

$$=4+\frac{a^2}{bc}\left(\frac{bc}{b^2}+\frac{bc}{c^2}\right)+\frac{bc}{a^2}\left(\frac{b^2}{bc}+\frac{c^2}{bc}\right)$$

$$=4+\frac{a^2}{bc}\left(\frac{c}{b}+\frac{b}{c}\right)+\frac{bc}{a^2}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right);$$

$\therefore$ the given expression

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a^2}{bc}+\frac{bc}{a^2}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\left(\frac{b}{c}+\frac{bc}{a^2}\right)+\left(\frac{c}{b}+\frac{a^2}{bc}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\frac{b}{a}\left(\frac{a}{c}+\frac{c}{a}\right)+\frac{a}{b}\left(\frac{c}{a}+\frac{a}{c}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

Example 7. If $2s=a+b+c$, show that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

We have $\quad \dfrac{1}{s-a}+\dfrac{1}{s-b}=\dfrac{2s-a-b}{(s-a)(s-b)}=\dfrac{c}{(s-a)(s-b)},$

and $\quad \dfrac{1}{s-c}-\dfrac{1}{s}=\dfrac{s-(s-c)}{s(s-c)}=\dfrac{c}{s(s-c)}.$

Hence, the given expression

$$=\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}=c.\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}$$

$$=c\cdot\frac{2s^2-s(a+b+c)+ab}{s(s-a)(s-b)(s-c)}$$

$$=\frac{abc}{s(s-a)(s-b)(s-c)}. \qquad [\because 2s^2-s(a+b+c)=2s^2-2s^2=0.]$$

Example 8. Show that

$$\frac{a+b}{ab}(a^2+b^2-c^2)+\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2)=2(a+b+c).$$

Putting $2s^2$ for $a^2+b^2+c^2$, we have

$$a^2+b^2-c^2=(a^2+b^2+c^2)-2c^2=2(s^2-c^2),$$
$$b^2+c^2-a^2=(a^2+b^2+c^2)-2a^2=2(s^2-a^2),$$
$$c^2+a^2-b^2=(a^2+b^2+c^2)-2b^2=2(s^2-b^2).$$

Hence, the given expression

$$=2\left(\frac{1}{b}+\frac{1}{a}\right)\left(s^2-c^2\right)+2\left(\frac{1}{c}+\frac{1}{b}\right)\left(s^2-a^2\right)+2\left(\frac{1}{a}+\frac{1}{c}\right)\left(s^2-b^2\right)$$
$$=2\left\{\frac{1}{a}\left(2s^2-b^2-c^2\right)+\frac{1}{b}\left(2s^2-c^2-a^2\right)+\frac{1}{c}\left(2s^2-a^2-b^2\right)\right\}$$
$$=2\left\{\frac{1}{a}\cdot a^2+\frac{1}{b}\cdot b^2+\frac{1}{c}\cdot c^2\right\}$$
$$=2(a+b+c).$$

Example 9. Show that

$$\frac{a}{a^2-1}+\frac{a^2}{a^4-1}+\frac{a^4}{a^8-1}=\frac{1}{2}\cdot\left(\frac{a+1}{a-1}-\frac{a^8+1}{a^8-1}\right).$$

$$\frac{a}{a^2-1}=\frac{1}{2}\cdot\frac{2a}{a^2-1}=\frac{1}{2}\cdot\frac{(a+1)^2-(a^2+1)}{a^2-1}$$
$$=\frac{1}{2}\cdot\left(\frac{a+1}{a-1}-\frac{a^2+1}{a^2-1}\right).$$

$$\frac{a^2}{a^4-1}=\frac{1}{2}\cdot\frac{2a^2}{a^4-1}=\frac{1}{2}\cdot\frac{(a^2+1)^2-(a^4+1)}{a^4-1}$$
$$=\frac{1}{2}\cdot\left(\frac{a^2+1}{a^2-1}-\frac{a^4+1}{a^4-1}\right);$$

$$\frac{a^4}{a^8-1}=\frac{1}{2}\cdot\frac{2a^4}{a^8-1}=\frac{1}{2}\cdot\frac{(a^4+1)^2-(a^8+1)}{a^8-1}$$
$$=\frac{1}{2}\cdot\left(\frac{a^4+1}{a^4-1}-\frac{a^8+1}{a^8-1}\right)$$

Hence, the given expression

$$=\frac{1}{2}\cdot\left\{\left(\frac{a+1}{a-1}-\frac{a^2+1}{a^2-1}\right)+\left(\frac{a^2+1}{a^2-1}-\frac{a^4+1}{a^4-1}\right)+\left(\frac{a^4+1}{a^4-1}-\frac{a^8+1}{a^8-1}\right)\right\}$$
$$=\frac{1}{2}\cdot\left\{\frac{a+1}{a-1}-\frac{a^8+1}{a^8-1}\right\}$$

Example 10. Show that

$$bc.\frac{a+d}{(a-b)(a-c)}+ac.\frac{b+d}{(b-a)(b-c)}+ab.\frac{c+d}{(c-a)(c-b)}=d.$$

Since, $\quad b-a=-(a-b),$

and $(c-a)(c-b)=[-(a-c)]\times[-(b-c)]=(a-c)(b-c)$;

$\therefore$ the given expression

$$= bc\,\frac{a+d}{(a-b)(a-c)} + ac\cdot\frac{-(b+d)}{(a-b)(b-c)} + ab\cdot\frac{c+d}{(a-c)(b-c)}$$

$$= \frac{bc(a+d)(b-c) - ac(b+d)(a-c) + ab(c+d)(a-b)}{(a-b)(a-c)(b-c)}.$$

Now, the numerator $= abc\{(b-c)-(a-c)+(a-b)\}$

$$+ d\{bc(b-c) - ac(a-c) + ab(a-b)\}$$

$$= d\{bc(b-c) - ac(a-c) + ab(a-b)\}$$

$$= d\{a^2(b-c) + b^2(c-a) + c^2(a-b)\}$$

$$= d(a-b)(a-c)(b-c).$$

Hence, the given expression $= d$.

Example 11. Simplify

$$\frac{a^2}{(a-b)(a-c)(x+a)} + \frac{b^2}{(b-a)(b-c)(x+b)} + \frac{c^2}{(c-a)(c-b)(x+c)}.$$

The given expression

$$= \frac{a^2}{(a-b)(a-c)(x+a)} + \frac{-b^2}{(a-b)(b-c)(x+b)} + \frac{c^2}{(a-c)(b-c)(x+c)}$$

$$= \frac{a^2(b-c)(x+b)(x+c) - b^2(a-c)(x+c)(x+a) + c^2(a-b)(x+a)(x+b)}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)}.$$

Now, the numerator

$$= a^2(b-c)\{x^2 + x(b+c) + bc\} + b^2(c-a)\{x^2 + x(c+a) + ca\}$$

$$+ c^2(a-b)\{x^2 + x(a+b) + ab\}$$

$$= x^2\{a^2(b-c) + b^2(c-a) + c^2(a-b)\}$$

$$+ x\{a^2(b^2-c^2) + b^2(c^2-a^2) + c^2(a^2-b^2)\}$$

$$+ abc\{a(b-c) + b(c-a) + c(a-b)\}$$

$$= x^2\{a^2(b-c) + b^2(c-a) + c^2(a-b)\} = x^2(a-b)(a-c)(b-c).$$

Hence, the given expression $= \dfrac{x^2}{(x+a)(x+b)(x+c)}$.

Example 12. Simplify

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)}. \qquad [\text{C. U. } 1887]$$

The given expression

$$= \frac{a^3}{(a-b)(a-c)} + \frac{-b^3}{(b-c)(a-b)} + \frac{c^3}{(a-c)(b-c)}$$

$$= \frac{a^3(b-c) - b^3(a-c) + c^3(a-b)}{(a-b)(a-c)(b-c)}.$$

Now, the numerator $=a^3(b-c)+b^3(c-a)+c^3(a-b)$

$$=(a-b)(a-c)(b-c)(a+b+c).$$

Hence, the given expression $=a+b+c$.

† ***Alternative Method*** :

Since, $$\frac{1}{(a-b)(a-c)}=\frac{1}{(a-b)(b-c)}-\frac{1}{(a-c)(b-c)},$$

$\therefore$ the given expression

$$=\left\{\frac{a^3}{(a-b)(b-c)}-\frac{a^3}{(a-c)(b-c)}\right\}+\frac{-b^3}{(a-b)(b-c)}+\frac{c^3}{(a-c)(b-c)}$$

$$=\frac{a^3-b^3}{(a-b)(b-c)}-\frac{a^3-c^3}{(a-c)(b-c)}=\frac{a^2+ab+b^2}{b-c}-\frac{a^2+ac+c^2}{b-c}$$

$$=\frac{a(b-c)+(b^2-c^2)}{b-c}=a+b+c.$$

Example 13. If $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{bc}-\frac{1}{ca}-\frac{1}{ab}=0$, prove that $a=b=c$.

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{bc}-\frac{1}{ca}-\frac{1}{ab}=0,$$

or, $$\frac{1}{2}\left\{\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{1}{c}-\frac{1}{a}\right)^2+\left(\frac{1}{a}-\frac{1}{b}\right)^2\right\}=0.$$

[Formula XXIV, Art. 133]

Now, as *none* of the terms of the left-hand expression is negative, this equation cannot hold unless *each* of those terms is zero.

Hence, $\frac{1}{b}-\frac{1}{c}=0$; $\therefore$ $b=c$,

$\frac{1}{c}-\frac{1}{a}=0$; $\therefore$ $c=a$,

and $\frac{1}{a}-\frac{1}{b}=0$; $\therefore$ $a=b$.

Thus, $a=b=c$.

EXERCISE 91

Prove that :

1. $\frac{a}{ax+x^2}+\frac{b}{bx+x^2}+\frac{c}{cx+x^2}=\frac{3}{x}-\frac{1}{a+x}-\frac{1}{b+x}-\frac{1}{c+x}$. [B. U. 1920]

2. $\frac{1+x^2}{(x+y)(x+z)}+\frac{1+y^2}{(y+z)(y+x)}+\frac{1+z^2}{(z+x)(z+y)}=3$, if $yz+zx+xy=1$.

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† This method is due to my friend and pupil Babu Bimala Charan Shome, Head Assistant, Forest Surveys, Dehra Dun.

3. $\left(a+b+c\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1+(b+c)(c+a)(a+b)$, if $abc=1$

4. $\frac{1}{a^2-bc}+\frac{1}{b^2-ca}+\frac{1}{c^2-ab}=0$, if $bc+ca+ab=0$.

5. $\frac{x+yz}{(y+x)(z+x)}+\frac{y+zx}{(y+z)(y+x)}+\frac{z+xy}{(z+x)(z+y)}=3$, if $x+y+z=1$.

6. $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$, if $x+y+z=xyz$.

7. $\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)=9$, if $x+y+z=0$.

8. $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{1}{a^3+b^3+c^3}=\frac{1}{(a+b+c)^3}$, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$

9. $\frac{(b-c)^2}{(c-a)(a-b)}+\frac{(c-a)^2}{(a-b)(b-c)}+\frac{(a-b)^2}{(b-c)(c-a)}=3$.

10. $\frac{(b^2-c^2)^3+(c^2-a^2)^3+(a^2-b^2)^3}{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}=\frac{(b+c)(c+a)(a+b)}{abc}$.

11. $\frac{x^6}{x^2+y^2}=x^4-x^2y^2+y^4-\frac{y^6}{x^2+y^2}$.

12. $\frac{x^6}{x^2-y^2}=x^4+x^2y^2+y^4+\frac{y^6}{x^2-y^2}$.

13. $\frac{x^2yz+xy^2z+xyz^2}{x^2y^2z^2}=\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}$.

14. $\frac{xy^2z^2+yz^2x^2+zx^2y^2}{x^2y^2z^2}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

15. $\frac{3a-6}{(a-1)(a-2)(a-3)}=\frac{1}{(a-2)(a-3)}+\frac{1}{(a-3)(a-1)}+\frac{1}{(a-1)(a-2)}$.

16. $\frac{3x^2-14}{(x+1)(x+2)(x+3)}=\frac{x-1}{(x+2)(x+3)}+\frac{x-2}{(x+3)(x+1)}+\frac{x-3}{(x+1)(x+2)}$.

17. $\frac{1-x}{1+x}=1-2x+2x^2-2x^3+\frac{2x^4}{1+x}$.

18. $\frac{a}{x^2-a^2}=\frac{a}{x^2}+\frac{a^3}{x^4}+\frac{a^5}{x^6}+\frac{a^7}{x^6(x^2-a^2)}$.

19. $\frac{a^3}{x^3+a^3}=\frac{a^3}{x^3}-\frac{a^6}{x^6}+\frac{a^9}{x^9}-\frac{a^{12}}{x^9(x^3+a^3)}=1-\frac{x^3}{a^3}+\frac{x^6}{a^6}-\frac{x^9}{a^9}+\frac{x^{12}}{a^9(x^3+a^3)}$.

20. $\frac{x^4-1}{x+a}=x^3-ax^2+a^2x-a^3+\frac{a^4-1}{x+a}$.

21. Find the value of $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}+\frac{4ab}{x^2-4b^2}$, when $x=\frac{ab}{a+b}$.

22. Show that $\frac{x+a}{x-a}+\frac{x+b}{x-b}+\frac{x+c}{x-c}-3$

$$=\frac{6abc}{(x-a)(x-b)(x-c)},\text{ if } x=\frac{2(ab+bc+ca)}{a+b+c}.$$

23. If $x=\frac{ab+bc+ca}{a+b+c}$, show that

$$\frac{a+2x}{a-2x}+\frac{b+2x}{b-2x}+\frac{c+2x}{c-2x}+3=\frac{6abc}{(a-2x)(b-2x)(c-2x)}.$$

24 Find the value of

$$\frac{x^2-(b+c)x}{(x-b)(x-c)}+\frac{x^2-(c+a)x}{(x-c)(x-a)}+\frac{x^2-(a+b)x}{(x-a)(x-b)},\text{ when } x=\frac{3abc}{ab+bc+ca}.$$

25. Find the value of

$$\frac{x^2-y^2+x}{y^2-x^2+y},\text{ when } x=\frac{a-b}{a+b}\text{ and } y=\frac{a+b}{a-b}.$$ [C. U. 1883]

$\left[\text{The given expression}=\frac{x(x+1)-y^2}{y(y+1)-x^2}=\text{\&c.}\right]$

26. Find the value of $\frac{x^4+3abx^2-10a^2b^2}{x^4+7abx^2+10a^2b^2}\times\frac{a^2+2ab+b^2}{a^2-2ab+b^2}$,

when $x^2=a^2+b^2$.

27. Find the value of $\frac{x^2y^2+3(2x^2-y^2)ab-18a^2b^2}{y^4+9aby^2+18a^2b^2}\times\frac{a^3-b^3}{a^3+b^3}$,

when $x=a+b$ and $y=a-b$.

28. Find the value of $\frac{x^4+abx^2-2a^2b^2}{x^2y^2+(x^2+2y^2)ab+2a^2b^2}-\frac{a^2+ab+b^2}{a^2-ab+b^2}$,

when $x=a+b$ and $y=a-b$.

29. Simplify $\frac{x^9}{x^3+1}+\frac{x^6}{x^3-1}+\frac{1}{x^3+1}-\frac{1}{x^3-1}$.

30. Simplify $\frac{x^2-(a-b)^2}{(x+b)^2-a^2}+\frac{a^2-(x-b)^2}{(x+a)^2-b^2}+\frac{b^2-(x-a)^2}{(u+b)^2-x^2}$.

31. Simplify $\frac{(a+2b)^2-b^2}{(a+b)^2-4b^2}+\frac{(a-b)^2-4b^2}{(a-2b)^2-b^2}+\frac{(2a+3b)^2-b^2}{(2a+b)^2-9b^2}$.

32. Simplify $\frac{x^4-(x-1)^2}{(x^2+1)^2-x^2}+\frac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\frac{x^2(x-1)^2-1}{x^4-(x+1)^2}$.

33. If $2s=a+b+c$, show that

$$1-\frac{a^2+b^2-c^2}{2ab}=\frac{2(s-a)(s-b)}{ab}.$$

34. Simplify $\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}$.

35. Simplify $\frac{a+b}{2ab}(a+b-c)+\frac{b+c}{2bc}(b+c-a)+\frac{c+a}{2ca}(c+a-b)$.

36. Simplify $\frac{x+y}{2xy}(x^2+y^2-z^2)+\frac{y+z}{2yz}(y^2+z^2-x^2)+\frac{z+x}{2zx}(z^2+x^2-y^2)$

37. Simplify $\frac{a+b}{2ab}(a^3+b^3-c^3)+\frac{b+c}{2bc}(b^3+c^3-a^3)+\frac{c+a}{2ca}(c^3+a^3-b^3)$.

38. If $x=\frac{b^2+c^2-a^2}{2bc}$, $y=\frac{a^2+c^2-b^2}{2ca}$ and $z=\frac{a^2+b^2-c^2}{2ab}$, find in its simplest form, the value of $(b+c)x+(c+a)y+(a+b)z$.

39. If $p=\frac{a-b}{x-c}$, $q=\frac{b-c}{x-a}$, $r=\frac{c-a}{x-b}$, find the value of $p+q+r+pqr$.

40. Show that $\left(\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right)^2=\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}$.

41. Show that $\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}=\frac{1}{1-x}-\frac{16x^{15}}{1-x^{16}}$.

Simplify :

42. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}$.

43. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}$ [A. U. 1925]

44. $\frac{x^2+yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y-z)(y-x)}+\frac{z^2+xy}{(z-x)(z-y)}$.

45. $\frac{2a^2-bc}{(a-b)(a-c)}+\frac{2b^2-ca}{(b-c)(b-a)}+\frac{2c^2-ab}{(c-a)(c-b)}$

46. $\frac{x^2-yz}{(x-y)(x-z)}+\frac{y^2+zx}{(y+z)(y-x)}+\frac{z^2+xy}{(z-x)(z+y)}$. [C. U. 1865]

47. $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}+\frac{1}{z(z-x)(z-y)}$. [C. U. 1872]

48. $\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-a)(b-c)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}$.

49. $\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}$.

50. $\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-a)(b-c)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}$.

51. $\dfrac{a^2+ha+k}{(a-b)(a-c)(x-a)}+\dfrac{b^2+hb+k}{(b-a)(b-c)(x-b)}+\dfrac{c^2+hc+k}{(c-a)(c-b)(x-c)}$

52. Show that $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=ab+bc+ca$

53. Show that $\dfrac{a(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{b(b+a)(b+c)}{(b-a)(b-c)}+\dfrac{c(c+a)(c+b)}{(c-a)(c-b)}=a+b+c.$

54. Prove that $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+\dfrac{ac}{b(b^2-a^2)(b^2-c^2)}$

$$+\frac{ab}{c(c^2-b^2)(c^2-a^2)}=\frac{1}{abc}.$$

55. Simplify $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}.$

MISCELLANEOUS EXERCISES V

I

1. Express the following as the difference of two squares:

   (i) $(x+7)(x+9)(x+11)(x+13)$;

   (ii) $(x+1)(x+2)(x+3)(x+4)-15.$

2. Factorise : (i) $7(z+x)^3-(x-y)^3-(y+z)^3.$

   (ii) $14a^3-4b^3+9a^2b.$

3. Simplify $(a-b)^2(a+b-2c)^2+(b-c)^2(b+c-2a)^2$
   $+(c-a)^2(c+a-2b)^2$, when $a+b+c=0.$

4. If $x+y+z=4xyz$, show that

$$\frac{x}{1-4x^2}+\frac{y}{1-4y^2}+\frac{z}{1-4z^2}=\frac{16xyz}{(1-4x^2)(1-4y^2)(1-4z^2)}.$$

5. If $2s=a+b+c$, show that

$$1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2=\frac{4s(s-a)(s-b)(s-c)}{b^2c^2}.$$

6. Show that $\dfrac{(b+c)(b^2+c^2-a^2)}{2bc}+\dfrac{(c+a)(c^2+a^2-b^2)}{2ca}$

$$+\frac{(a+b)(a^2+b^2-c^2)}{2ab}=a+b+c.$$

7. Find the value of $\frac{qr}{(a-q)(a-r)}+\frac{rp}{(a-r)(a-p)}+\frac{pq}{(a-p)(a-q)}$,

when $\frac{1}{a}=\frac{1}{3}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$.

8. Show that $(x^2-y^2)^3+(y^2-z^2)^3+(z^2-x^2)^3$ is divisible by each of the expressions x^2-y^2, y^2-z^2 and z^2-x^2.

II

1. If $x+y+z=15$, $xy+yz+zx=75$, find the value of $x^3+y^3+z^3-3xyz$.

2. Show that $(a+b-2c)^3+(b+c-2a)^3+(c+a-2b)^3$
$=3(a+b-2c)(b+c-2a)(c+a-2b)$.

3. Show that $(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2$
$+(a-b)(a+b-2c)^2=9(a-b)(b-c)(a-c)$.

4. Simplify $\frac{1}{bc(b-a)(c-a)}+\frac{1}{ca(c-b)(a-b)}+\frac{1}{ab(a-c)(b-c)}$.

5. Find the value of $\frac{y}{x}+\frac{y-1}{x+1}$, when $x=\frac{b}{a-b}$ and $y=\frac{a}{a+b}$.

6. Find the H.C.F. of $ab+2a^2-3b^2-4bc-ac-c^2$ and $9ac+2a^2-5ab+4c^2+8bc-12b^2$.

7. Find the L.C.M. of $6x^3-11x^2+5x-3$ and $9x^3-9x^2+5x-2$.

8. Resolve into factors : (i) $x^3+(x-1)^3+(1-2x)^3$.
(ii) $(a-b)(b+c)(c+a)+(b-c)(c+a)(a+b)+(c-a)(a+b)(b+c)$.

III

1. Expand $\left(x+\frac{2}{x}\right)^5$ in a series of descending powers of x.

2. Show that $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.

Hence, prove that

$$(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3=24xyz.$$

3. Find the value of $a^3-b^3+c^3+3abc$, when $a=\cdot4278$, $b=1\cdot2345$ and $c=\cdot8067$.

4. Show that $(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)$
$=(a-b)(a-c)(b-c)$.

5. Find the H.C.F. and L.C.M. of
$6x^3-25x^2+23x-6$, $2x^2-7x+3$ and $6x^2-7x+2$.

6. Find the H.C.F. of $x^5+11x-12$ and x^5+11x^3+54.

7. Simplify $\frac{a^3(b+c)}{(c-a)(b-a)}+\frac{b^3(c+a)}{(a-b)(c-b)}+\frac{c^3(a+b)}{(a-c)(b-c)}$.

8. Show that $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$ is exactly divisible by each of $b-c$, $c-a$ and $a-b$.

IV

1. If $a+b=2$, $ab=7$, find the value of a^5+b^5!

2. Resolve $2(a^6+b^6)-ab(a^2+b^2)(2ab-3a^2+3b^2)$ into five factors.

3. Find the value of $a^3+b^3+c^3-3abc$, when $a=2658$, $b=2664$ and $c=2678$.

4. If $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$, prove that $x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc)$.

5. Find the value of $\dfrac{2x^2+5xy+3y^2}{2x^2+xy-3y^2}$, when $x=\dfrac{a}{a+b}$ and $y=\dfrac{b}{a-b}$

6. Show that $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$
$$=3(2a+b+c)(a+2b+c)(a+b+2c).$$

7. Find the L.C.M. of $x^2-3xy-10y^2$, $x^2+2xy-35y^2$ and $x^2-8xy+15y^2$; and resolve into simple factors the quotient when the L.C.M. of the above expressions is divided by their H.C.F.

8. Find, without direct substitution, the value of $x^5-18x^4+47x^3-31x^2+19x-60$, when $x=15$.

V

1. If $x=\dfrac{a-b}{m-c}$, $y=\dfrac{b-c}{m-a}$ and $z=\dfrac{c-a}{m-b}$, show that
$$x+y+z+xyz=0.$$

2. If $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{b}{a}+\dfrac{d}{c}$, prove that $\dfrac{a^3}{b^3}+\dfrac{c^3}{d^3}=\dfrac{b^3}{a^3}+\dfrac{d^3}{c^3}$.

3. Find the value of $\dfrac{(x-a)(x-b)}{(x-a-b)^2}$, when $x=\dfrac{a^2+ab+b^2}{a+b}$.

4. If $x=\dfrac{2ac}{a+c}$, show that the value of $\dfrac{(x-a)^2+(x-c)^2}{a^2+c^2}+\dfrac{4ac}{(a+c)^2}$ is the same for all values of a and c.

5. Resolve the following into factors :
   (i) $6a^4+43a^3b-56a^2b^2+43ab^3+6b^4$;
   (ii) $12x^4-37x^3+45x^2-37x+12$;
   (iii) $abx^4+(ac+b^2)x^3+(2ab+bc)x^2+(ac+b^2)x+ab$.

6. Show that $(x+y)^3-(y+z)^3+(z-x)^3=3(x+y)(y+z)(x-z)$.

7. Find the H.C.F. of :
   (i) $x^3-(a+p)x^2+(q+ap)x-aq$ and $x^3+ax^2-3a^2x+a^3$;
   (ii) $x^3-y^3-z^3-3xyz$ and $x^2-2xy+y^2-2xz+2yz+z^2$.

8. Show that, if a rational and integral expression in x vanishes when 'a' is put for x, the expression contains $x-a$ as a factor.

VI

1. Show that $(a^2-a+1)(b-c)+(b^2-b+1)(c-a)+(c^2-c+1)(a-b)$

$$=(a^2-a+1)(b^2-c^2)+(b^2-b+1)(c^2-a^2)+(c^2-c+1)(a^2-b^2)$$

2. Show that $\dfrac{ab}{(x-a)(x-b)}+\dfrac{bc}{(x-b)(x-c)}+\dfrac{ca}{(x-c)(x-a)}=0,$

when $\dfrac{1}{x}=\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right).$

3. Prove that $a(b-c)^3+b(c-a)^3+c(a-b)^3=0$, when $a+b+c=0$.

4. Express $(x^2+y^2+z^2+2xy)^2-2(x+y)^2z^2$ as the sum of two perfect squares.

5. Simplify $\left\{\dfrac{y^2-yz+z^2}{x}+\dfrac{x^2}{y+z}-\dfrac{3}{\dfrac{1}{y}+\dfrac{1}{z}}\right\}\cdot\dfrac{\dfrac{2}{y}+\dfrac{2}{z}}{\dfrac{1}{yz}+\dfrac{1}{zx}+\dfrac{1}{xy}}+(x+y+z)^2.$

6. Find the H.C.F. of

(i) $x^3+(5m-3)x^2+3m(2m-5)x-18m^2$

and $x^3+(m-3)x^2-m(2m+3)x+6m^2$.

(ii) $10x^3-54x^2+87x-45$ and $5x^4-36x^3+87x^2-90x+54$.

7. Find the H.C.F. and L.C.M. of

$2x^4+x^3-9x^2+8x-2$ and $2x^4-7x^3+11x^2-8x+2$.

8. Show, without actual division, that $x^{55}-y^{55}$ is divisible by $x-y$; and that the remainder when it is divided by $x+y$ is $-2y^{55}$.

VII

1. Divide the continued product of $1+x+y$, $1-x+y$, $1+x-y$ and $x+y-1$ by $1+2xy-x^2-y^2$.

2. Simplify $\dfrac{bc(x-a)}{(a-b)(a-c)}+\dfrac{ca(x-b)}{(b-a)(b-c)}+\dfrac{ab(x-c)}{(c-a)(c-b)}$. [C. U. 1896]

3. Prove that $2\{(b+c-2a)^4+(c+a-2b)^4+(a+b-2c)^4\}$

$$=\{(b+c-2a)^2+(c+a-2b)^2+(a+b-2c)^2\}^2.$$

4. Reduce the following to their lowest terms :

(i) $\dfrac{a^2}{\dfrac{a^2+b^2}{a+b}-b}+\dfrac{b^2}{\dfrac{a^2+b^2}{a+b}-a}$; (ii) $\dfrac{\dfrac{1-x+x^2}{1+x+x^2}+\dfrac{1-x}{1+x}}{\dfrac{1-x+x^2}{1+x+x^2}+\dfrac{1+x}{1-x}}$.

Express $41x^2 - 60xy + 104y^2$ in the form of $(px + qy)^2 + 4(qx - py)^2$, finding the numerical values of p and q.

6. Find the H.C.F. and L.C.M. of

$$6x^3 - 17x^2 + 11x - 2 \text{ and } 12x^3 - 4x^2 - 3x + 1.$$

7. Show that $m - n$ is a factor of

$$(a + b)(m^2 + n^2) + am(n - 3m) + bn(m - 3n).$$

For what value of a is $x^3 + 5x + a$ divisible by $x - 3$?

8. Show that the last digit in $3^{2n+1} + 2^{2n+1}$ is 5, if n be any positive integer. [M. M. 1868]

VIII

1. Show that

$$(a - b)(x - a)(x - b) + (b - c)(x - b)(x - c) + (c - a)(x - c)(x - a)$$
$$= (a - b)(b - c)(a - c).$$

2. Show that $4(a^2 + ab + b^2)^3 - (a - b)^2(a + 2b)^2(2a + b)^2$ $= 27a^2b^2(a + b)^2$. [M. M. 1888]

3. If $2s = a + b + c$, show that $16s(s - a)(s - b)(s - c)$ $= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$. [C. U. 1867]

4. Resolve into factors

$$(a^2 - b^2)^2 + (c^2 - d^2)^2 - (a + b)^2(c - d)^2 - (a - b)^2(c + d)^2.$$

[M. M. 1876]

5. Simplify $\dfrac{(y - z)(y + z)^3 + (z - x)(z + x)^3 + (x - y)(x + y)^3}{(y + z)(y - z)^3 + (z + x)(z - x)^3 + (x + y)(x - y)^3}$.

[M. M. 1892 ; B. M. 1888]

6. Simplify $\dfrac{x^2 - yz}{(x - y)(x - z)} + \dfrac{y^2 + zx}{(y + z)(y - x)} + \dfrac{z^2 + xy}{(z - x)(z + y)}$. [C. U. 1865]

7. Show that $2^{4n} - 1$ is divisible by 15, if n be a positive integer. [M. M. 1875]

8. Find the H.C.F. and L.C.M. of

$x^4 + 2x^2 + 1$, $x^6 + x^4 - x^2 - 1$ and $x^4 - 1$. [C. U. 1869]

CHAPTER XXVI

SIMPLE EQUATIONS AND PROBLEMS

I. Simple Equations

172. We have already explained the process of solving easy simple equations in Chapters V and XVII and shall now consider the subject more fully.

173. Solution of equations facilitated by suitable transposition and combination of terms.

The following are typical examples.

Example 1. Solve $4(x+1)^2+9(x+2)^2=13(x+3)^2$.

Simplifying the sides, we have

$$4(x^2+2x+1)+9(x^2+4x+4)=13(x^2+6x+9),$$

or, $13x^2+44x+40=13x^2+78x+117$,

or, $13x^2+44x-13x^2-78x=117-40$, [transposing]

i.e., $-34x=77$; $\therefore\ x=-\frac{77}{34}=-2\frac{9}{34}$.

Example 2. Solve $(x-2)^3+(x-6)^3+(x-10)^3=3(x-2)(x-6)(x-10)$.

Transposing, we have

$$(x-2)^3+(x-6)^3+(x-10)^3-3(x-2)(x-6)(x-10)=0,$$

or, $\frac{1}{2}\{(x-2)+(x-6)+(x-10)\}[\{(x-6)-(x-10)\}^2$
$+\{(x-10)-(x-2)\}^2+\{(x-2)-(x-6)\}^2]=0,$

[factorising the left side by Art. 134]

or, $\frac{1}{2}(3x-18)\{(10-6)^2+(-10+2)^2+(-2+6)^2\}=0$,

or, $\frac{1}{2}(3x-18).96=0$; $\therefore\ 3x-18=0$, or, $x=6$.

174. Fractional Equations.

Example 1. Solve $\dfrac{7x-11}{6}=\dfrac{31x-41}{24}-\dfrac{7x^2-4}{56x-47}$.

By transposition, we have

$$\frac{7x^2-4}{56x-47}=\frac{31x-41}{24}-\frac{7x-11}{6}=\frac{(31x-41)-(28x-44)}{24}=\frac{3(x+1)}{24}=\frac{x+1}{8}.$$

Multiplying both sides by $8(56x-47)$, we have

$$8(7x^2-4)=(x+1)(56x-47),$$

or, $56x^2-32=56x^2+9x-47$; $\therefore$ $-32=9x-47$.

Hence, $9x=-32+47=15$; $\therefore$ $x=\frac{15}{9}=1\frac{2}{3}$.

Example 2. Solve $\dfrac{25-\frac{1}{3}x}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=\dfrac{23}{x+1}+5$.

By transposition, we have

$$\frac{16x+4\frac{1}{5}}{3x+2}-5=\frac{23}{x+1}-\frac{25-\frac{1}{3}x}{x+1}, \quad \text{or,} \quad \frac{x-5\frac{4}{5}}{3x+2}=\frac{\frac{1}{3}x-2}{x+1}.$$

Hence, $(x-5\frac{4}{5})(x+1)=(\frac{1}{3}x-2)(3x+2)$,

or, $x^2-(4\frac{4}{5})x-5\frac{4}{5}=x^2-(5\frac{1}{3})x-4$.

Hence, $(5\frac{1}{3}-4\frac{4}{5})x=5\frac{4}{5}-4$,

or, $\frac{8}{15}x=1\frac{4}{5}=\frac{9}{5}$; $\therefore$ $x=\frac{9}{5}\times\frac{15}{8}=\frac{27}{8}=3\frac{3}{8}$.

Example 3. Solve $\dfrac{3}{x-2}+\dfrac{5}{x-6}=\dfrac{8}{x+3}$.

Since, $\dfrac{8}{x+3}=\dfrac{3}{x+3}+\dfrac{5}{x+3}$, we have $\dfrac{3}{x-2}+\dfrac{5}{x-6}=\dfrac{3}{x+3}+\dfrac{5}{x+3}$.

Hence, by transposition, $\dfrac{3}{x-2}-\dfrac{3}{x+3}=\dfrac{5}{x+3}-\dfrac{5}{x-6}$,

or, $$\frac{15}{(x-2)(x+3)}=\frac{-45}{(x+3)(x-6)}.$$

Multiplying both sides by $x+3$, and dividing by 15,

we have $\dfrac{1}{x-2}=\dfrac{-3}{x-6}$.

Hence, $x-6=-3(x-2)$;

$\therefore$ $4x=12$, or, $x=3$.

Example 4. Solve $\dfrac{8}{2x-1}+\dfrac{9}{3x-1}=\dfrac{7}{x+1}$.

We have $\dfrac{8}{2x-1}+\dfrac{9}{3x-1}=\dfrac{4}{x+1}+\dfrac{3}{x+1}$.

Hence, $\left\{\dfrac{8}{2x-1}-\dfrac{4}{x+1}\right\}+\left\{\dfrac{9}{3x-1}-\dfrac{3}{x+1}\right\}=0$. [by transposition]

or, $$\frac{12}{(2x-1)(x+1)}+\frac{12}{(3x-1)(x+1)}=0.$$

Hence, $\dfrac{1}{2x-1}+\dfrac{1}{3x-1}=0$.

Multiplying both sides by $(2x-1)(3x-1)$, we have

$$(3x-1)+(2x-1)=0.$$

Therefore, $5x=2$, or, $x=\frac{2}{5}$.

Example 5. Solve $\dfrac{a-c}{2b+x}+\dfrac{b-c}{2a+x}=\dfrac{a+b-2c}{a+b+x}.$

We have $\dfrac{a-c}{2b+x}+\dfrac{b-c}{2a+x}=\dfrac{(a-c)+(b-c)}{a+b+x}=\dfrac{a-c}{a+b+x}+\dfrac{b-c}{a+b+x}.$

Hence, by transposition,

$$(a-c)\left\{\frac{1}{2b+x}-\frac{1}{a+b+x}\right\}=(b-c)\left\{\frac{1}{a+b+x}-\frac{1}{2a+x}\right\},$$

or, $(a-c)\cdot\dfrac{a-b}{(2b+x)(a+b+x)}=(b-c)\cdot\dfrac{a-b}{(a+b+x)(2a+x)}.$

Hence, $\dfrac{a-c}{2b+x}=\dfrac{b-c}{2a+x};$

$$\therefore \quad (a-c)(2a+x)=(b-c)(2b+x);$$

$$\therefore \quad x\{(a-c)-(b-c)\}=2b(b-c)-2a(a-c),$$

$$\begin{aligned} \text{or,} \quad x(a-b)&=2(b^2-a^2)-2c(b-a)\\ &=2(b-a)(b+a-c)\\ &=2(a-b)(c-a-b);\end{aligned}$$

$$\therefore \quad x=2(c-a-b).$$

EXERCISE 92

Solve the following equations:

1. $3(x+1)^2+4(x+3)^2=7(x+2)^2.$ 2. $(x-a)(x-b)=(x-a-b)^2.$

3. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c).$

4. $(x+a)^2+(x+b)^2+(x+c)^2=(x-2a)^2+(x-2b)^2+(x-2c)^2.$

5. $\dfrac{98x-73}{21}=\dfrac{14x-9}{3}-\dfrac{13x-16}{15x-9}.$

6. $\dfrac{95x-159}{35}=\dfrac{19x-29}{7}-\dfrac{17x-47}{23x-59}.$ 7. $\dfrac{91x-21}{56}+\dfrac{24x-93}{35x-138}=\dfrac{18x+9}{8}.$

8. $\dfrac{117x-26}{135}+\dfrac{16x-77}{23x-110}=\dfrac{13x+4}{15}+\dfrac{3\frac{4}{5}}{27}.$

9. $\dfrac{6x-7\frac{1}{3}}{13-2x}+2x+\dfrac{1+16x}{24}=4\frac{5}{12}-\dfrac{12\frac{5}{6}-8x}{3}.$

10. $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}.$

11. $\dfrac{41-35x}{105}-\dfrac{7-2x^2}{14(x-1)}=\dfrac{1+3x}{21}-\dfrac{2x-2\frac{1}{2}}{6}.$

12. $\dfrac{1}{x-1}-\dfrac{2}{x+7}=\dfrac{1}{7(x-1)}$. 13. $\dfrac{2}{5(3x+4)}+\dfrac{4}{2x+3}=\dfrac{6}{3x+4}$.

14. $\dfrac{3}{3x-5}-\dfrac{6}{7(4x-7)}=\dfrac{7}{9(3x-5)}+\dfrac{2}{4x-7}$.

15. $\dfrac{11}{12(14x-19)}+\dfrac{7}{9(13x-14)}=\dfrac{3}{14x-19}-\dfrac{2}{13x-14}$.

16. $\dfrac{50}{3x-1}+\dfrac{37-\frac{1}{3}x}{12x-1}=\dfrac{35}{12x-1}+\dfrac{49-\frac{1}{12}x}{3x-1}$.

17. $\dfrac{(1\text{[illegible]})x+19\text{[illegible]}}{2x+5}-\dfrac{\text{[illegible]}x+8}{x+8}=\dfrac{20\text{[illegible]}-(1\text{[illegible]})x}{2x+5}+\dfrac{(1\text{[illegible]})x-9}{2(x+8)}$.

18. $\dfrac{(9\text{[illegible]})x-32}{4x+7}+\dfrac{65x+4\text{[illegible]}}{8x+29}=\dfrac{75x+5\text{[illegible]}}{8x+29}+\dfrac{(4\text{[illegible]})x-29}{4x+7}$.

19. $\dfrac{1}{x-1}+\dfrac{2}{x-2}=\dfrac{3}{x-3}$. 20. $\dfrac{3}{4x+1}+\dfrac{4}{4x+5}=\dfrac{7}{4x+3}$.

21. $\dfrac{15}{3x+11}-\dfrac{8}{3x+17}=\dfrac{7}{3x+5}$. 22. $\dfrac{6}{5x+7}-\dfrac{4}{5x+13}=\dfrac{9}{5x+13}-\dfrac{7}{5x+19}$.

23. $\dfrac{8}{2x+17}-\dfrac{12}{2x+25}=\dfrac{5}{2x+25}-\dfrac{9}{2x+33}$.

24. $\dfrac{5}{3-4x}+\dfrac{9}{4x+13}-\dfrac{4}{4x+5}=0$. 25. $\dfrac{6}{5-6x}+\dfrac{13}{6x+19}=\dfrac{7}{6x+7}$.

26. $\dfrac{9}{3-7x}+\dfrac{1}{7x+15}=\dfrac{8}{12-7x}$. 27. $\dfrac{10}{2x-5}+\dfrac{1}{x+5}=\dfrac{18}{3x-5}$.

28. $\dfrac{9}{3x-4}+\dfrac{20}{4x+1}=\dfrac{8}{x+7}$. 29. $\dfrac{12}{3x-8}=\dfrac{20}{4x-13}-\dfrac{1}{x+9}$.

30. $\dfrac{a+b}{x-c}=\dfrac{a}{x-a}+\dfrac{b}{x-b}$. 31. $\dfrac{a^2}{ax-b}+\dfrac{b^2}{bx-a}=\dfrac{a+b}{x+c}$.

32. $\dfrac{m(x+a)}{x+b}+\dfrac{n(x+b)}{x+a}=m+n$. 33. $\dfrac{b-c}{x+a}+\dfrac{a-b}{x+b}=\dfrac{a-c}{x+c}$.

34. $\dfrac{2a-3b}{x-a+b}-\dfrac{2b-3a}{x+a-b}=\dfrac{5(a-b)}{x+a+b}$. 35. $\dfrac{1}{x-6a}+\dfrac{2}{x+3a}+\dfrac{3}{x-2a}=\dfrac{6}{x-a}$.

175. Solution of fractional equations facilitated by the division of each numerator by its denominator.

Example 1. Solve $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=\dfrac{22x+30}{11x-18}$.

We have $\dfrac{(x-1)+2}{x-1}+\dfrac{(x-2)+4}{x-2}=\dfrac{2(11x-18)+66}{11x-18}$,

or, $\left\{1+\dfrac{2}{x-1}\right\}+\left\{1+\dfrac{4}{x-2}\right\}=2+\dfrac{66}{11x-18}$,

or, $\dfrac{2}{x-1}+\dfrac{4}{x-2}=\dfrac{66}{11x-18}$.

Hence, by transposition,

$$\frac{2}{x-1}-\frac{22}{11x-18}=\frac{44}{11x-18}-\frac{4}{x-2},$$

or,
$$\frac{-14}{(x-1)(11x-18)}=\frac{-16}{(11x-18)(x-2)}.$$

Therefore, $\dfrac{7}{x-1}=\dfrac{8}{x-2}$,

or, $7x-14=8x-8$; $\therefore$ $x=-6$.

Example 2. Solve $\dfrac{4x^2+7}{2x-1}+\dfrac{6x^2-8x+11}{3x-1}=\dfrac{4x^2+3x+6}{x+1}$.

We have

$$\frac{(4x^2-1)+8}{2x-1}+\frac{2x(3x-1)-2(3x-1)+9}{3x-1}=\frac{4x(x+1)-(x+1)+7}{x+1},$$

or,
$$\left\{2x+1+\frac{8}{2x-1}\right\}+\left\{2x-2+\frac{9}{3x-1}\right\}=4x-1+\frac{7}{x+1}.$$

Hence, $\dfrac{8}{2x-1}+\dfrac{9}{3x-1}=\dfrac{7}{x+1}$.

For the subsequent part of the solution the student is referred to xample 4 worked out in Art. 174.

Example 3. Solve $\dfrac{7x-55}{x-8}+\dfrac{2x-17}{x-9}=\dfrac{6x-71}{x-12}+\dfrac{3x-14}{x-5}$.

We have

$$\frac{7(x-8)+1}{x-8}+\frac{2(x-9)+1}{x-9}=\frac{6(x-12)+1}{x-12}+\frac{3(x-5)+1}{x-5},$$

or,
$$\left\{7+\frac{1}{x-8}\right\}+\left\{2+\frac{1}{x-9}\right\}=\left\{6+\frac{1}{x-12}\right\}+\left\{3+\frac{1}{x-5}\right\};$$

$$\therefore\ \frac{1}{x-8}+\frac{1}{x-9}=\frac{1}{x-12}+\frac{1}{x-5}.$$

Hence, by transposition,

$$\frac{1}{x-8}-\frac{1}{x-5}=\frac{1}{x-12}-\frac{1}{x-9},$$

or,
$$\frac{3}{(x-8)(x-5)}=\frac{3}{(x-12)(x-9)};$$

$\therefore$ $(x-8)(x-5)=(x-12)(x-9)$,

or, $x^2-13x+40=x^2-21x+108$;

$\therefore$ $8x=68$, or, $x=8\frac{1}{2}$.

EXERCISE 93

Solve the following equations :

1. $\frac{2x-1}{x-1}+\frac{3x-4}{x-2}=\frac{5x-12}{x-3}$.

2. $\frac{2x+7}{x+2}+\frac{4x+29}{x+6}-\frac{6x-10}{x-3}=0$.

3. $\frac{25x-40}{5x-6}-\frac{7x+9}{x+2}+\frac{6x-1}{3x+4}=0$.

4. $2+\cfrac{1}{2+\cfrac{3}{2+\frac{x}{2}}}=\frac{7}{3}$.

5. $8+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{x+2}}}=\frac{214}{25}$. [See Ex. 3 worked out in Art. 168.]

6. $2+\cfrac{1}{1+\cfrac{1}{1+x}}=\frac{2x+7}{2+x}$.

7. $\frac{15x-7}{5x-4}+\frac{4x+3}{4x-3}=\frac{8x+1}{2x-1}$.

8. $\frac{4x-7}{4x+5}+\frac{15x+11}{5x+7}=\frac{12x+1}{3x+4}$.

9. $\frac{4x^3+4x^2+8x+1}{2x^2+2x+3}=\frac{2x^2+2x+1}{x+1}$.

10. $\frac{12x^3+16x^2+29x-1}{3x^2+4x+8}=\frac{4x^2+20x-1}{x+5}$.

11. $\frac{x^2-x+1}{x-1}+\frac{x^2-2x+1}{x-2}=2x+\frac{2}{x-3}$.

12. $\frac{x^2+3}{x-1}+\frac{x^2-x+1}{x-2}=\frac{2x^2-4x+1}{x-3}$.

13. $\frac{2x^2-3x+7}{2x-1}+\frac{6x^2+2x+21}{3x+1}=\frac{3x^2+8x+7}{x+3}$.

14. $\frac{3+2x}{1+2x}-\frac{5+2x}{7+2x}=1-\frac{4x^2-2}{7+16x+4x^2}$.

15. $\frac{2x-3}{x-2}+\frac{3x-20}{x-7}=\frac{x-3}{x-4}+\frac{4x-19}{x-5}$.

16. $\frac{3x-8}{x-3}+\frac{4x-35}{x-9}=\frac{2x-9}{x-5}+\frac{5x-34}{x-7}$.

17. $\frac{3x-13}{x-4}+\frac{4x-41}{x-10}=\frac{2x-13}{x-6}+\frac{5x-41}{x-8}$.

18. $\frac{4x+21}{x+5}+\frac{5x-69}{x-14}=\frac{3x-5}{x-2}+\frac{6x-41}{x-7}$.

19. $\frac{5-6x}{3x-1}+\frac{2x+7}{x+3}=\frac{31-12x}{3x-7}+\frac{4x+21}{x+5}$.

20. $\frac{x^2+3x+3}{x+2}+\frac{x^2-15}{x-4}=\frac{x^2+7x+11}{x+5}+\frac{x^2-4x-20}{x-7}$.

21. $\dfrac{2x+11}{x+5}-\dfrac{9x-9}{3x-4}=\dfrac{4x+13}{x+3}-\dfrac{15x-47}{3x-10}.$ [C. U. 1880]

22. $\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}=\dfrac{x-1}{x-2}+\dfrac{x-4}{x-5}.$ [C. U. 1887]

176. Miscellaneous Examples.

Example 1. Solve $\dfrac{3abc}{a+b}+\dfrac{a^2b^2}{(a+b)^3}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=3cx+\dfrac{bx}{a}.$

By transposition, we have

$$\frac{ab}{a+b}\left\{3c+\frac{ab}{(a+b)^2}\right\}=x\left\{3c+\frac{b}{a}-\frac{(2a+b)b^2}{a(a+b)^2}\right\}$$
$$=x\left\{3c+\frac{b}{a}\left[1-\frac{(2a+b)b}{(a+b)^2}\right]\right\}$$
$$=x\left\{3c+\frac{b}{a}\cdot\frac{a^2}{(a+b)^2}\right\}$$
$$=x\left\{3c+\frac{ab}{(a+b)^2}\right\}.$$

Therefore, $x=\dfrac{ab}{a+b}.$

Example 2. Solve $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}.$

We have $\dfrac{x(ax+b)+c}{x(px+q)+r}=\dfrac{ax+b}{px+q}.$

Hence, putting m for $ax+b$ and n for $px+q$, we have

$$\frac{mx+c}{nx+r}=\frac{m}{n};$$

$\therefore\ mnx+cn=mnx+rm$; $\therefore\ cn=rm$,

or, $c(px+q)=r(ax+b)$; $\therefore\ x(cp-ar)=br-cq$;

$$\therefore\quad x=\frac{br-cq}{cp-ar}.$$

Example 3. Solve $(x-2a)^3+(x-2b)^3=2(x-a-b)^3.$

By transposition, we have

$$(x-2a)^3-(x-a-b)^3=(x-a-b)^3-(x-2b)^3.$$

Putting X for $x-2a$, Y for $x-2b$, and Z for $x-a-b$, we have

$$X^3-Z^3=Z^3-Y^3,$$

or, $$(X-Z)(X^2+XZ+Z^2)=(Z-Y)(Z^2+ZY+Y^2).$$

But, $X-Z=Z-Y$, because each of them $=b-a$;

$\therefore\quad X^2+XZ+Z^2=Z^2+ZY+Y^2.$

Hence, by transposition, $X^2 - Y^2 = Z(Y - X)$.

Removing the common factor $X - Y$, which $= 2b - 2a$, we have

$$X + Y = -Z,$$

$$i.e.,\quad (x - 2a) + (x - 2b) = -(x - a - b).$$

Hence, $3x = 3(a + b)$, and $\therefore\ x = a + b$.

Example 4. Solve $\dfrac{x+a}{x+b} = \left(\dfrac{2x+a+c}{2x+b+c}\right)^2$.

Since, $\dfrac{x+a}{x+b} = \dfrac{(x+b)+(a-b)}{x+b} = 1 + \dfrac{a-b}{x+b}$,

and $\dfrac{2x+a+c}{2x+b+c} = \dfrac{(2x+b+c)+(a-b)}{2x+b+c} = 1 + \dfrac{a-b}{2x+b+c}$,

we have $1 + \dfrac{a-b}{x+b} = \left\{1 + \dfrac{a-b}{2x+b+c}\right\}^2 = 1 + \dfrac{2(a-b)}{2x+b+c} + \dfrac{(a-b)^2}{(2x+b+c)^2}$.

Hence, transposing and dividing both sides by $a - b$, we have

$$\frac{1}{x+b} - \frac{2}{2x+b+c} = \frac{a-b}{(2x+b+c)^2},$$

$$\text{or,}\quad \frac{c-b}{(x+b)(2x+b+c)} = \frac{a-b}{(2x+b+c)^2};$$

$$\therefore\quad \frac{c-b}{x+b} = \frac{a-b}{2x+b+c};$$

$$\therefore\quad 2x(c-b) + (c^2 - b^2) = x(a-b) + b(a-b);$$

$$\therefore\quad x(a+b-2c) = c^2 - ab;$$

$$\therefore\quad x = \frac{c^2 - ab}{a+b-2c}.$$

Example 5. Solve $\dfrac{4x}{3} - \dfrac{125x^2-5}{(5x-1)(x+5)} = 5x - \dfrac{5}{3}.\dfrac{3x^2-1}{x+5} - \dfrac{95-4x}{3}$.

Since, $\dfrac{125x^2-5}{(5x-1)(x+5)} = \dfrac{5(25x^2-1)}{(5x-1)(x+5)} = \dfrac{5(5x+1)}{x+5}$,

and $\dfrac{5}{3}.\dfrac{3x^2-1}{x+5} = \dfrac{\tfrac{5}{3}(3x^2-1)}{x+5} = \dfrac{5x^2-\tfrac{5}{3}}{x+5}$,

we have $\dfrac{4x}{3} - \dfrac{5(5x+1)}{x+5} = 5x - \dfrac{5x^2-\tfrac{5}{3}}{x+5} - \dfrac{95}{3} + \dfrac{4x}{3}$.

Hence, transposing and dividing both sides by 5, we have

$$\frac{x^2 - \tfrac{1}{3} - (5x+1)}{x+5} = x - 6\tfrac{1}{3}.$$

Hence, $x^2 - 5x - 1\tfrac{1}{3} = x^2 - (1\tfrac{1}{3})x - 31\tfrac{2}{3}$;

$\therefore\quad (3\tfrac{2}{3})x = 30\tfrac{1}{3}$; $\quad\therefore\ x = \tfrac{91}{11} = 8\tfrac{3}{11}$.

EXERCISE 94

Solve the following equations :

1. $\dfrac{2x}{x-4}+\dfrac{7x-3}{x+1}=9.$

2. $\dfrac{x+4a+b}{x+a+b}+\dfrac{4x+a+2b}{x+a-b}=5.$

3. $\dfrac{3x+5}{x+1}=\dfrac{4x+8}{3x+3}+\dfrac{10x+1}{6x+3}.$

4. $\dfrac{6x+8}{2x+1}-\dfrac{2x+38}{x+12}=1.$

5. $\dfrac{x+18}{x-2}-\dfrac{27-3x}{3x-19}=2.$

6. $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}.$

7. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}=0.$

8. $\dfrac{(x-a)(x-b)}{x-a-b}=\dfrac{(x-c)(x-d)}{x-c-d}.$

9. $\dfrac{1}{x^2+3x+2}+\dfrac{2x}{x^2+4x+3}+\dfrac{1}{x^2+5x+6}=14-\dfrac{60+4x}{x+3}.$

10. $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}=\dfrac{3a}{x(a^4+a^2x^2+x^4)}.$

11. $\dfrac{x}{x-2}+\dfrac{x-9}{x-7}=\dfrac{x+1}{x-1}+\dfrac{x-8}{x-6}.$

12. $\dfrac{1}{(x+a)^2-b^2}+\dfrac{1}{(x+b)^2-a^2}=\dfrac{1}{x^2-(a+b)^2}+\dfrac{1}{x^2-(a-b)^2}.$

13. $\dfrac{3x^2+5x+8}{5x^2+6x+12}=\dfrac{3x+5}{5x+6}.$

14. $\dfrac{58x^2+87x+7}{87x^2+145x+11}=\dfrac{2x+3}{3x+5}.$

15. $\dfrac{a^2(a-2b)}{b(a-b)^2}\cdot x+\dfrac{2abc}{a-b}-\dfrac{ax}{b}=2cx-\dfrac{a^2b^2}{(a-b)^3}.$

16. $(x-23)^3+(x-27)^3=2(x-25)^3.$

17. $\dfrac{4x-17}{9}-\dfrac{3\frac{2}{3}-22x}{33}=x-\dfrac{6}{x}\left(1-\dfrac{x^2}{54}\right).$

18. $\left(\dfrac{x-2a}{x+2b}\right)^2=\dfrac{x-2a-2b}{x+2a+2b}.$

19. $\dfrac{x+19}{x+10}=\left(\dfrac{2x+33}{2x+24}\right)^2.$

20. $\left(\dfrac{x-a}{x+b}\right)^3=\dfrac{x-2a-b}{x+a+2b}.$

177. A simple equation cannot have more than one root. If terms containing the unknown quantity be transferred to one side of the equation and those involving known quantities to the other side, every simple equation can ultimately be reduced to the form $ax=b$.

Thus, to make the equation true, x must be equal to $\dfrac{b}{a}$ and to nothing else.

Hence, a simple equation cannot have more than one root.

Otherwise : Every simple equation is ultimately reducible to the orm $ax=b$. Let this equation, if possible, have two different roots α and β.

Thus, we must have $\left.\begin{array}{l} a\alpha=b \\ a\beta=b \end{array}\right\}$
and also

Hence, by subtraction, $a(\alpha-\beta)=0$.

But this is impossible because a is not zero and by supposition $\alpha-\beta$ also is not zero.

Thus, a simple equation cannot have more than one root.

178. Two exceptions in the solution of a simple equation.

(1) If a simple equation reduces to the form

$$0\times x=0, \quad i.e., \quad 0=0.$$

Evidently, *the equation is identically true and has, therefore, any number of roots.*

Example. The equation

$$x+2=\frac{x}{2}+\frac{x+4}{2}$$

gives, on transposition, $(1-\frac{1}{2}-\frac{1}{2})x=\frac{4}{2}-2$,

$$\text{or,} \quad 0\times x=0, \quad \text{or,} \quad 0=0.$$

The equation is, therefore, an identity and is true for every value of x.

(2) The equation

$$\left(\frac{x+5}{3}\right)=\frac{x+4}{2}-\frac{x-4}{6}$$

leads on simplification and transposition to

$$(\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{6})x=\tfrac{4}{2}+\tfrac{4}{6}-\tfrac{5}{3},$$

or, $0\times x=1$, or, $0=1$, which is absurd.

This equation is, therefore, absurd and has consequently no root.

Generally, *if a simple equation reduces to the form* $0\times x=b$, where b is *not* zero, *the equation is absurd and cannot, therefore, have any root.*

II. Problems leading to Simple Equations

179. The general process of solving such problems has been explained in Chapter XVII. We shall in the present section consider a few problems of a harder type than those treated of previously.

The following examples will serve as further illustrations.

Example 1. At what time between 1 o'clock and 2 o'clock is there exactly one minute-division between the hands of a clock ?

Suppose it is x minutes past one when the hands are one minute division apart from each other.

Then, at the required instant the minute-hand is at a distance of x minute-divisions from the 12 o'clock mark ; and since the minute-hand moves twelve times as fast as the hour-hand, the hour-hand moves over $\frac{x}{12}$ths of a minute-division whilst the minute-hand moves over

x minute-divisions ; therefore, at the required instant the hour-hand is at a distance of $\left(5+\frac{x}{12}\right)$ minute-divisions from the 12 o'clock mark.

Hence, as the minute-hand is at the required instant one minute-division apart from the hour-hand, we must have

$$x=\left(5+\frac{x}{12}\right)\pm 1$$

The upper sign being taken when the minute-hand is ahead of the hour-hand, and the lower when behind it.

$$\therefore \quad \tfrac{11}{12}x=5\pm 1=6, \text{ or, } 4\,;$$

$$\therefore \quad x=\tfrac{72}{11}=6\tfrac{6}{11}, \text{ or, } =\tfrac{48}{11}=4\tfrac{4}{11}.$$

Thus, the hands are one minute-division apart at $4\frac{4}{11}$ or $6\frac{6}{11}$ minutes past one.

Example 2. A gentleman went out for evening walk between 5 P.M. and 6 P.M., and came back between 6 P.M. and 7 P.M. He found that the hands of his watch interchanged their places. When did he go out ?

[Interchange of places by the hands of a watch means that the minute-hand has come to the place where the hour-hand was and the hour-hand to the place where the minute-hand was.]

From the condition of the problem it is evident that the hour-hand was in between the marks V and VI and the minute-hand in between VI and VII when the gentleman went out.

Suppose the gentleman went out at x minutes past 5 P.M.

At 5 P.M. the hour-hand was 25 minute-divisions ahead of the minute-hand and in x minutes it has moved $\frac{x}{12}$ minute-divisions. So the hour-hand was $\left(25+\frac{x}{12}\right)$ minute-divisions away from the mark XII when the gentleman went out.

So, when the gentleman returned, the hour-hand was 30 minute-divisions (when it is six, the hour-hand is 30 minute-divisions away from the mark XII) plus the minute-divisions which are passed by the hour-hand during the period in which the minute-hand passes $\left(25+\frac{x}{12}\right)$ minute-divisions, away from the mark XII.

$$\therefore \; x \text{ minute-divisions}=30+\frac{1}{12}\left(25+\frac{x}{12}\right) \text{ minute-divisions}\,;$$

or, $12x=360+25+\frac{x}{12}$; or, $\frac{143}{12}x=385$;

$\therefore\quad x=385\times\frac{12}{143}=32\frac{4}{13}$.

The gentleman went out at $32\frac{4}{13}$ minutes past 5 P.M

Example 3. The distance from a place P to another place Q is $3\frac{1}{2}$ kilometres. Two persons, A and B, start together from P to go to Q, the former by carriage which travels at the rate of 6 kilometres an hour, the latter walking at the rate of 3 kilometres an hour. If A remains at Q for 15 minutes, and then returns by the carriage to P, find where he will meet B. [C. U. 1882 (*adapted*)]

Let x kilometres be the distance of the place of meeting from P.

Then during the time that B travels x kilometres, A finishes the journey, remains at Q for 15 minutes, and then travels back $(3\frac{1}{2}-x)$ kilometres.

Now, the time in which A does all these

$$=\left(\frac{3\frac{1}{2}}{6}+\frac{1}{4}+\frac{3\frac{1}{2}-x}{6}\right)\text{ hours};$$

and the time in which B travels x kilometres $=\frac{x}{3}$ hours;

$$\therefore\quad \frac{3\frac{1}{2}}{6}+\frac{1}{4}+\frac{3\frac{1}{2}-x}{6}=\frac{x}{3},$$

or, $7+3+(7-2x)=4x$; $\therefore\ 6x=17$; $\therefore\ x=2\frac{5}{6}$.

Thus, A will meet B at a distance of $2\frac{5}{6}$ kilometres from P.

Example 4. A landlord let his farm for Rs. 200 a year in money and a corn-rent. When corn sold at Rs. 10 a bushel, he received at the rate of Rs. 10 an acre for his land ; but when it sold at Rs. 13 50 paise a bushel, Rs. 13 an acre. Of how many bushels did the corn-rent consist ?

Let $x=$ the number of bushels the corn-rent consisted of.

Then when corn sold at Rs. 10 a bushel, the annual income was $(200+10x)$ rupees ; hence, as the income in this case was at the rate of Rs. 10 an acre, the number of acres must evidently be $\frac{200+10x}{10}$, or, $20+x$.

In the second case (*i.e.*, when corn sold at Rs. 13 50 paise a bushel) the annual income amounted to Rs. 200 + Rs. $(13\frac{1}{2})x$, or, Rs. $\frac{400+27x}{2}$ rupees ; but now the income was at the rate of Rs. 13 an acre. Hence, the number of acres must also be equal to $\frac{400+27x}{26}$.

Hence, $20+x=\dfrac{400+27x}{26}$,

or, $520+26x=400+27x$; $\therefore\ x=120$.

Thus, the corn-rent consisted of 120 bushels.

Example 5. A hare is eighty of her own leaps before a greyhound; she takes three leaps for every two that he takes, but he covers as much ground in one leap as she does in two. How many leaps will the hare have taken before she is caught?

Let $3x=$ the number of leaps the hare takes.

Then $2x=$ the number of leaps the greyhound takes in the same time.

The distance of the place where the hare is caught from the first position of the greyhound $=(80+3x)$ leaps of the hare and is also $=2x$ leaps of the greyhound.

But, 1 leap of the greyhound being equal to 2 leaps of the hare, $2x$ leaps of the greyhound $=4x$ leaps of the hare.

$$\therefore\ 80+3x=4x;\quad \therefore\ x=80.$$

Hence, the number of leaps which the hare takes before she is caught $=3\times 80=240$.

Example 6. A banker has two kinds of money, silver and gold and a pieces of silver or b pieces of gold, make up the same sum s. A person comes and wishes to be paid the sum s with c pieces of money; how many of each must the banker give him?

Let $x=$ the number of silver pieces required;

then $c-x=$ " " " gold " "

$$\left.\begin{array}{l}\text{The value of one piece of silver}=\dfrac{s}{a}\\[2ex]\text{and that of one piece of gold}=\dfrac{s}{b}\end{array}\right\}$$

Hence, since by supposition x pieces of silver and $(c-x)$ pieces of gold are together equal in value to s, we must have

$$s=x\cdot\frac{s}{a}+(c-x)\cdot\frac{s}{b};$$

$$\therefore\ 1=\frac{x}{a}+\frac{c-x}{b}.$$

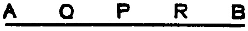
or, $x\left(\frac{1}{b}-\frac{1}{a}\right)=\frac{c}{b}-1$;

$$\therefore \quad x=\frac{a(c-b)}{a-b},$$

and $\quad\therefore\quad c-x=c-\frac{a(c-b)}{a-b}=\frac{b(a-c)}{a-b}.$

Thus, $\frac{a(c-b)}{a-b}$ pieces of silver and $\frac{b(a-c)}{a-b}$ pieces of gold will be required.

Example 7. *AB* is a railway 220 kilometres long, and three trains (*P*, *Q*, *R*) travel upon it at the rate of 25, 20 and 30 kilometres per hour respectively; *P* and *Q* leave *A* at 7 A.M. and 8-15 A.M. respectively and *R* leaves *B* at 10-30 A.M. When and where will *P* be equidistant from *Q* and *R*?



Let *P*, *Q*, *R*, as in the figure, be the respective positions of the trains at the instant when *P* is equidistant from *Q* and *R*.

Let this happen x hours after *R* has left *B*, *i.e.*, x hours after 10-30 A.M.

Then, since *P* left *A* 3½ hours before 10-30 A.M., it has evidently been travelling for $(3\frac{1}{2}+x)$ hours up to the instant in question.

Hence, clearly $AP=(3\frac{1}{2}+x).25$ kilometres,

and $AQ=(2\frac{1}{4}+x).20$ kilometres;

also $BR=30x$ kilometres.

Hence, $PQ=AP-AQ$

$=\{(3\frac{1}{2}+x).25-(2\frac{1}{4}+x).20\}$ kilometres,

and $PR=AB-AP-BR$

$=\{220-(3\frac{1}{2}+x).25-30x\}$ kilometres.

But $PQ=PR$;

$\therefore\quad (3\frac{1}{2}+x).25-(2\frac{1}{4}+x).20=220-(3\frac{1}{2}+x).25-30x$;

$\therefore\quad 50(3\frac{1}{2}+x)-(2\frac{1}{4}+x).20=220-30x$;

$\therefore\quad 60x=220-175+45=90$;

$\therefore\quad x=1\frac{1}{2}$.

Thus, *P* will be equally distant from *Q* and *R* at 1½ hours after 10-30 A.M., *i.e.*, at 12 A.M.

Also, as *P* left *A* at 7 A.M., its distance from *A* at that instant will be 5 × 25, or, 125 kilometres.

Example 8. Two passengers have together 5 cwt. of luggage and are charged for the excess above the weight allowed 5*s*. 2*d*. and 9*s*. 10*d* respectively ; but if the luggage had all belonged to one of them he would have been charged 19*s*. 2*d*. How much luggage is each passenger allowed to carry free of charge ? And how much luggage had each passenger ? [O. U. 1877]

Let x cwt. = weight of luggage that each passenger is allowed to carry free of charge.

Then, (5*s*. 2*d*.) + (9*s*. 10*d*.) = charge for $(5-2x)$ cwt.

$$\therefore \quad \frac{15 \times 12}{5-2x} d. = \text{charge for 1 cwt.}$$

Also, 19*s*. 2*d*. = charge for $(5-x)$ cwt.,

$$\therefore \quad \frac{230}{5-x} d. = \text{charge for 1 cwt.}$$

$$\text{Hence,} \quad \frac{15 \times 12}{5-2x} = \frac{230}{5-x} ;$$

$$\therefore \quad 18(5-x) = 23(5-2x),$$

$$\text{or,} \quad 28x = 115 - 90 = 25 ; \qquad \therefore \quad x = \tfrac{25}{28},$$

i.e., weight of luggage allowed free of charge $= \frac{25}{28}$ cwt. $= \frac{25}{28} \times 4 \times 28$ lbs.
= 100 lbs.

$$\text{Now, charge for 1 cwt.} = \frac{230}{5-x} d. = \frac{230}{5-\frac{25}{28}} d. = \frac{230 \times 28}{5 \times 23} d. = 56d.$$

And since charge for excess luggage of the first passenger = 5*s*. 2*d*. = 62*d*., and charge for excess luggage of the second passenger = 9*s*. 10*d*. = 118*d*.,

$\therefore$ weight of excess luggage of the first passenger
$= \frac{62}{56}$ cwt. $= \frac{62}{56} \times 4 \times 28$ lbs. = 124 lbs. ;

and weight of excess luggage of the second passenger
$= \frac{118}{56}$ cwt. $= \frac{118}{56} \times 4 \times 28$ lbs. = 236 lbs.

Hence, the whole luggage of the first passenger
= (100 + 124) lbs. = 224 lbs. ;

and the whole luggage of the second passenger
= (100 + 236) lbs. = 336 lbs.

Example 9. A person buys some tea at 3 rupees a kilogram and some at 5 rupees a kilogram ; he wishes to mix them, so that by selling the mixture at $3\frac{3}{4}$ rupees a kilogram, he may gain 10 *per cent.* on each kilogram sold. Find how many kilograms of the inferior tea he must mix with each kilogram of the superior.

Suppose x kilograms of the inferior tea are mixed with each kilogram of the superior.

The price of x kilograms of the inferior tea and one kilogram of the superior

$$=(3x+5) \text{ rupees};$$

$$\therefore \text{ the average cost per kilogram} = \frac{3x+5}{x+1} \text{ rupees.}$$

But by selling the mixture at $3\frac{2}{3}$ rupees a kilogram, he *gains* 10 *per cent.* on each kilogram, *i.e.*, realises 110 rupees, for every 100 rupees, or $\frac{11}{10}$ rupees for every rupee.

Hence, $3\frac{2}{3}$ rupees $= \frac{11}{10}$ of the cost per kilogram ;

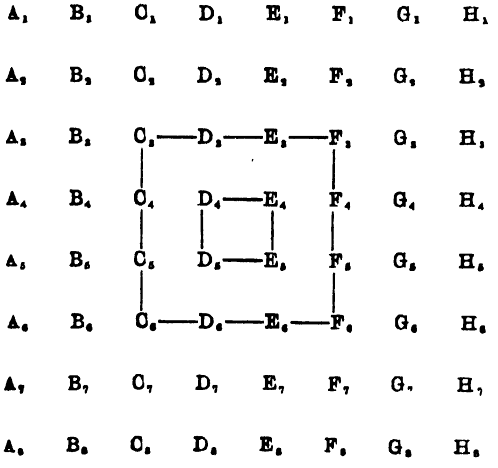
$$\therefore \quad 3\frac{2}{3} = \frac{11}{10} \times \frac{3x+5}{x+1}, \qquad \text{or,} \quad \frac{11}{3} = \frac{11}{10} \times \frac{3x+5}{x+1};$$

$$\therefore \quad 10(x+1) = 3(3x+5); \qquad \therefore \quad x = 5.$$

Thus, 5 kilograms of the inferior tea must be mixed with each kilogram of the superior.

Example 10. An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former ; find the number of men. [C. U. 1887]

[A number of men are said to be arranged in a *solid* square when they are arranged in parallel rows and the number of rows is equal to the number of men in each row. The following diagram, in which A_1, B_1, C_1, &c., represent men, will give the student a correct notion of such arrangement.



$A_1 \quad B_1 \quad C_1 \quad D_1 \quad E_1 \quad F_1 \quad G_1 \quad H_1$

$A_2 \quad B_2 \quad C_2 \quad D_2 \quad E_2 \quad F_2 \quad G_2 \quad H_2$

$A_3 \quad B_3 \quad C_3 \quad D_3 \quad E_3 \quad F_3 \quad G_3 \quad H_3$

$A_4 \quad B_4 \quad C_4 \quad D_4 \quad E_4 \quad F_4 \quad G_4 \quad H_4$

$A_5 \quad B_5 \quad C_5 \quad D_5 \quad E_5 \quad F_5 \quad G_5 \quad H_5$

$A_6 \quad B_6 \quad C_6 \quad D_6 \quad E_6 \quad F_6 \quad G_6 \quad H_6$

$A_7 \quad B_7 \quad C_7 \quad D_7 \quad E_7 \quad F_7 \quad G_7 \quad H_7$

$A_8 \quad B_8 \quad C_8 \quad D_8 \quad E_8 \quad F_8 \quad G_8 \quad H_8$

The above diagram represents an arrangement in which there are 8 rows, each containing 8 men. This is a *solid* square. If the square $C_3F_3F_6C_6$ be removed from inside, the remainder will be a *hollow* square *two deep*, having 8 men in the front rank; if, however, the square $D_4E_4E_8D_8$ be removed, the remainder will be a *hollow* square *three deep*, having the same 8 men in the front rank.

Hence, the number of men in a *hollow* square *two deep* having x men in the front rank $= x^2 - (x-4)^2$; in one *three deep* $= x^2 - (x-6)^2$; and so on; thus, the number of men in a hollow square n deep having x men in the front row $= x^2 - (x-2n)^2$.]

Let $x =$ the number of men in the front row of the first arrangement.

Then, $x-4 =$ the number of men in the front row of the second arrangement

Hence, the number of men in the first square

$$= x^2 - (x-10)^2 \quad \cdots \quad (1)$$

and the number of men in the second square

$$= (x-4)^2 - \{(x-4)-12\}^2.$$

But the men that form the first square are exactly those that form the second;

$$\therefore \quad x^2 - (x-10)^2 = (x-4)^2 - \{(x-4)-12\}^2,$$

$$\text{or,} \quad 20x - 100 = 24(x-4) - 144,$$

$$\therefore \quad 4x = 144 + 96 - 100 = 140,$$

$$\therefore \quad x = 35.$$

Hence, from (1), the total number of men

$$= (35)^2 - (25)^2 = 60 \times 10 = 600.$$

EXERCISE 95

1. Find the time between 3 and 4 o'clock, when the two hands of a watch are coincident.

2. At what time are the hands of a watch together between 5 and 6 o'clock? [C. U. 1886]

3. Find the respective times between 7 and 8 o'clock, when the hour and minute-hands of a watch are (i) exactly opposite to each other; (ii) at right angles to each other; (iii) coincident.

4. What is the *first* hour after 6 o'clock, at which the two hands of a watch are (i) directly opposite, and (ii) at right angles to each other?

5. Two men set out at the same time to walk, one from A to B and the other from B to A, a distance of a kilometres. The former walks at the rate of p kilometres and the latter at the rate of q kilometres an hour; at what distance from A will they meet?

6. Two persons walk at the rate of 5 and 6 kilometres an hour respectively. They set out to meet each other from two places 22 kilometres apart. Having passed each other once, find the place of their *second* meeting, supposing them to continue their journey between the two places. Also find the time when the second meeting takes place.

7. A man rides one-third of the distance from A to B at the rate of a kilometres per hour and the remainder at the rate of $2b$ kilometres per hour. If he had travelled at a uniform rate of $3c$ kilometres per hour, he could have ridden from A to B and back again in the same time.

Prove that $\frac{2}{c} = \frac{1}{a} + \frac{1}{b}$. [C. U. 1888]

8. A and B start to run a race. At the end of 5 minutes, when A has run 900 metres and has outstripped B by 75 metres, he falls; but though he loses ground by the accident, and for the rest of the course makes 20 metres a minute less than before, he comes in only half a minute behind B. How long did the race last?

9. A person sets out to walk from a certain town; but when he has accomplished a quarter of his journey, he finds that if he continues at the same pace he will have gone only $\frac{5}{8}$ths of the whole distance when he ought to be at his destination. He, therefore, increases his speed by a kilometre an hour, and arrives just in time. Find the rate of walking.

10. A tenant hired his farm for Rs. 1600 a year in money and a corn-rent in rice. When rice sold at rupees twenty-five a bushel, he paid at the rate of Rs. 35 an acre for his land; when it sold at Rs. 30 a bushel, he paid at the rate of Rs. 40 an acre. Find the number of bushels of rice in the rent.

11. A footman who contracted for Rs. 240 a year and a livery suit, was turned away at the end of 7 months and received only Rs. 65 and his livery. What was its value?

12. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare?

13. A greyhound spying a hare at a distance of 60 of his own leaps from him, pursues her, making 4 leaps for every 5 leaps of the hare; but he passes over as much ground in 3 leaps as the hare does in 4. How many leaps did each make during the whole course?

14. The St. John's boat is ahead of the Caius by a distance equivalent to 30 strokes of the former. The Johnians pull 4 strokes to 3 strokes of the Caius, but 2 of the latter are equivalent to 3 of the former. How many strokes must the Caius take to bump the St. John's boat?

15. A and B find a purse with rupees in it. A takes out two rupees and one-sixth of what remains; then B takes out three rupees and one-sixth of what remains; and then they find that they have taken out equal shares. How many rupees were in the purse, and how many did each take?

16. A ship sails with a supply of biscuit for 60 days at a daily allowance of 1 kilogram a head; after being at sea for 20 days she encounters a storm in which 5 men are washed overboard and damage

sustained, that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ths of a kilogram. Find the original number of the crew.

17. If 19 kilograms of gold weigh 18 kilograms in water, and 10 kilograms of silver weigh 9 kilograms in water, find the quantity of gold and silver in a mass of gold and silver weighing 106 kilograms in air and 99 kilograms in water.

18. A person rows from Cambridge to Ely, a distance of 32 kilometres and back again in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row $3\frac{1}{8}$ kilometres against the stream in the same time that he rows $4\frac{3}{8}$ kilometres with it. Find the time of his going and returning.

19. A person passed $\frac{1}{6}$th of his age in childhood, $\frac{1}{12}$th in youth, $\frac{1}{7}$th + 5 years in matrimony; he had then a son, whom he survived 4 years, and who reached only one-half the age of his father. Find the son's age when he died.

20. There are two bars of metal, the first containing 14 grams of silver and 6 of tin, the second containing 8 of silver and 12 of tin. How much must be taken from each to form a bar of 20 grams containing equal weights of silver and tin?

21. Divide Rs. 11000 into two sums, such that the simple interest of the greater sum for two years, at $3\frac{1}{2}$ per cent. shall exceed that of the less for $2\frac{1}{2}$ years, at $3\frac{1}{4}$ per cent. by Rs. 165.

22. To remove four articles of furniture, I required for the 1st article two coolies, for the 2nd three, for the 3rd four, and for the 4th five. After giving the 1st set of men one group of paise and one paisa more, to the 2nd set an equal group and four paise more, to the 3rd an equal group and five paise more and to the 4th an equal group and nine paise more, I found that each man of the 3rd and 4th sets had received the same number of paise. How many paise were there in each group; how many paise did each man receive, and how many paise did I distribute?

23. Fifteen current guineas should weigh 4 ounces; but a parcel of light gold being weighed and counted, was found to contain 9 more guineas than was supposed from the weight; and a part of the whole exceeding the half by 10 guineas and a half, was found to be $1\frac{1}{8}$ oz. deficient in weight. What was the number of guineas in the parcel?

24. A silversmith received in payment for a certain weight of wrought plate, the price of which was £10, the same weight of unwrought plate, and £3. 15*s*. besides. At another time he exchanged 12 oz. of wrought plate of the same workmanship as before for 8 oz. of unwrought (for which he allowed the same price as before), and £2. 16*s*. in money. What was the price of wrought plate per ounce, and the weight of the first sold?

25. Two passengers are charged for excess of luggage Rs. 34 and Rs. 90 respectively; had the luggage all belonged to one of them, he would have been charged for excess Rs. 174; how much would they have been charged if none had been allowed free?

26. How many bundles of hay, at Rs. 5 per thousand, must a *ghaswalla* mix with 5600 bundles at Rs. 6 per thousand, in order that he may gain 20 per cent. by selling the whole at $\frac{11}{16}$th rupee per hundred? [C. U. 1875]

27. A boy buys a certain number of oranges at 3 for 2*d.* and one-third of that number at 2 for 1*d.*; at what price must he sell them to get 20 per cent. profit; if his profit be 5*s.* 4*d.*, find the number bought. [C. U. 1885]

28. *A* went out after 3 P.M. and returning half an hour later found that the minute-hand was as much in advance of the hour-hand as it was behind the hour-hand when he went out. Find at what time he went out. [W. B. C. S. 1955]

29. A man went out between 4 P.M. and 5 P.M. and returning home between 5 P.M. and 6 P.M. he found that the hands of his watch interchanged their places. When did he go out?

30. From each of a number of foreign gold coins a person filed a fifth part, and had passed two-thirds of them, when the rest were seized as light coins except one, with which the man decamped, having lost upon the whole half as much as he had gained before. How many coins were there at first?

31. Find a number of three digits, each greater by unity than that which follows it, so that its excess above one-fourth of the number formed by inverting the digits shall be 36 times the sum of the digits.

32. A number of troops being formed into a solid square, it was found there were 60 over; but when formed into a column with 5 men more in front than before and 3 less in depth, there was just one man wanting to complete it. Find the number.

33. An officer can form the men of his regiment into a hollow square 10 deep. The number of men in the regiment is 2800. Find the number of men in the front of the hollow square.

34. A company of men is formed into a hollow square 4 deep and also into a hollow square 8 deep; the front in the latter formation contains 19 men fewer than that in the former formation; find the number of men.

35 A detachment from an army was marching in regular column with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. Find the number of men in the detachment.

CHAPTER XXVII

HARDER SIMULTANEOUS EQUATIONS AND PROBLEMS

I. Harder Simultaneous Equations

180. The process of solving easy simultaneous equations in two variables has already been explained in Chapter XVIII. We propose now to consider the subject more fully.

181. Method of Cross Multiplication.

If $a_1x+b_1y+c_1z=0$, and $a_2x+b_2y+c_2z=0$, † to prove that

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Multiplying the 1st equation by c_2, and the 2nd by c_1, we have

$$a_1c_2x+b_1c_2y+c_1c_2z=0,$$

and $a_2c_1x+b_2c_1y+c_2c_1z=0.$

Hence, by subtraction,

$$(c_1a_2-c_2a_1)x+(b_2c_1-b_1c_2)y=0,$$

$$\therefore\quad (c_1a_2-c_2a_1)x=(b_1c_2-b_2c_1)y\,;$$

$$\therefore\quad \frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}. \qquad \ldots \qquad \ldots \qquad (1)$$

Again, multiplying the 1st equation by a_2, and the 2nd by a_1, we have

$$a_1a_2x+b_1a_2y+c_1a_2z=0,$$

and $a_2a_1x+b_2a_1y+c_2a_1z=0.$

Hence, by subtraction,

$$(a_1b_2-a_2b_1)y+(c_2a_1-c_1a_2)z=0\,;$$

$$\therefore\quad (a_1b_2-a_2b_1)y=(c_1a_2-c_2a_1)z\,;$$

$$\therefore\quad \frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}. \qquad \ldots \qquad \ldots \qquad (2)$$

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† It is necessary to point out to the student the notation here used. The letter a_1 is as different from a_2 as c is from d, or as any letter of the alphabet from any other; a similar remark applies to the pairs of letters (b_1, b_2) and (c_1, c_2). But it is very convenient as an aid to memory to use the same letter with different suffixes to denote corresponding coefficients in different equations; thus, whilst a_1 denotes the coefficient of x in the 1st equation; a_2 denotes the coefficient of x in the 2nd equation; and precisely a similar meaning is attached to the letters b_1, b_2 and c_1, c_2. Sometimes, however, letters with accents serve the same purpose; thus, if a, b, c denote the coefficients of x, y, z in one equation, the corresponding coefficients in a second equation are denoted by a', b', c'; in a third equation by a'', b'', c''; and so on.

Hence, from (1) and (2),

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

Note. *This result can be easily remembered ; writing down the equations one above the other,*

$$\left.\begin{array}{l}a_1x+b_1y+c_1z=0\\a_2x+b_2y+c_2z=0\end{array}\right\},\text{ we find that}$$

(i) *the quantity under* x = *coefficient of* y *in the 1st equation* × *coefficient of* z *in the 2nd minus coefficient of* y *in the 2nd* × *coefficient of* z *in the 1st ;*

(ii) *the quantity under* y = *coefficient of* z *in the 1st equation* × *coefficient of* x *in the 2nd minus coefficient of* z *in the 2nd* × *coefficient of* x *in the 1st ;*

(iii) *the quantity under* z = *coefficient of* x *in the 1st equation* × *coefficient of* y *in the 2nd minus coefficient of* x *in the 2nd* × *coefficient of* y *in the 1st.*

Cor. In the above equations, if we put $z=1$, we have

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1},$$

which gives the solution of the equations

$$\left.\begin{array}{rl}&a_1x+b_1y+c_1=0\\\text{and}&a_2x+b_2y+c_2=0\end{array}\right\}$$

Note. *The above results should be thoroughly committed to memory, as ready applications of them will enable the student to solve with neatness not only simple equations involving two unknown quantities, but also a certain class of equations involving three unknown quantities. The following examples are intended for illustration.*

Example 1. Solve $\left.\begin{array}{l}3x-5y+9=0\\5x-3y-1=0\end{array}\right\}$

Here $a_1=3,\ b_1=-5,\ c_1=\ \ 9;$

$\qquad a_2=5,\ b_2=-3,\ c_2=-1.$

Hence, we must have

$$\frac{x}{(-5)(-1)-(-3).9}=\frac{y}{9\times5-(-1).3}=\frac{1}{3.(-3)-5.(-5)},$$

$$\text{or, }\frac{x}{5+27}=\frac{y}{45+3}=\frac{1}{-9+25},\quad\text{or, }\frac{x}{32}=\frac{y}{48}=\frac{1}{16};$$

$\therefore\quad x=\frac{32}{16}=2$, and $y=\frac{48}{16}=3$.

Thus, we have $x=2$, and $y=3$.

Example 2. Solve $\left.\begin{array}{rlr}-7x+8y=&9&\cdots(1)\\5x-4y=&-3&\cdots(2)\end{array}\right\}.$

From (1), $-7x+8y-9=0$

From (2), $\ \ 5x-4y+3=0$

Hence $$\frac{x}{8\times3-(-4)(-9)}=\frac{y}{(-9).5-3.(-7)}=\frac{1}{(-7)(-4)-5\times8},$$

or, $$\frac{x}{24-36}=\frac{y}{-45+21}=\frac{1}{28-40},$$

or, $$\frac{x}{-12}=\frac{y}{-24}=\frac{1}{-12};$$

$$\therefore \quad x=\frac{-12}{-12}=1, \text{ and } y=\frac{-24}{-12}=2.$$

Thus, we have $x=1$, and $y=2$.

Example 3. Solve

$$\left.\begin{aligned}(x+7)(y-3)+7&=(y+3)(x-1)+5 \quad \cdots \ (1)\\ 5x-11y+35&=0 \quad \cdots \quad \cdots \ (2)\end{aligned}\right\}$$

[C. U. 1888]

From (1), $xy+7y-3x-14=xy+3x-y+2$.

$\therefore \quad 6x-8y+16=0$;

$$\left.\begin{aligned}\therefore \quad 3x-4y+8&=0\\ \text{also } 5x-11y+35&=0\end{aligned}\right\}$$

Hence, $$\frac{x}{(-4).35-(-11).8}=\frac{y}{8\times5-35\times3}=\frac{1}{3.(-11)-5.(-4)},$$

or, $$\frac{x}{-140+88}=\frac{y}{40-105}=\frac{1}{-33+20},$$

or, $$\frac{x}{-52}=\frac{y}{-65}=\frac{1}{-13}.$$

Hence, $x=4$, and $y=5$.

Example 4. Solve
$$\left.\begin{aligned}2x-3y+4z&=0 \cdots (1)\\ 7x+2y-6z&=0 \cdots (2)\\ 4x+3y+z&=37 \cdots (3)\end{aligned}\right\}$$

From (1) and (2), we have,

$$\frac{x}{(-3)(-6)-2\times4}=\frac{y}{4\times7-(-6).2}=\frac{z}{2\times2-7.(-3)},$$

or, $\frac{x}{10}=\frac{y}{40}=\frac{z}{25}$, or, $\frac{x}{2}=\frac{y}{8}=\frac{z}{5}$.

Now, let k denote the common value of these fractions which is at present unknown.

Then, we have $\frac{x}{2}=\frac{y}{8}=\frac{z}{5}=k$.

$\therefore \quad x=2k, y=8k, z=5k. \quad \cdots \quad \cdots \quad \cdots$ (A)

Substituting these values of x, y, z in (3), we have

$k(8+24+5)=37$,

or, $\quad 37k=37$; $\quad \therefore \quad k=1$.

Hence, from (A), $x=2$, $y=8$, and $z=5$.

Example 5. Solve

$$\left.\begin{aligned} x+6y&=5z \quad \cdots \quad (1)\\ 7x+z&=6y \quad \cdots \quad (2)\\ 5x+6y-4z&=24 \quad \cdots \quad (3)\end{aligned}\right\}$$

$$\left.\begin{aligned} &\text{From (1),} \quad x+6y-5z=0\\ &\text{From (2),} \quad 7x-6y+z=0\end{aligned}\right\}$$

Hence, $\dfrac{x}{6\times1-(-6).(-5)}=\dfrac{y}{(-5).7-1\times1}=\dfrac{z}{1.(-6)-7\times6}$,

or, $\dfrac{x}{6-30}=\dfrac{y}{-35-1}=\dfrac{z}{-6-42}$,

or, $\dfrac{x}{-24}=\dfrac{y}{-36}=\dfrac{z}{-48}$;

$\therefore \quad \dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{4}$. [Multiplying each fraction by -12]

Supposing each of these fractions $=k$, we have

$x=2k,\ y=3k,\ z=4k. \quad \cdots \quad \therefore \ (A)$

Substituting these values of x, y, z in (3), we have

$k(10+18-16)=24$,

or, $12k=24$; $\therefore \ k=2$.

Hence, from (A), $x=4$, $y=6$, and $z=8$.

EXERCISE 96

Solve the following equations :

1. $\left.\begin{aligned} 2x+3y-8&=0\\ 3x-4y+5&=0\end{aligned}\right\}$

2. $\left.\begin{aligned} 3x-5y+9&=0\\ 5x+2y-16&=0\end{aligned}\right\}$

3. $\left.\begin{aligned} 4x-5y+8&=0\\ 2x-3y+6&=0\end{aligned}\right\}$

4. $\left.\begin{aligned} -3x+2y+2&=0\\ 5x-3y-5&=0\end{aligned}\right\}$

5. $\left.\begin{aligned} 6x-7y+12&=0\\ -7x+4y+11&=0\end{aligned}\right\}$

6. $\left.\begin{aligned} 7x-8y&=-14\\ 5x-3y&=9\end{aligned}\right\}$

7. $\left.\begin{aligned} -6x+5y+2&=0\\ 13x-9y&=19\end{aligned}\right\}$

8. $\left.\begin{aligned} -7x+5y+11&=0\\ 8x-5y&=19\end{aligned}\right\}$

9. $\left.\begin{aligned} 4x-11y+6&=0\\ 9x-13y&=10\end{aligned}\right\}$

10. $\left.\begin{aligned} 8x-7y&=19\\ 10x-9y&=23\end{aligned}\right\}$

11. $\left.\begin{aligned} -12x+17y+16&=0\\ 9x-13y&=11\end{aligned}\right\}$

12. $\left.\begin{aligned} 14x-11y+18&=0\\ 11x-7y+1&=0\end{aligned}\right\}$

13. $\left.\begin{aligned} 17x-7y&=52\\ 3x&=2y\end{aligned}\right\}$

    [From the 2nd equation, $\frac{x}{2}=\frac{y}{3}=k$ (suppose)]

14. $\left.\begin{aligned} 9x+5y&=124\\ 7x&=3y\end{aligned}\right\}$

15. $$\left.\begin{aligned} 15x+7y &= 246 \\ 9x &= 4y \end{aligned}\right\}$$

16. $$\left.\begin{aligned} 9x &= 8y \\ 10x+23y-287 &= 0 \end{aligned}\right\}$$

17. $$\left.\begin{aligned} 4x-3y &= 0 \\ 7x-11y+92 &= 0 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} 4x-7y &= 0 \\ 10x-9y-102 &= 0 \end{aligned}\right\}$$

19. $$\left.\begin{aligned} 13x-12y+15 &= 0 \\ 8x-7y &= 0 \end{aligned}\right\}$$

20. $$\left.\begin{aligned} 11x-10y+82 &= 0 \\ 14x-9y &= 0 \end{aligned}\right\}$$

21. $$\left.\begin{aligned} \tfrac{1}{3}(x+y)+\tfrac{1}{4}(x-y) &= 59 \\ 5x-33y &= 0 \end{aligned}\right\}$$

22. $$\left.\begin{aligned} \frac{4x+5y}{40} &= x-y \\ \frac{2x-y}{3}+2y &= 20 \end{aligned}\right\}$$

23. $$\left.\begin{aligned} y(3+x) &= x(7+y) \\ 4x+9 &= 5y-14 \end{aligned}\right\}$$

24. $$\left.\begin{aligned} \frac{4y-6}{x+y} &= 2 \\ \frac{8x-5}{y-x} &= 9 \end{aligned}\right\}$$

25. $$\left.\begin{aligned} (x+5)(y+7) &= (x+1)(y-9)+112 \\ 2x+10 &= 3y+1 \end{aligned}\right\}$$

26. $$\left.\begin{aligned} 4x-5y+2z &= 0 \\ 2x-7y+4z &= 0 \\ x+y+z &= 6 \end{aligned}\right\}$$

27. $$\left.\begin{aligned} 5x+6y+8z &= 0 \\ 3x+4y+6z &= 0 \\ x+5y+16z &= 3 \end{aligned}\right\}$$

28. $$\left.\begin{aligned} 2x-7y+11z &= 0 \\ 6x-8y+7z &= 0 \\ 3x+4y+5z &= 35 \end{aligned}\right\}$$

29. $$\left.\begin{aligned} 7x+3y-8z &= 0 \\ 5x-7y+8z &= 0 \\ 3x+5y+7z &= 64 \end{aligned}\right\}$$

30. $$\left.\begin{aligned} x-2y+z &= 0 \\ 9x-8y+3z &= 0 \\ 2x+3y+5z &= 36 \end{aligned}\right\}$$

[C. U. 1887]

31. $$\left.\begin{aligned} 2(4x+9y) &= 7(2y+z) \\ 7(x+2y) &= 8(y+z) \\ 3x+4y+5z &= 38 \end{aligned}\right\}$$

32. $$\left.\begin{aligned} 4(x+y) &= 3(2z-y) \\ 5(x-2y) &= 3(2y-3z) \\ 6(x-2)+7(y-3)+8(z-4) &= 67 \end{aligned}\right\}$$

33. $$\left.\begin{aligned} 5x=2y,\ 7y &= 5z \\ 4x+5y+6z &= 150 \end{aligned}\right\}$$

34. $$\left.\begin{aligned} 15x=10y &= 6z \\ 7x+8y+9z &= 332 \end{aligned}\right\}$$

35. $$\left.\begin{aligned} 4x-13y+8z &= 0 \\ 7x+6y-9z &= 0 \\ \frac{5}{x}+\frac{8}{y}+\frac{15}{z} &= 6\tfrac{2}{3} \end{aligned}\right\}$$

182. Equations of the form $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$, $a_3x+b_3y+c_3z=d_3$.

Multiply the first equation by c_2 and the 2nd by c_1; then by subtraction, we have

$$(a_1c_2-a_2c_1)x+(b_1c_2-b_2c_1)y=d_1c_2-d_2c_1. \quad \cdots \quad (1)$$

Similarly, multiplying the first equation by c_3 and the 3rd by c_1, we have

$$(a_1c_3-a_3c_1)x+(b_1c_3-b_3c_1)y=d_1c_3-d_3c_1. \quad \cdots \quad (2)$$

Now, from (1) and (2), the values of x and y can be at once found by cross multiplication. Then substituting the values of x and y thus found in any of the given equations, the value of z will be obtained.

Otherwise: Multiply the 1st equation by d_2 and the 2nd by d_1, then by subtraction, we have

$$(a_1d_2-a_2d_1)x+(b_1d_2-b_2d_1)y+(c_1d_2-c_2d_1)z=0. \quad \cdots \quad (\alpha)$$

Similarly, multiplying the 1st equation by d_3 and the 3rd by d_1, we have

$$(a_1d_3-a_3d_1)x+(b_1d_3-b_3d_1)y+(c_1d_3-c_3d_1)z=0. \quad \cdots \quad (\beta)$$

Now, evidently (α) and (β) together with any one of the given equations form a group which can be easily solved by the method illustrated in the last article.

Example 1. Solve
$$\left.\begin{aligned} 4x-3y+2z&=40 \quad \cdots \quad (1)\\ 5x+9y-7z&=47 \quad \cdots \quad (2)\\ 9x+8y-3z&=97 \quad \cdots \quad (3)\end{aligned}\right\}$$

Multiplying (1) by 7, and (2) by 2, we have

$$\left.\begin{aligned} 28x-21y+14z&=280\\ \text{and}\quad 10x+18y-14z&=94\end{aligned}\right\}$$

Hence, by addition, $38x-3y=374.\quad \cdots \quad (4)$

Again, multiplying (1) by 3, and (3) by 2, we have

$$\left.\begin{aligned} 12x-9y+6z&=120\\ \text{and}\quad 18x+16y-6z&=194\end{aligned}\right\}$$

Hence, by addition, $30x+7y=314.\quad \cdots \quad (5)$

Now, from (4) and (5), we have

$$\left.\begin{aligned} 38x-3y-374&=0\\ \text{and}\quad 30x+7y-314&=0\end{aligned}\right\}$$

Hence,

$$\frac{x}{(-3)(-314)-7.(-374)}=\frac{y}{(-374).30-(-314).38}=\frac{1}{38\times 7-30.(-3)},$$

or,
$$\frac{x}{942+2618}=\frac{y}{-11220+11932}=\frac{1}{266+90},$$

or,
$$\frac{x}{3560}=\frac{y}{712}=\frac{1}{356}.$$

Therefore, $x=10$, and $y=2$.

Substituting these values of x and y in (1), we have

$$40-6+2z=40, \qquad \text{whence } z=3.$$

Thus, we have $x=10$, $y=2$, and $z=3$.

Example 2. Solve
$$\left.\begin{aligned} 2x-\ \ 4y+\ \ 9z&=28 \quad \cdots \quad (1)\\ 7x+\ \ 3y-\ \ 5z&=\ \ 3 \quad \cdots \quad (2)\\ 9x+10y-11z&=\ \ 4 \quad \cdots \quad (3)\end{aligned}\right\}$$

Multiplying (1) by 3, and (2) by 4, we have

$$\left.\begin{aligned} &6x-12y+27z=84\\ \text{and}\quad &28x+12y-20z=12\end{aligned}\right\}$$

Hence, by addition, $34x+7z=96 \quad \cdots \quad (4)$

Again, multiplying (2) by 10, and (3) by 3, we have

$$\left.\begin{aligned} &70x+30y-50z=30\\ \text{and}\quad &27x+30y-33z=12\end{aligned}\right\}$$

Hence, by subtraction, $43x-17z=18. \quad \cdots \quad (5)$

Now, from (4) and (5), we have

$$\left.\begin{aligned} &34x+\ \ 7z-96=0\\ \text{and}\quad &43x-17z-18=0\end{aligned}\right\}$$

Hence,

$$\frac{x}{7.(-18)-(-17).(-96)}=\frac{z}{(-96).43-(-18).34}=\frac{1}{34.(-17)-43\times 7},$$

$$\text{or,}\qquad \frac{x}{-126-1632}=\frac{z}{-4128+612}=\frac{1}{-578-301},$$

$$\text{or,}\qquad \frac{x}{-1758}=\frac{z}{-3516}=\frac{1}{-879}.$$

Therefore, $x=\dfrac{-1758}{-879}=2$ and $z=\dfrac{-3516}{-879}=4$.

Substituting these values of x and z in (2), we have

$$14+3y-20=3,$$

$$\text{whence}\qquad 3y=9, \quad \text{and} \quad \therefore\ y=3.$$

Thus, we have $x=2$, $y=3$, and $z=4$.

Example 3. Solve
$$\left.\begin{aligned} 12x+\ \ 9y-\ \ 7z&=2 \quad \cdots \quad (1)\\ 8x-26y+\ \ 9z&=1 \quad \cdots \quad (2)\\ 23x+21y-15z&=4 \quad \cdots \quad (3)\end{aligned}\right\}$$

Multiplying (2) by 2, we have

$$16x-52y+18z=2,$$

$$\text{also,}\qquad 12x+\ \ 9y-\ \ 7z=2. \quad \cdots \quad (1)$$

Hence, by subtraction, $4x-61y+25z=0. \quad (4)$

Again, multiplying (1) by 2, we have

$$24x+18y-14z=4,$$

also, $23x+21y-15z=4.$ ... (3)

Hence, by subtraction, $x-3y+z=0.$ ... (5)

$$\left.\begin{array}{lr} \text{Now, since we have } 4x-61y+25z=0, & \cdots\ (4) \\ \text{and } x-3y+z=0. & \cdots\ (5) \end{array}\right\}$$

Therefore, by cross multiplication,

$$\frac{x}{-61+75}=\frac{y}{25-4}=\frac{z}{-12+61},$$

$$\text{or, } \frac{x}{14}=\frac{y}{21}=\frac{z}{49}, \quad \text{or, } \frac{x}{2}=\frac{y}{3}=\frac{z}{7}.$$

Supposing each of these fractions $=k$, we have

$$x=2k,\ y=3k,\ z=7k.$$

Hence, from (1), $k(24+27-49)=2,$

or, $2k=2$; $\therefore$ $k=1$.

Therefore, $x=2$, $y=3$, and $z=7$.

EXERCISE 97

Solve the following equations :

1. $\left.\begin{array}{r} 2x-3y+5z=11 \\ 5x+2y-7z=-12 \\ -4x+3y+z=5 \end{array}\right\}$

2. $\left.\begin{array}{r} 3x+2y+5z=32 \\ 2x+5y+3z=31 \\ 5x+3y+2z=27 \end{array}\right\}$

3. $\left.\begin{array}{r} x+y-z=1 \\ 8x+3y-6z=1 \\ 3z-4x-y=1 \end{array}\right\}$

4. $\left.\begin{array}{r} 2x+3y+4z=29 \\ 3x+2y+5z=32 \\ 4x+3y+2z=25 \end{array}\right\}$

5. $\left.\begin{array}{r} 2x+3y+4z=16 \\ 3x+2y-5z=8 \\ 5x-6y+3z=6 \end{array}\right\}$

6. $\left.\begin{array}{r} 4x-3y+2z=8 \\ 3x-4y+5z=6 \\ -6x+5y+7z=-1 \end{array}\right\}$

7. $\left.\begin{array}{r} 8x-7y-5z=1 \\ -7x+5y+6z=-1 \\ 12x-8y-11z=2 \end{array}\right\}$

8. $\left.\begin{array}{r} x+5y-4z=5 \\ 3x-2y+2z=14 \\ -10x+8y+z=6 \end{array}\right\}$

   [C. U. 1867]

9. $\left.\begin{array}{r} 2x+4y+5z=49 \\ 3x+5y+6z=64 \\ 4x+3y+4z=55 \end{array}\right\}$

10. $\left.\begin{array}{r} x+3y+5z=10 \\ 3x+5y+7z=14 \\ 5x+7y+8z=15 \end{array}\right\}$

11. $\left.\begin{array}{r} 12x+8y-11z=-3 \\ 11x-13y-z=2 \\ 8x+17y-12z=-2 \end{array}\right\}$

12. $\left.\begin{array}{r} 5x-4y+9z=19 \\ 7x+6y-12z=16 \\ -9x+8y+15z=-13 \end{array}\right\}$

13. $$\left.\begin{aligned} x-y-z&=-15\\ y+x+2z&=40\\ 4z-5x-6y&=-150 \end{aligned}\right\}$$
[C. U. 1886]

14. $$\left.\begin{aligned} 2(x-y)&=3z-2\\ x-3z&=3y-1\\ 2x+3z&=4(1-y) \end{aligned}\right\}$$

15. $$\left.\begin{aligned} 3x+2y-z&=20\\ 2x+3y+6z&=70\\ x-y+6z&=41 \end{aligned}\right\}$$

16. $$\left.\begin{aligned} 4(y-x)&=5z-22\\ 3z+4x&=6y+2\\ z-3y&=14-10x \end{aligned}\right\}$$

17. $$\left.\begin{aligned} 5x+2y+z&=30\\ \tfrac{1}{2}x+\tfrac{4}{5}y-\tfrac{1}{10}z&=4\\ 2x+5y+10z&=129 \end{aligned}\right\}$$

18. $$\left.\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y&=12-\tfrac{1}{6}z\\ \tfrac{1}{2}y+\tfrac{1}{3}z-\tfrac{1}{6}x&=8\\ \tfrac{1}{2}x+\tfrac{1}{3}z&=10 \end{aligned}\right\}$$
[C. U. 1868]

19. $$\left.\begin{aligned} \frac{1}{x}+\frac{5}{y}-\frac{4}{z}&=\frac{1}{12}\\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}&=\frac{19}{24}\\ -\frac{4}{x}+\frac{5}{y}+\frac{6}{z}&=\frac{1}{2} \end{aligned}\right\}$$

20. $$\left.\begin{aligned} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z}&=7\tfrac{3}{5}\\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}&=10\tfrac{1}{6}\\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}&=16\tfrac{1}{10} \end{aligned}\right\}$$

21. $$\left.\begin{aligned} 5x+3y&=65\\ 2y-z&=11\\ 3x+4z&=57 \end{aligned}\right\}$$

22. $$\left.\begin{aligned} \frac{2}{x}+\frac{1}{y}&=\frac{3}{2}\\ \frac{3}{z}-\frac{2}{y}&=2\\ \frac{1}{x}+\frac{1}{z}&=\frac{4}{3} \end{aligned}\right\}$$

23. $$\left.\begin{aligned} ay+bx&=c\\ cx+az&=b\\ bz+cy&=a \end{aligned}\right\}$$

24. $$\left.\begin{aligned} 3x+4y-11&=0\\ 5y-6z&=-8\\ 7z-8x-13&=0 \end{aligned}\right\}$$
[C. U. 1877]

25. $$\left.\begin{aligned} 3y+x-2&=0\\ 3z-4y&=x+15\\ 2x+7z&=7 \end{aligned}\right\}$$
[C. U. 1883]

183. Miscellaneous Examples.

Example 1. Solve $\dfrac{a}{x}+\dfrac{b}{y}=1,\ \dfrac{b}{y}+\dfrac{c}{z}=1,\ \dfrac{c}{z}+\dfrac{a}{x}=1.$

Adding together the given equations, we have

$$2\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)=3, \quad \text{or,} \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{3}{2}. \qquad \cdots (a)$$

Subtracting the 2nd equation from (a), we have

$$\frac{a}{x}=\frac{1}{2}; \qquad \therefore\ x=2a.$$

Similarly, we have $y=2b$ and $z=2c$.

Example 2. Solve

$$\text{(i) } \frac{xy}{x+y}=1; \quad \text{(ii) } \frac{xz}{x+z}=2; \quad \text{(iii) } \frac{yz}{y+z}=3.$$

From (i), we have $\frac{x+y}{xy}=1,$ or, $\frac{1}{y}+\frac{1}{x}=1$ ... (4)

" (ii), " " $\frac{x+z}{xz}=\frac{1}{2},$ or, $\frac{1}{z}+\frac{1}{x}=\frac{1}{2}$ ... (5)

" (iii), " " $\frac{y+z}{yz}=\frac{1}{3},$ or, $\frac{1}{z}+\frac{1}{y}=\frac{1}{3}.$ ... (6)

From (4), (5) and (6), by addition, we have

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6};$$

$$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{12}. \quad \cdots \quad \cdots \quad (7)$$

Subtracting (6) from (7), we have

$$\frac{1}{x}=\frac{11}{12}-\frac{1}{3}=\frac{7}{12}; \qquad \therefore \quad x=\tfrac{12}{7}.$$

Subtracting (5) from (7), we have

$$\frac{1}{y}=\frac{11}{12}-\frac{1}{2}=\frac{5}{12}; \qquad \therefore \quad y=\tfrac{12}{5}.$$

Subtracting (4) from (7), we have

$$\frac{1}{z}=\frac{11}{12}-1=-\frac{1}{12}; \qquad \therefore \quad z=-12.$$

Example 3. Solve $xyz=a(yz-zx-xy)$
$=b(zx-xy-yz)=c(xy-yz-zx).$

Since, $xyz=a(yz-zx-xy)$, we have

$$\frac{1}{a}=\frac{1}{x}-\frac{1}{y}-\frac{1}{z} \quad \cdots \quad (1) \text{ [Dividing both sides by } a\times xyz \text{]}$$

Similarly, we have $\frac{1}{b}=\frac{1}{y}-\frac{1}{z}-\frac{1}{x},$ ... ... (2)

and $\frac{1}{c}=\frac{1}{z}-\frac{1}{x}-\frac{1}{y};$ ... ... (3)

Adding together (2) and (3), we have

$$-\frac{2}{x}=\frac{1}{b}+\frac{1}{c}=\frac{b+c}{bc}; \qquad \therefore \quad x=\frac{-2bc}{b+c}.$$

Similarly,

$$-\frac{2}{y}=\frac{1}{c}+\frac{1}{a}=\frac{a+c}{ac}; \qquad \therefore \quad y=\frac{-2ca}{c+a}.$$

and

$$-\frac{2}{z}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}; \qquad \therefore \quad z=\frac{-2ab}{a+b}.$$

Example 4. Solve
$$\left.\begin{aligned} x+y+z&=0\\ (b+c)x+(c+a)y+(a+b)z&=0\\ bcx+cay+abz&=1 \end{aligned}\right\}$$

Since
$$\left.\begin{aligned} (b+c)x+(c+a)y+(a+b)z&=0\\ \text{and}\qquad x+y+z&=0 \end{aligned}\right\},$$
therefore, by cross-multiplication,
$$\frac{x}{(c+a)-(a+b)}=\frac{y}{(a+b)-(b+c)}=\frac{z}{(b+c)-(c+a)},$$
or,
$$\frac{x}{c-b}=\frac{y}{a-c}=\frac{z}{b-a}.$$

Supposing each of these fractions $=k$, we have
$$x=k(c-b),\ y=k(a-c),\ z=k(b-a).$$

Substituting these values of x, y, z in the third equation, we have
$$k\{bc(c-b)+ca(a-c)+ab(b-a)\}=1.$$

But
$$\begin{aligned} &bc(c-b)+ca(a-c)+ab(b-a)\\ &=bc(c-b)+a^2(c-b)-a(c^2-b^2)\\ &=(c-b)\{bc+a^2-a(c+b)\}\\ &=(c-b)(a-c)(a-b). \end{aligned}$$

Thus, $k(c-b)(a-c)(a-b)=1$; $\therefore\ k=\dfrac{1}{(c-b)(a-c)(a-b)}$.

Hence,
$$\begin{aligned} x&=k(c-b)=\frac{1}{(a-c)(a-b)};\\ y&=k(a-c)=\frac{1}{(c-b)(a-b)};\\ z&=k(b-a)=\frac{1}{(c-b)(c-a)}. \end{aligned}$$

EXERCISE 98

Solve the following equations :

1. $\dfrac{x}{a}+\dfrac{y}{b}=1,\quad \dfrac{x}{a}+\dfrac{z}{c}=1,\quad \dfrac{y}{b}+\dfrac{z}{c}=1.$

2. $\dfrac{1}{x}+\dfrac{1}{y}=a,\quad \dfrac{1}{x}+\dfrac{1}{z}=b,\quad \dfrac{1}{y}+\dfrac{1}{z}=c.$

3. $\dfrac{yz}{y+z}=a,\quad \dfrac{zx}{z+x}=b,\quad \dfrac{xy}{x+y}=c.$

4. $\left.\begin{aligned} axy&=c(bx+ay)\\ bxy&=c(ax-by) \end{aligned}\right\}$

5. $3xy=4(x+y),\ 2xz=3(x+z),\ 5yz=12(y+z).$

6. $y+z=4,\quad z+x=6,\quad x+y=8.$

7. $y+z-x=6,\quad z+x-y=10,\quad x+y-z=14,$

8. $\left.\begin{aligned} x-4y+z&=-10\\ y-4z+x&=-15\\ z-4x+y&=-35 \end{aligned}\right\}$

9. $\left.\begin{aligned} y+z-7x+16&=0\\ z+x-7y+24&=0\\ x+y-7z+40&=0 \end{aligned}\right\}$

10. $\left.\begin{aligned} a^2x+b^2y&=2ab(a+b)\\ b(2a+b)x+a(a+2b)y&=a^3+a^2b+ab^2+b^3 \end{aligned}\right\}$

11. $\left.\begin{aligned} x+y+z&=A\\ ax+by+cz&=0\\ a^2x+b^2y+c^2z&=0 \end{aligned}\right\}$

12. $\left.\begin{aligned} x+y+z&=0\\ (a+b)x+(a+c)y+(b+c)z&=0\\ abx+acy+bcz&=1 \end{aligned}\right\}$

13. $\left.\begin{aligned} x+y+z&=0\\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}&=0\\ \frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}&=1 \end{aligned}\right\}$

14. $\left.\begin{aligned} x-ay+a^2z&=a^3\\ x-by+b^2z&=b^3\\ x-cy+c^2z&=c^3 \end{aligned}\right\}$

15. $\left.\begin{aligned} ax+by+cz&=0\\ (b+c)x+(c+a)y+(a+b)z&=0\\ a^2x+b^2y+c^2z&=a^2(b-c)+b^2(c-a)+c^2(a-b) \end{aligned}\right\}$

16. Find the condition that the three equations,

$$a_1x+b_1y+c_1=0,\quad a_2x+b_2y+c_2=0,\quad a_3x+b_3y+c_3=0,$$

may be consistent.

17. Find the value of a so that the four equations,

$$2x-3y+5z=18,\quad 3x-y+4z=20,\quad 4x+2y-z=5,$$

$(a+1)x+(a+2)y+(a+3)z=76$, may be consistent.

18. $\left.\begin{aligned} 3w-2y&=2\\ 5x-7z&=11\\ 2x+3y&=39\\ 4y+3z&=41 \end{aligned}\right\}$

19. $\left.\begin{aligned} 9x-2z+w&=41\\ 7y-5z-t&=12\\ 4y-3x+2w&=5\\ 3y-4w+3t&=7\\ 7z-5w&=11 \end{aligned}\right\}$

20. $\left.\begin{aligned} x+y+z&=ab+bc+ca\\ \frac{x}{ab}+\frac{y}{bc}+\frac{z}{ca}&=3\\ (c-b)x+(a-b)y+(c-a)z&=2abc-ab^2-b^2c+ac^2-a^2c \end{aligned}\right\}$

II. Problems producing Simple Equations with more than One Unknown Quantity

184. In this section we shall consider a few problems of a harder than those treated of in Chapter XVIII.

The following examples will serve as illustrations.

Example 1. A cask P contains 12 litres of wine and 18 litres of water, and another cask Q contains 9 litres of wine and 3 litres of water. How many litres must be drawn from each cask so as to produce by their mixture 7 litres of wine and 7 litres of water ?

Out of 30 litres of the mixture of wine and water in P, there are 12 litres of wine ; hence, $\frac{12}{30}$ or $\frac{2}{5}$ths of the mixture consists of wine and $\therefore$ $\frac{3}{5}$ths of water.

Hence, for every litre drawn from P, there are taken out $\frac{2}{5}$ths of a litre of wine and $\frac{3}{5}$ths of a litre of water.

Similarly, for every litre drawn from Q, there are taken out $\frac{3}{4}$ths of a litre of wine and $\frac{1}{4}$th of a litre of water.

Let x = the number of litres to be drawn from P,

and y = " " " " " " " " Q.

Then, since x litres from P contain $\frac{2}{5}x$ litres of wine and $\frac{3}{5}x$ litres of water, and y litres from Q contain $\frac{3}{4}y$ litres of wine and $\frac{1}{4}y$ litres of water, $\therefore$ in the new mixture there are $(\frac{2}{5}x+\frac{3}{4}y)$ litres of wine and $(\frac{3}{5}x+\frac{1}{4}y)$ litres of water.

Hence, by the conditions of the problem,

$$\left.\begin{aligned} &\tfrac{2}{5}x+\tfrac{3}{4}y=7 \quad \cdots \quad \cdots \quad (1)\\ \text{and}\quad &\tfrac{3}{5}x+\tfrac{1}{4}y=7. \quad \cdots \quad \cdots \quad (2)\end{aligned}\right\}$$

Multiplying (2) by 3, and subtracting (1) from the resulting equation, we have

$$\tfrac{7}{5}x=14 ; \qquad \therefore \quad x=10.$$

Hence, from (2), $\quad y=4(7-\frac{3}{5}\times 10)=4.$

Thus, 10 litres must be drawn from P, and 4 litres from Q.

Example 2. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 metres ; if the circumference of the fore-wheel be increased by one-fourth of its present size, and the circumference of the hind-wheel by one-fifth of its present size, the *six* will be changed to *four*. Required the circumference of each wheel.

Let x metres be the circumference of the fore-wheel,

and " y " " " " " " hind-wheel.

Then the numbers of revolutions made by the wheels in going 120 metres are respectively $\dfrac{120}{x}$ and $\dfrac{120}{y}$.

When the circumference of the fore-wheel is increased by one-fourth, and that of the hind-wheel by one-fifth, the circumferences respectively become

$$\left(x+\frac{x}{4}\right) \text{ and } \left(y+\frac{y}{5}\right) \text{ metres, or, } \frac{5x}{4} \text{ and } \frac{6y}{5} \text{ metres.}$$

Therefore, the numbers of revolutions made by the wheels respectively will be

$$120+\frac{5x}{4} \text{ and } 120+\frac{6y}{5}, \qquad \text{or, } \frac{96}{x} \text{ and } \frac{100}{y}.$$

Hence, from the conditions of the problem,

$$\left.\begin{aligned} &\frac{120}{x}=\frac{120}{y}+6 \quad \cdots \quad \cdots \quad (1)\\ \text{and} \quad &\frac{96}{x}=\frac{100}{y}+4 \quad \cdots \quad \cdots \quad (2) \end{aligned}\right\}$$

Multiplying (1) by 5 and (2) by 6, we have

$$\frac{600}{x}=\frac{600}{y}+30,$$

$$\text{and} \quad \frac{576}{x}=\frac{600}{y}+24\,;$$

$$\therefore \text{ by subtraction, } \frac{24}{x}=6 \qquad \therefore \quad x=4.$$

$$\text{Hence, from (1), } \frac{120}{y}=\frac{120}{4}-6=24\,; \quad \therefore \quad y=5$$

Thus, the circumferences of the wheels are respectively 4 and 5 metres.

Example 3. A kilogram of tea and three kilograms of sugar cost twelve rupees; but if sugar were to rise 50 per cent., and tea 10 per cent., they would cost fourteen rupees. Find the price of tea and sugar.

Let x rupees be the price of a kilogram of tea, and y rupees, the price of a kilogram of sugar; then, we must have

$$x+3y=12. \quad \cdots \quad \cdots \quad (1)$$

When the price of tea *rises* 10 *per cent.*, the price of a kilogram of tea becomes $\left(x+\frac{x}{10}\right)$, or, $\frac{11}{10}x$ rupees; and the price of sugar *rising* 50 *p.c.*, the price of a kilogram of sugar becomes $\left(y+\frac{y}{2}\right)$, or, $\frac{3y}{2}$ rupees

$$\text{Hence,} \quad \frac{11}{10}x+3.\frac{3y}{2}=14. \quad \cdots \quad \cdots \quad (2)$$

$$\text{From (2),} \quad \frac{11}{5}x+9y=28,$$

$$\text{and from (1),} \quad 3x+9y=36\,; \qquad \therefore \quad \left(3-\frac{11}{5}\right)x=8\,;$$

$$\text{or,} \quad \frac{4x}{5}=8\,; \qquad \therefore \quad x=10.$$

$$\text{Hence, from (1),} \quad y=\frac{12-10}{3}=\frac{2}{3}.$$

Thus, the price of a kilogram of tea = Rs. 10, and that of a kilogram of sugar = Rs. $\frac{2}{3}$.

Example 4. A certain sum of money is to be divided among a certain number of men; if there were 3 men fewer, each man would have Rs. 150 more; but if there were 6 men more, each man would have Rs. 120 less. Find the sum of money and the number of men.

Let $x =$ the sum of money in rupees,

and $y =$ the number of men.

Therefore, each man gets Rs. $\frac{x}{y}$; if there were 3 men fewer, each would get Rs. $\frac{x}{y-3}$; and if there were 6 men more, each would get Rs. $\frac{x}{y+6}$.

Hence, from the conditions of the problem,

$$\frac{x}{y-3} = \frac{x}{y} + 150, \qquad \cdots (1)$$

and

$$\frac{x}{y+6} = \frac{x}{y} - 120. \qquad \cdots (2)$$

From (1), $150 = x\left(\frac{1}{y-3} - \frac{1}{y}\right) = \frac{3x}{y^2-3y}$; $\therefore\ x = 50(y^2-3y)$.

From (2), $120 = x\left(\frac{1}{y} - \frac{1}{y+6}\right) = \frac{6x}{y^2+6y}$; $\therefore\ x = 20(y^2+6y)$.

Hence, $50(y^2-3y) = 20(y^2+6y)$,

or, $30y^2 = (150+120)y = 270y$; $\therefore\ y = 9$.

$\therefore\ x = 20(81+54) = 20 \times 135 = 2700$.

Thus, there are 9 men and a sum of Rs. 2700.

Example 5. A man has to travel a certain distance. When he has travelled 40 kilometres, he increases his speed 2 kilometres per hour. If he had travelled with his increased speed during the whole of his journey, he would have arrived 40 minutes earlier; but if he had continued at his original speed, he would have arrived 20 minutes later. How far had he to travel?

Let $x =$ the number of kilometres the man had to travel; and suppose his original speed was y kilometres an hour.

Hence, the time actually taken to complete the journey

$$= \left(\frac{40}{y} + \frac{x-40}{y+2}\right) \text{hours} = \frac{80+xy}{y(y+2)} \text{ hours.}$$

The time he would have taken if he had travelled at the increased speed during the whole of his journey $= \frac{x}{y+2}$ hours.

And the time he would have taken if he had travelled all the way at his original speed $=\frac{x}{y}$ hours.

Hence, from the conditions of the problem,

$$\frac{x}{y+2}=\frac{80+xy}{y(y+2)}-\frac{2}{3}, \qquad \dots \qquad \dots \quad (1)$$

and
$$\frac{x}{y}=\frac{80+xy}{y(y+2)}+\frac{1}{3}. \qquad \dots \qquad \dots \quad (2)$$

Subtracting (1) from (2),

$$x\left(\frac{1}{y}-\frac{1}{y+2}\right)=1, \text{ or, } 2x=y(y+2). \qquad \dots \quad (3)$$

Also, from (2), $3x(y+2)=3(80+xy)+y(y+2)$,

$$\text{or, } \quad 6x-240=y(y+2). \qquad \dots \qquad \dots \quad (4)$$

Hence, from (3) and (4), $6x-240=2x$,

$$\text{or, } \quad 4x=240\ ; \ \therefore \ x=60.$$

Thus, the man had to travel 60 kilometres.

Example 6. If there were no accidents, it would take half as long to travel the distance from A to B by rail-road as by coach ; but three hours being allowed for accidental stoppages by the former, the coach will travel the distance all but fifteen kilometres in the same time ; if the distance were two-thirds as great as it is, and the same time allowed for railway stoppages, the coach would take exactly the same time. Required the distance.

Let x kilometres be the distance from A to B.

Suppose the coach travels at the rate of y kilometres an hour, then evidently, the rate of the train is $2y$ kilometres an hour.

The time in which the train can travel the distance *plus* 3 hours = the time in which the coach travels only $(x-15)$ kilometres.

Hence,
$$\frac{x}{2y}+3=\frac{x-15}{y}\ ; \qquad \dots \qquad \dots \quad (1)$$

and
$$\frac{\frac{2}{3}x}{2y}+3=\frac{\frac{2}{3}x}{y}, \quad \text{or, } \quad \frac{x}{3y}+3=\frac{2x}{3y}. \qquad \dots \quad (2)$$

From (2), $\frac{x}{3y}=3$, or, $x=9y$. $\qquad \dots \quad (3)$

From (1), $x+6y=2x-30$, or, $6y=x-30$. $\qquad \dots \quad (4)$

Hence, from (3) and (4), $6y=9y-30$, whence $y=10$;

and $\therefore \quad x=9\times10=90$.

Then, the required distance = 90 kilometres.

Example 7. A boat goes upstream 30 kilometres and downstream 44 kilometres in 10 hours; it also goes upstream 40 kilometres and downstream 55 kilometres in 13 hours; find the rate of the stream and of the boat.

Suppose the boat will travel x kilometres per hour if there were no current, and that the current flows at the rate of y kilometres per hour.

Then, it is clear that *with the current*, the boat travels $(x+y)$ kilometres per hour, and *against the current*, $(x-y)$ kilometres per hour.

Hence, the time taken to travel 30 kilometres upstream $=\frac{30}{x-y}$ hours, and the time taken to travel 44 kilometres downstream $=\frac{44}{x+y}$ hours; and ∴ by the 1st condition of the problem, we must have

$$\frac{30}{x-y}+\frac{44}{x+y}=10. \qquad \cdots \quad (1)$$

Similarly, by the 2nd condition, we have

$$\frac{40}{x-y}+\frac{55}{x+y}=13. \qquad \cdots \quad (2)$$

Multiplying (1) by 4, and (2) by 3, we have

$$\frac{120}{x-y}+\frac{176}{x+y}=40,$$

and

$$\frac{120}{x-y}+\frac{165}{x+y}=39.$$

Therefore, by subtraction,

$$\frac{11}{x+y}=1; \qquad \therefore \quad x+y=11.$$

Hence, from (1), $\frac{30}{x-y}=10-4=6;$ $\qquad \therefore \quad x-y=5.$

Thus, we have

$$\left.\begin{aligned} x+y&=11 \\ \text{and} \quad x-y&=5 \end{aligned}\right\}$$

Hence, by addition, $2x=16$; ∴ $x=8$

and by subtraction, $2y=6$; ∴ $y=3$

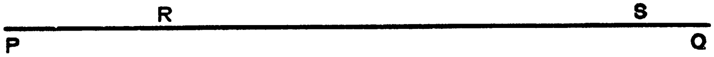
Thus, the rates of the stream and the boat are respective 3 kilometres and 8 kilometres per hour.

Example 8. *A* challenged *B* to ride a bicycle race of 1040 metre He first gave *B*, 120 metres' start, but lost by 5 seconds; he then ga *B*, 5 seconds' start, and won by 40 metres. How long does each ta to ride the distance?

Let the times which *A* and *B* take to ride the distance x seconds and y seconds respectively.

Then, the times they take to travel one metre are respectively $\frac{x}{1040}$ and $\frac{y}{1040}$ seconds.

Let PQ represent the given distance, and let PR, SQ on it respectively represent 120 metres and 40 metres.



In the first race B is at R, and A at P when they start, but B reaches Q, 5 seconds earlier than A; therefore, the time taken by B to travel $RQ = (x-5)$ seconds.

Hence, $$x-5=(1040-120)\times\frac{y}{1040}$$
$$=(1-\tfrac{3}{26})y=\tfrac{23}{26}y. \qquad \cdots \qquad (1)$$

In the second race B starts from P, 5 seconds earlier than A, but arrives at S when A arrives at Q; therefore, the time taken by B to travel $PS=(x+5)$ seconds.

Hence, $$x+5=(1040-40)\times\frac{y}{1040}$$
$$=(1-\tfrac{1}{26})y=\tfrac{25}{26}y. \qquad \cdots \qquad (2)$$

Subtracting (1) from (2), we have
$$\tfrac{2}{26}y=10; \qquad \therefore \quad y=130.$$

Hence, from (1), $$x=5+\tfrac{23}{26}\times 130$$
$$=5+115=120.$$

Thus, the times required by A and B to ride the distance are respectively 2 minutes, and 2 minutes 10 seconds.

Example 9. An author started writing between 2 P.M. and 3 P.M., stopped writing between 6 P.M. and 7 P.M., and found that the hands of the clock interchanged their places. When did he start writing? When did he stop writing? How long did he write?

Suppose the author started writing at x minutes past 2 P.M. and stopped writing at y minutes past 6 P.M.

At x minutes past 2 P.M. the minute-hand and the hour-hand were x minute-divisions and $\left(10+\frac{x}{12}\right)$ minute-divisions ahead of the mark XII respectively.

At y minutes past 6 P.M. the minute-hand and the hour-hand were y minute-divisions and $\left(30+\frac{y}{12}\right)$ minute-divisions ahead of the mark XII respectively.

From the conditions of the problem,

$$x = 30 + \frac{y}{12} \quad \text{and} \quad y = 10 + \frac{x}{12}.$$

$$x = 30 + \frac{y}{12}$$

or, $12x = 360 + y$,

or, $12x - y = 360$ ... ... (i)

$$y = 10 + \frac{x}{12},$$

or, $12y = 120 + x$,

or, $12y - x = 120$ ... ... (ii)

Solving the equations (i) and (ii), we get

$$x = 31\tfrac{7}{143} \text{ and } y = 12\tfrac{84}{143}.$$

Therefore, the author started writing at $31\frac{7}{143}$ minutes past 2 P.M., stopped writing at $12\frac{84}{143}$ minutes past 6 P.M. and wrote for 3 hours $41\frac{7}{13}$ minutes.

Example 10. If the sum of the digits of a number is divisible by 9, so is the number. [B. C. S. 1923]

If the number consists of one digit it must evidently be 9. Thus, the problem is true for a number of one digit.

If the number consists of two digits, let x and y be the digits in the units' place and tens' place respectively.

$\therefore$ the number $= 10y + x$.

Now, $$\frac{10y + x}{9} = y + \frac{y + x}{9}.$$

Hence, the number is divisible by 9 if $x + y$ is divisible by 9, *i.e.*, if the sum of the digits is divisible by 9.

Proceeding similarly, the proof follows for a number with more digits.

EXERCISE 99

1. There is a certain number consisting of 3 digits which is equal to 25 times the sum of the digits, and if 198 be added to the number, the digits will be reversed ; also the sum of the extreme digits exceeds the middle digit by unity ; find the number.

2. A shopkeeper, on account of bad book-keeping knows neither the weight nor the prime cost of a certain article which he purchased. He only recollects that if he had sold the whole at Rs. 30 per Kg., he would have gained Rs. 100 by it, and if he had sold it at Rs. 22 per Kg., he would have lost Rs. 300 by it. What was the weight and prime cost of the article ?

3. Two persons A and B, played cards. After a certain number of games, A had won half as much as he had at first and found that if he had 15 rupees more, he would have had just three times as much as B. But B afterwards won 10 rupees back, and he had then twice as much as A. What had each at first ?

4. A and B can do a piece of work together in 12 days, which B working for 15 days and C for 30 would together complete ; in 10 days they would finish it, working all three together ; in what time could they separately do it ?

5. A has twice as many pennies as shillings ; B, who has 8*d*. more than A, has twice as many shillings as pennies ; together they have one more penny than they have shillings. How much has each ?

6. Two persons, A and B could finish a work in m days ; they worked together n days when A was called off, and B finished it in p days. In what time could each do it ?

7. A, B, C compare their fortunes ; A says to B, 'give me Rs. 700 of your money, and I shall have twice as much as you retain' ; B says to C, 'give me Rs. 1400, and I shall have thrice as much as you have remaining' ; C says to A, 'give me Rs. 420, and then I shall have five times as much as you retain'. How much has each ?

8. A man walks 35 kilometres partly at the rate of 4 kilometres an hour and partly at 5 ; if he had walked at 5 kilometres an hour when he walked at 4, and *vice versa*, he would have covered two kilometres more in the same time. Find the time he was walking.

9. A train travelled a certain distance at a uniform rate. Had the speed been 6 kilometres an hour more, the journey would have occupied 4 hours less ; and had the speed been 6 kilometres an hour less, the journey would have occupied 6 hours more. Find the distance.

10. Two vessels contain mixtures of milk and water ; in one there is three times as much milk as water, in the other five times as much water as milk. Find how much must be drawn off from each to fill a third vessel which holds seven litres, in order that its contents may be half milk and half water.

11. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two ; and the number will be increased by 99 if its digits be reversed. Find the number.

12. A man has one pound's worth of silver in half crowns, shillings and six-pences; and he has in all 20 coins. If he changed the six-pences for pennies, and the shillings for six-pences, he would have 73 coins. How many coins of each kind has he?

13. A sum of money is divided equally among a certain number of persons; if there had been four more, each would have received one rupee less than he did; if there had been five fewer, each would have received two rupees more than he did; find the number of persons and what each received.

14. There is a cistern, into which water is admitted by three cocks, two of which are exactly of the same dimensions. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks is stopped, seven-ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

15. A person exchanged 12 bushels of wheat for 8 bushels of barley, and Rs. 56; offering at the same time to sell a certain quantity of wheat for an equal quantity of barley, and Rs. 75 in money, or for Rs. 200 in money. Required the prices of the wheat and barley per bushel.

16. A wine-merchant has two sorts of wine, one sort worth 6 rupees a litre and the other worth 10 rupees a litre; from these he wants to make a mixture of 100 litres worth rupees seven a litre. How many litres must he take from each sort?

17. The rent of a farm is paid in certain fixed numbers of quarters of wheat and barley; when wheat is at 55*s.* and barley at 33*s.* per quarter, the portions of rent by wheat and barley are equal to one another; but when wheat is at 65*s.* and barley at 41*s.* per quarter, the rent is increased by £7. What is the corn-rent?

18. A train 60 metres long passed another train 72 metres long which was travelling in the same direction on a parallel line of rails, in 12 seconds. Had the slower train been travelling half as fast again, it would have been passed in 24 seconds. Find the rates at which the trains were travelling.

19. A farmer with 28 bushels of barley at 2*s.* 4*d.* a bushel, would mix rye at 3*s.* per bushel, and wheat 4*s.* per bushel, so that the whole mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* per bushel. How many bushels of rye, and how many of wheat must he mix with the barley?

20. A person has Rs. 62 50 p. in rupees and twenty-five paise, out of which he pays a debt of Rs. 35; and finds that he has exactly as many rupees left as he has paid away twenty-five paise and as many twenty-five paise as he has paid away rupees. How many of each had he at first and how many of each had he left?

21. A waterman finds that he can row with the tide from A to B a distance of 18 kilometres, in an hour and a half, and that to return from B to A against the same tide, though he rows back along the shore where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. Find the rate at which the tide runs in the middle where it is the strongest.

22. A and B run 1760 metres. First A gives B a start of 44 metres, and beats him by 51 seconds; at the second hit A gives B a start of 1 minute 15 seconds, and is beaten by 88 metres. Find the times in which A and B can run the race separately.

23. A and B run a race round a two-kilometre course. In the first hit B reaches the winning post 2 minutes before A. In the second hit A increases his speed by 2 kilometres an hour, and B diminishes his by the same quantity, and A then arrives at the winning post 2 minutes before B. Find at what rate each ran in the first hit.

24. A railway train running from London to Cambridge meets on the way with an accident, which causes it to diminish its speed to $\frac{1}{n}$th of what it was before, and it is in consequence a hours late. If the accident had happened b kilometres nearer Cambridge, the train would have been c hours late. Find the rate of the train before the accident occurred.

25. A railway train after travelling for one hour, meets with an accident, which delays it one hour, after which it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occurred 50 kilometres further on, the train would have arrived 1 hour 20 minutes sooner. Required the length of the journey.

26. A boy started and stopped reading between 3 P.M. and 4 P.M. and 5 P.M. and 6 P.M. respectively and found that the hands of the watch interchanged their places. When did the boy start reading? When did he stop reading? How long did he read?

27. If the difference between the sums of the odd and even digits of a number is zero or divisible by 11, the number is divisible by 11.
[B. C. S. 1923]

28. If the sum of the digits of a number is divisible by 3, so is the number.

CHAPTER XXVIII

GRAPHS AND THEIR APPLICATIONS

185. We have explained in Chapters VII and XIX how algebraic expression can be represented graphically by points and lines.

We shall now give some illustrations of the way in which graphs may be used to solve algebraic equations and problems. Graphical solutions are generally in the nature of approximation, but in many cases they are obtained more easily than the corresponding exact solutions by algebraic processes explained previously.

186. Graphical Solutions of Equations.

Example 1. Solve graphically,

$$\left.\begin{aligned} 2x-7y+12&=0\\ 3x+2y&=32 \end{aligned}\right\}$$

Let us draw the graphs of the two equations.

From the 1st equation, we get

x	-6	1	15	-13
y	0	2	6	-2

From the 2nd equation, we get

x	0	2	4	6	10
y	16	13	10	7	1

Hence, taking three-times the length of a side of a small square a the unit of length, the two graphs are as shown on the next page.

Let P be the point where the two graphs intersect, P bein common to the graphs, its co-ordinates will satisfy both the give equations.

Now, the co-ordinates of P are found to be 8 and 4.

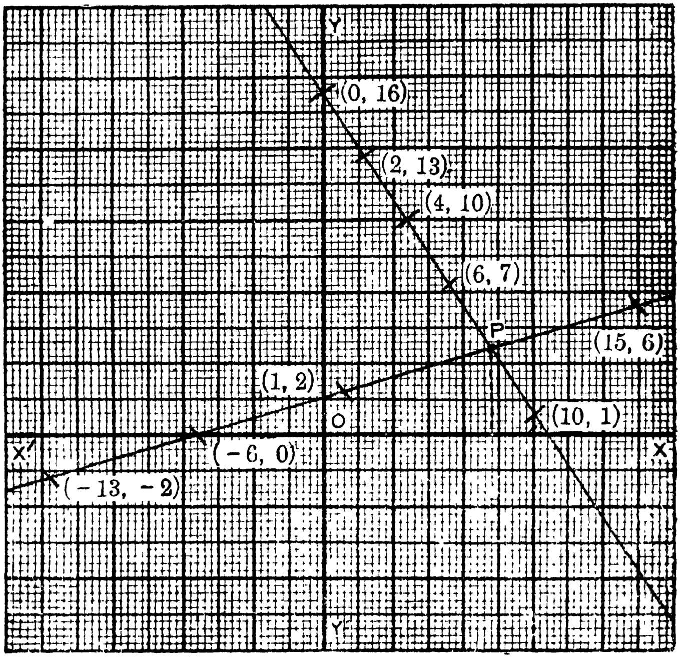
Hence, $\left.\begin{aligned} x&=8\\ y&=4 \end{aligned}\right\}$ is the required solution.

Verification : Substituting $x=8$ and $y=4$ in the given equations, we have

$$2x-7y+12=2\times 8-7\times 4+12=0,$$

and $$3x+2y-32=3\times 8+2\times 4-32=0\ ;$$

$\therefore$ both the equations are satisfied when $x=8$ and $y=4$.



Example 2 Solve graphically $\dfrac{2x+12}{7}=\dfrac{32-3x}{2}$

All that we have to do is to draw the graphs of the expressions $\dfrac{2x+12}{7}$ and $\dfrac{32-3x}{2}$; and take the *abscissa* of the point common to the two graphs.

The graph of the function $\frac{2x+12}{7}$ is the same as the graph of $y=\frac{2x+12}{7}$, *i.e.*, $2x-7y+12=0$; and graph of the function $\frac{32-3x}{2}$ is the same as that of $y=\frac{32-3x}{2}$, *i.e.*, $3x+2y=32$.

Drawing the graphs of $2x-7y+12=0$ and $3x+2y=32$ (see example 1 above), we find that the abscissa of the common point, *P*, of the graphs $=8$;

$\therefore$ $x=8$ is the required solution.

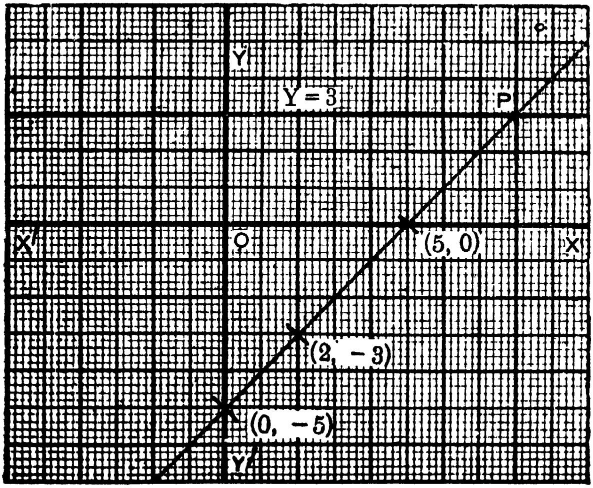
Example 3. Solve graphically $x-5=3$.

Let us draw the graphs of the expressions $x-5$ and 3. The abscissa of the point common to the two graphs is the required solution.

Now, the graph of the expression $x-5$ is the same as the graph of $y=x-5$; and we find that

$$\left.\begin{array}{l}x=\ \ 0\\ y=-5\end{array}\right\},\ \left.\begin{array}{l}x=\ \ 2\\ y=-3\end{array}\right\}\ \text{and}\ \left.\begin{array}{l}x=5\\ y=0\end{array}\right\}\ \text{are points on this graph.}$$

Also, the graph of the expression 3 is the same as the graph of $y=3$, which is a straight line parallel to x-axis at a distance of 3 units from the origin.



Hence, taking five times the length of a side of a small square as the unit of length the two graphs are as shown in the figure at page 365.

Let P be the point where the two graphs intersect. We find that the abscissa of $P=8$;

$\therefore\quad x=8$ is the required solution.

Example 4. Find the co-ordinates of the vertices of the triangle whose sides are given by the equations $x-2y+12=0$, $x+y+3=0$ and $5x-y-21=0$, and calculate its area.

We find that $\left.\begin{array}{l}x=0\\y=6\end{array}\right\}$, $\left.\begin{array}{l}x=-2\\y=5\end{array}\right\}$ and $\left.\begin{array}{l}x=-12\\y=0\end{array}\right\}$

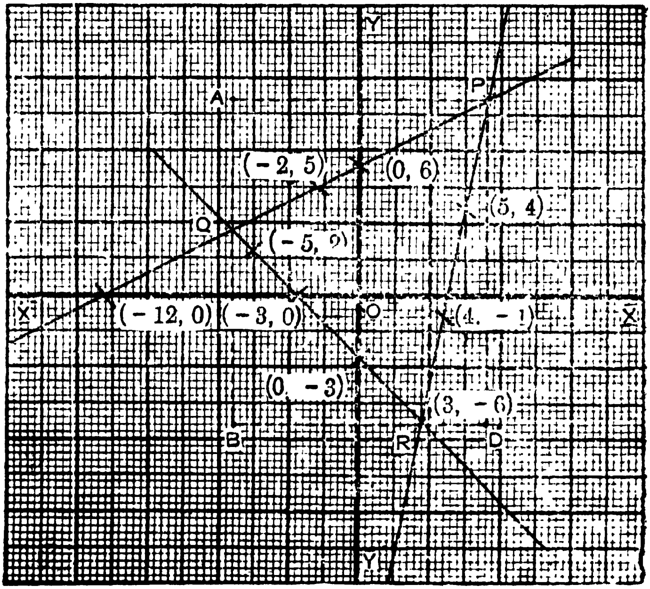
are points on the graph of $x-2y+12=0$;

whilst $\left.\begin{array}{l}x=0\\y=-3\end{array}\right\}$, $\left.\begin{array}{l}x=-3\\y=0\end{array}\right\}$ and $\left.\begin{array}{l}x=-5\\y=2\end{array}\right\}$

are points on the graph of $x+y+3=0$;

and $\left.\begin{array}{l}x=3\\y=-6\end{array}\right\}$, $\left.\begin{array}{l}x=4\\y=-1\end{array}\right\}$ and $\left.\begin{array}{l}x=5\\y=4\end{array}\right\}$

are points on the graph of $5x-y-21=0$



Hence, taking three times the length of a side of a small square as the unit of length, the straight lines PQ, QR and RP represent the graphs of the 1st, 2nd and 3rd equations respectively.

We find from the diagram that the co-ordinates of the vertex

P, are $\left.\begin{matrix} x=6 \\ y=9 \end{matrix}\right\}$; of Q, $\left.\begin{matrix} x=-6 \\ y=3 \end{matrix}\right\}$; and of R, $\left.\begin{matrix} x=3 \\ y=-6 \end{matrix}\right\}$

Drawing lines parallel to the axes (as shown in the diagram by dotted lines), we have

$$\begin{aligned}\triangle PQR &= \text{the rect. } ABDP - \triangle QAP - \triangle QBR - \triangle RDP \\ &= AB \times BD - \frac{AP \times AQ}{2} - \frac{QB \times BR}{2} - \frac{RD \times DP}{2} \\ &= 15 \times 12 - \frac{12 \times 6}{2} - \frac{9 \times 9}{2} - \frac{3 \times 15}{2} \\ &= 180 - 36 - \tfrac{81}{2} - \tfrac{45}{2} = 81 \text{ units of area.}\end{aligned}$$

[*Otherwise* : Count the small squares within the triangle PQR and the small squares through which the three sides of the triangle pass and of which half or more than half is within the triangle ; and they are found to be 729 (nearly). Since three times the length of a side of a small square represents the unit of length, $(3^2=)$ 9 small squares represent the unit of area.

Therefore, the area of the triangle PQR is 81 units of area (nearly).]

Example 5. Find graphically the co-ordinates of the vertices of the quadrilateral whose sides are $x+y-10=0$, $x-y+10=0$, $x+y+10=0$ and $x-y-10=0$. Prove that the quadrilateral is a square and find its area.

We find that $\left.\begin{matrix} x=10 \\ y=0 \end{matrix}\right\}$, $\left.\begin{matrix} x=5 \\ y=5 \end{matrix}\right\}$ and $\left.\begin{matrix} x=0 \\ y=10 \end{matrix}\right\}$

are points on the graph of $x+y-10=0$;

$\left.\begin{matrix} x=0 \\ y=10 \end{matrix}\right\}$, $\left.\begin{matrix} x=-5 \\ y=5 \end{matrix}\right\}$ and $\left.\begin{matrix} x=-10 \\ y=0 \end{matrix}\right\}$

are points on the graph of $x-y+10=0$;

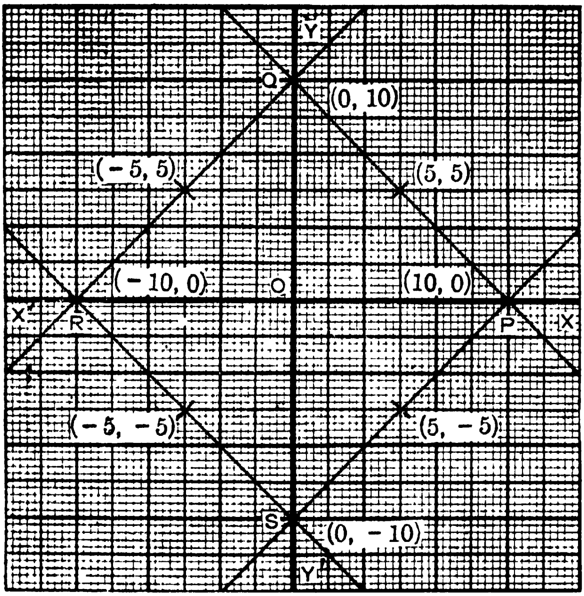
$\left.\begin{matrix} x=0 \\ y=-10 \end{matrix}\right\}$, $\left.\begin{matrix} x=-5 \\ y=-5 \end{matrix}\right\}$ and $\left.\begin{matrix} x=-10 \\ y=0 \end{matrix}\right\}$

are points on the graph of $x+y+10=0$;

whilst $\left.\begin{matrix} x=0 \\ y=-10 \end{matrix}\right\}$, $\left.\begin{matrix} x=5 \\ y=-5 \end{matrix}\right\}$ and $\left.\begin{matrix} x=10 \\ y=0 \end{matrix}\right\}$

are points on the graph of $x-y-10=0$.

Hence, taking three times the length of a side of a small square as the unit of length the four graphs are represented by the straight lines *PQ*, *QR*, *RS* and *SP*.



We notice that the co-ordinates of the vertices *P*, *Q*, *R* and *S* are

$$\left.\begin{matrix}x=10\\ y=0\end{matrix}\right\}, \quad \left.\begin{matrix}x=0\\ y=10\end{matrix}\right\}, \quad \left.\begin{matrix}x=-10\\ y=0\end{matrix}\right\} \text{ and } \left.\begin{matrix}x=0\\ y=-10\end{matrix}\right\} \text{ respectively.}$$

It is obvious from the diagram that $OP=OQ=OR=OS$, each being 10 units long and the diagonal *PR* is perp. to *QS*.

Hence, it follows from geometry that the quadrilateral *PQRS* is a square.

The area required $=\triangle PQR+\triangle PSR$

$$=\frac{PR\times OQ}{2}+\frac{PR\times OS}{2}$$

$$=\frac{20\times 10}{2}+\frac{20\times 10}{2}=200 \text{ units of area.}$$

Example 6. Draw the graphs of the equations $x+y=2$ and $x=y$. Find the co-ordinates of their point of intersection and measure the angle formed by them at the point of intersection.

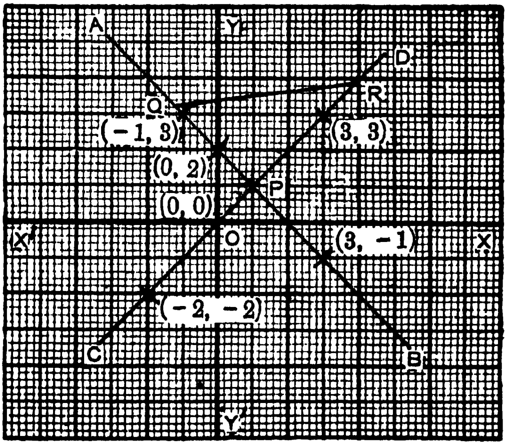
From the equation $x+y=2$,

or, $y=2-x$, we get

x	0	3	-1
y	2	-1	3

From the equation $x=y$, we get

x	0	-2	3
y	0	-2	3



Hence taking five times the length of a side of a small square as the unit of length, the two graphs are as shown above.

Let P be the point where the two graphs intersect. The co-ordinates of P are found to be 1 and 1.

Measure the angle formed by the graphs AB and CD at P with a protractor. It is found to be 90°.

It can be verified as follows :

Take two points Q and R on AP and DP respectively.

Join QR. Now measure QP, PR and RQ. It is found that $QP^2+PR^2=QR^2$.

$\therefore \angle APD =$ a right angle $= 90^\circ$.

EXERCISE 100

Solve the following equations graphically :

1. $x+y=9$, $3x-2y=7$.
2. $4x+3y=13$, $3x+2y=11$.
3. $\frac{x}{4}+\frac{y}{3}=4$, $4x-5y=2$.
4. $y-x=2$, $3x-2y=5$.
5. $5x-3y=11$, $2y-3x+4=0$.
6. $\frac{x-2}{2}=\frac{-5x+4}{5}$.
7. $\frac{2x+7}{3}=\frac{3x-7}{2}$.
8. $\frac{4x-3}{5}=\frac{6x}{7}-1$.
9. $x-12=-3$.
10. $5x-13=7$.

11. Find the vertices of the triangle whose sides are given by $-x+3y=18$, $x+7y=22$ and $y+3x=26$ and calculate its area.

12. Show that the straight lines $4x-y=16$, $3x-2y=7$ and $x+y=9$ meet at a point. Find its co-ordinates.

13. Find the vertices and the areas of the quadrilaterals whose sides are given by (i) $x+y=3$, $\frac{x}{3}-\frac{y}{3}=1$, $\frac{x+y}{3}=-1$, and $x-y+3=0$ (ii) $x=1$, $y=5$, $x=12$ and $y=10$; (iii) $x=0$, $y=0$, $\frac{x}{3}+\frac{y}{5}=1$, $\frac{x}{8}+\frac{y}{12}=1$.

14. Find the vertices and the areas of the triangles whose sides are given by (i) $x=0$, $y=0$, $\frac{x}{5}+\frac{y}{6}=1$; (ii) $x-2=0$, $y-1=0$, $x+y=6$; (iii) $x-2y+8=0$, $x+y+2=0$, $5x-y-14=0$.

In each of the following examples, show by solving the equations that they are satisfied by the same values of x and y.

Find these values and verify graphically :

15. $x+y=2$, $x=1$, $y=1$.
16. $7x+5y=24$, $x+y=2$, $2x+y=9$
17. $2x-y=7$, $y-x=2$, $11x=9y$.
18. Solve graphically :

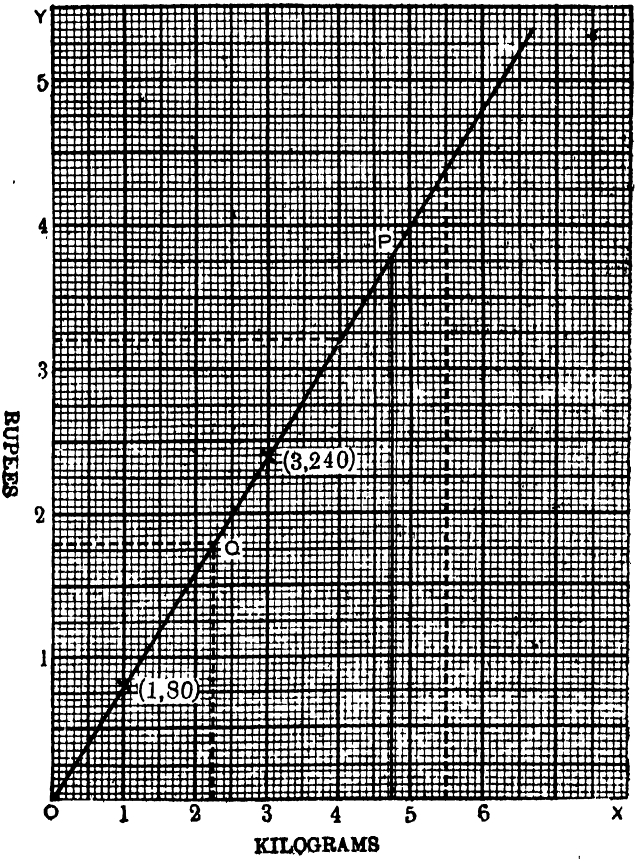
$$3x=17-2y,\ 3y=2x+6. \qquad \text{[A. U. 1927]}$$

Measure the angle formed by the two graphs at the point of intersection.

187. Applications of Graphs to Problems.

Example 1. Given that the price of a kilogram of rice is 80 paise show that a graph in the form of a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the quantity of rice of which the price is represented by the ordinate.

Determine from the graph (i) the price of 4 kilograms of rice and (ii) the number of kilograms of rice that can be had for Rs. 4 40 P.



From the given condition, the price of x kilograms of rice is $80x$ paise. If this is denoted by y, the graph of the equation $y=80x$ will be the graph to determine the quantity of rice and its price.

Suppose ten times the length of a side of a small square along OX indicates one kilogram of rice and the length of a side of a small square along OY indicates 5 paise. Then the meaning of the figures along OX and OY in the diagram on the pre-page is clear.

From the equation $y=80x$, we get

x	0	1	3
y	0	80	240

Plotting the points tabulated above we get the figure (OP) at page 371

Let us take a point Q on the straight line OP. Evidently its abscissa is 2·25 and ordinate is 180. Since the price of one kilogram of rice is 80 paise, so the price of 2·25 Kgs. of rice is rupee one and eighty paise. Therefore Q is such a point that its abscissa represents a quantity of rice of which the price is represented by its ordinate.

Let us take the point P (4·75, 380) on the straight line. Its abscissa represents the quantity of rice of which the price is represented by its ordinate. Similarly this is true for every point on the line OP.

So OP is the required graph.

With the help of this straight line the price of any quantity of rice can be found out, viz. ordinate of the point of which the abscissa is 4 is 320. So the price of 4 kilograms of rice is Rs. 3·20.

The graph also enables us to determine quickly the number of kilograms of rice that can be had for any given price. For instance, if the ordinate is taken to be 440, the corresponding abscissa is immediately found to be 5·5, which shows that we can have 5·5 kilograms of rice for 440 paise, *i.e.*, Rs. 4 40 paise,

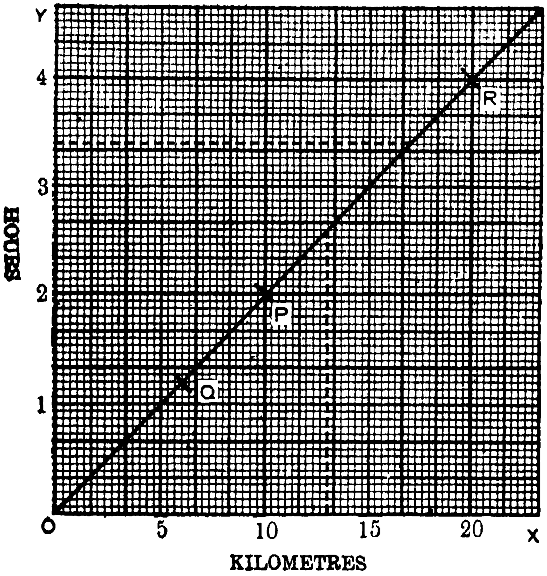
Note. *The line OP is called the graph of the price of rice, or more simply the price-graph of rice.*

Example 2. A person, named B, starting from a given place, travels at the rate of 5 kilometres an hour. Show that a graph in the form of a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the number of kilometres that B travels in the time represented by the ordinate.

Determine from the graph (i) the distance travelled in 3 hours 24 minutes and (ii) the time to travel 13 kilometres.

In the figure on the next page let three times the length of a side of a small square measured along OX represent one kilometre and let the length of a side of a small square along OY represent 4 minutes. Then the meaning of the figures along OX and OY is clear.

Since *B* travels 5 kilometres in one hour, he travels 10 kilometres in 2 hours. Clearly, therefore, *P* is a point such that its abscissa represents the number of kilometres that the person travels in the time represented by its ordinate.



Join *OP*, and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by *P*.

Let *Q* be any point on the line. Its abscissa represents 6 kilometres and ordinate represents 1 hour 12 minutes; but we *know* that the person travels 6 kilometres in 1 hour 12 minutes.

Hence, *Q* satisfies the condition mentioned above.

Let *R* be some other point on the line. Its abscissa represents 20 kilometres and ordinate represents 4 hours; but we *know* that the person travels 20 kilometres in 4 hours.

Hence, *R* also satisfies the proposed condition.

Similarly it is true for any other point on the line.

Hence, *OP* is the requird straight line.

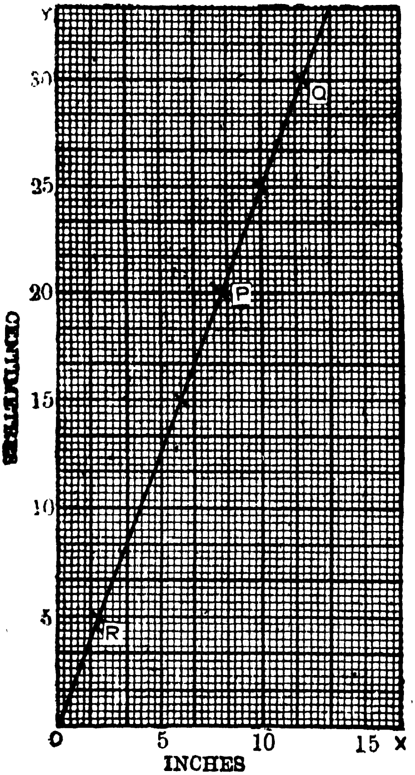
The graph enables us to determine readily the time in which *B* travels any given number of kilometres. For instance, if the abscissa be taken which represents 13 kilometres, the corresponding ordinate is immediately found to be that which represents 2 hours 36 minutes;

thus, it is known that the time taken by the person to travel 13 kilometres is 2 hours 36 minutes.

The graph also enables us to determine readily the number of kilometres that the person travels in any given time. For instance, if the ordinate be taken which repesents 3 hours 24 minutes, the corresponding abscissa is immediately found to be that which represents 17 kilometres; thus, it is known that in 3 hours and 24 minutes the person travels 17 kilometres

Note *The line OP is called the graph of B's motion, or the motion-graph of B.*

Example 3. If one inch be equal in length to 2·5 centimetres show that a straight line can be drawn such that the abscissa of any point on the line will represent the number of inches that are equivalent to the number of centimetres represented by the ordinate.



Determine from the graph (i) the number of centimetres in 10 inches and (ii) the number of inches in 15 centimetres.

In the figure let three times the length of a side of a small square measured along OX represent one inch, and let an equal length measured along OY represent one centimetre. Then the meaning of the figures along OX and OY is clear.

Since 1 inch = 2·5 centimetres, we have 8 inches = 20 centimetres. Clearly, therefore, P is a point such that its abscissa represents the number of inches that are equivalent to the number of centimetres represented by its ordinate.

Join OP, and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P.

Let Q be any point on the line. Its abscissa represents 12 inches, whilst its ordinate represents 30 centimetres; but we *know* that these two are equivalent. Hence, Q satisfies the condition mentioned above.

Let R be some other point on the line. Its abscissa represents 2 inches, whilst its ordinate represents 5 centimetres; but we *know* that these two are equivalent. Hence, R also satisfies the proposed condition

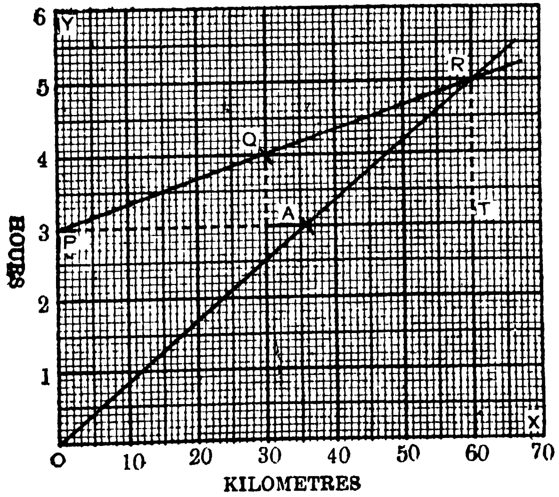
Similarly it is true for any other point on the line. Hence, OP is the required straight line.

The graph enables us to determine readily the number of centimetres that are equivalent to any given number of inches. For instance, if the abscissa be taken which represents 10 inches, the corresponding ordinate is immediately found to be that which represents 25 centimetres; thus, it is known that 10 inches are equivalent to 25 centimetres.

The graph also enables us to determine readily the number of inches that are equivalent to any given number of centimetres. For instance, if the ordinate be taken which represents 15 centimetres, the corresponding abscissa is immediately found to be that which represents 6 inches; thus, it is known that 15 centimetres are equivalent to 6 inches.

Note. *The line OP is called the graph for converting inches into centimetres and vice versa, or more briefly, the conversion-graph for inches and centimetres.*

Example 4. A man is cycling at the rate of 12 kilometres per hour.



A motor car starts 3 hours later and travels at the rate of 30 kilometres per hour. Find *graphically* when the car will overtake the cyclist.

[W. B. C. S. 1956 (adapted)]

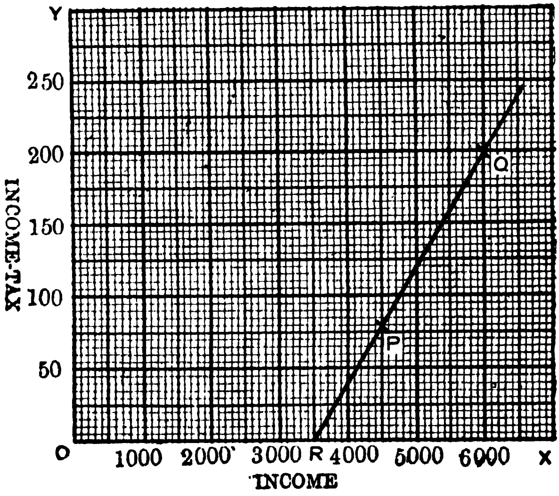
In the figure on the pre-page let the length of a side of a small square measured along OX represent a kilometre and let an equal length measured along OY represent 6 minutes. Then the meaning of the figures along OX and OY is clear.

Suppose the cyclist was at O at the time of starting. Since he travels at the rate of 12 kilometres per hour, he travels 36 kilometres in 3 hours. Hence if the point A is so taken that its co-ordinates respectively represent 36 kilometres and 3 hours, OA is the graph of the cyclist's motion for the first 3 hours.

The motor car starts from O 3 hours after the cyclist leaves O. Hence, if OP be measured vertically to represent 3 hours, OP may be regarded as the graph of rest of the motor car at O. If the point Q be so taken that its co-ordinates respectively represent 30 kilometres and 1 hour when referred to P as origin, *i.e.*, 30 kilometres and 4 hours when referred to O as origin, the straight line passing through P and Q will be the motion-graph of the motor car.

Suppose the two graphs intersect at R. The ordinate of R represents 5 hours. Since the motor car started 3 hours later, so it will overtake the cyclist after 2 hours indicated by RT.

Example 5. The tax on income is directly proportional to income above a certain value. If the tax changes from Rs. 80 to Rs. 200 as the



income changes from Rs. 4,500 to Rs. 6,000, find *graphically* the highest limit of income for which no tax is charged. [W. B. C. S. 1955]

Let the length of a side of a small square measured along OX represent an income of Rs. 100 and an equal length measured along OY represent income-tax of Rs. 5. Then the meaning of the figures along OX and OY is clear.

If P and Q are such two points that their co-ordinates represent respectively an income of Rs. 4,500 and income-tax of Rs. 80, and an income of Rs. 6,000 and income-tax of Rs. 200, the straight line PQ will indicate income-tax directly proportional to income.

Let QP intersect OX at R when produced. OR represents the highest limit of income for which no tax is charged. But OR represents an income of Rs. 3,500.

$\therefore$ the highest limit is Rs. 3,500.

Example 6. A and B are two stations 30 kilometres apart. P starts from A and travels towards B at the rate of 5 kilometres an hour ; at the end of 2 hours he takes rest for one hour, and then resumes his journey at the rate of 3 kilometres an hour. Q leaves B, 2 hours 40 minutes after P leaves A, and travels towards A, without stoppage, at the rate of 4 kilometres an hour. When and where will the two travellers meet ?

Suppose the straight line AB represents x-axis and the straight line AY passing through A and perpendicular to AB represents y-axis.

Let $2\frac{1}{2}$ times the length of a side of a small square measured horizontally represent one kilometre, and let an equal length measured vertically represent 10 minutes Then the meaning of the figures along the lines in the diagram on the next page is clear.

(i) P starts from A, and travelling at the rate of 5 kilometres an hour completes 10 kilometres in 2 hours. Hence, if the point C be taken such that its co-ordinates respectively represent 10 kilometres and 2 hours, AC is the graph of P's motion for the first two hours.

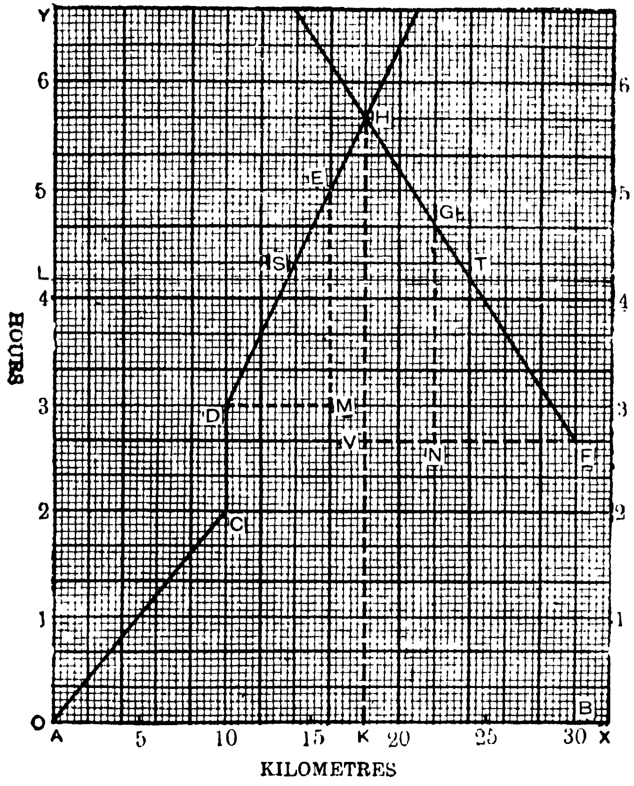
The graph for the 3rd hour must be such that the abscissa of any point on it may represent 10 kilometres, because P is supposed to be at rest throughout this hour. Hence, CD drawn vertically to represent one hour, as in the diagram, will be the graph of P's rest.

After the third hour, P travels at the rate of 3 kilometres an hour. Hence, if DM be taken to represent 6 kilometres and ME to represent 2 hours, the straight line DE is the graph of P's motion after the 3rd hour.

Thus, the broken line $ACDE$ is the complete graph of P's motion.

(ii) Q starts from B, 2 hours 40 minutes after P leaves A. Hence, if BF be measured vertically to represent 2 hours 40 minutes, BF may be regarded as the graph of Q's rest at B.

When Q leaves B, he moves towards A at the rate of 4 kilometres an hour. Hence, if FN be taken to represent 8 kilometres and NG to represent 2 hours, the straight line FG will be the graph of Q's motion.



(iii) Let the two graphs intersect at H, and draw HK perpendicular to AB. Produce FN to meet HK at V.

Now it is clear that at the end of time indicated by HK, P will have gone a distance AK towards B, and Q will have gone a distance BK (*i.e.*, FV) towards A. Hence, they will meet at this instant. Thus the required time of meeting = that represented by HK = 5 hours 40 minutes after the commencement of P's motion.

Also, the distance of the place of meeting from A = that represented by AK = 18 kilometres.

Note 1. *As HV represents 3 hours, it is clear that P and Q meet at the end of 3 hours after Q starts from B.*

Note 2. *The horizontal line through L meets the graphs at the points S and T. As AL represents 4 hours 10 minutes and ST represents 10$\frac{1}{2}$ kilometres, it is clear that at the end of 4 hours 10 minutes from the commencement of P's motion, P and Q are at a distance of 10$\frac{1}{2}$ kilometres from each other.*

EXERCISE 101

1. If potato sells for 80 paise per kilogram, construct the price-graph of potato, giving the price of any quantity of potato up to 10 kilograms. From the graph read off the price of 4 kilograms and 250 grams of potato and also the quantity of potato that can be had for rupees two and eighty paise.

2. If *Fazli* mangoes be worth three rupees a dozen, construct a price-graph for mangoes, giving the price of any number up to 30. Read off from the graph the price of 17 mangoes and also the number of mangoes that can be had for Rs. 4 75 P.

3. If a man walks at the rate of 4 kilometres an hour, construct a graph of his motion. Read off from the graph the time in which he travels 13 kilometres, and also the number of kilometres he travels in 4$\frac{3}{4}$ hours.

4. If one mile be equal to 1·6 kilometres, construct a conversion-graph for miles and kilometres. Read off from the graph the number of kilometres that are equivalent to 5 miles and also the number of miles that are equivalent to 11·2 kilometres.

5. A starts from a place and walks in a given direction at the rate of 3 kilometres an hour; B starts from the same place one hour later and moves in the same direction at the rate of 5 kilometres an hour. Draw the motion-graphs of A and B, and find when and where B overtakes A.

6. A and B are two stations 40 kilometres apart. P starts from A and travels towards B at the rate of 6 kilometres an hour; whilst Q starting from B travels towards A at the rate of 4 kilometres an hour. Construct the motion-graphs of P and Q, and find when and where they meet.

7. Fifty articles of the same kind cost Rs. 3. Construct a graph from which you can read off the cost of any number of articles up to 50. Hence, find the cost of 19 articles, and the number of articles that you would get for Rs. 2 34 P.

8. Given that 1 kilogram = 2·2 lbs., construct a graph which will enable you to read off the number of kilograms that are equivalent to any given number of lbs. up to 15 lbs. Read off the number of kilograms in 11 lbs.

9. A man travels for 3 hours at the rate of 4 kilometres an hour, at the end of which he takes rest for an hour and a half, and then starts to walk at the rate of 5 kilometres per hour. Construct the graph of his motion.

10. A man starts from a place *B* to walk towards *C* at the rate of 8 kilometres an hour. After 3 hours he changes his mind and walks back towards *B* at the rate of 6 kilometres an hour. At the end of 2 hours again he suddenly changes his mind and begins to run towards *C* at the rate of 14 kilometres an hour. Draw a graph of his motion.

11. *A*, *B* and *C* are three stations in order on the same road, the distance between *A* and *B* being 6 kilometres. *Q* starts from *B* at noon to walk towards *C* at the rate of 3 kilometres an hour, and at 1-30 P.M. *P* starts from *A* to run towards *C* at the rate of $6\frac{1}{2}$ kilometres an hour. Draw graphs of their motion, and find when and where *P* will overtake *Q*.

12. A man spends Rs. 620 in 40 days. Draw a graph to give his expenditure in any number of days up to 90. Determine from the graph the amount spent in 28 days.

13. At what time between 3 and 4 o'clock are the two hands of a watch together ?

14. An income-tax of 3 paise per rupee is in force. Draw a graph to show the tax on all incomes from Rs. 3,000 to Rs. 5,000 and determine the income corresponding to a tax of Rs. 96 and the tax corresponding to an income of Rs. 4,350.

15. The tax on income is directly proportional to income above a certain value. If tax changes from Rs. 18 to Rs. 39 as the income changes from Rs. 4,200 to Rs. 4,900, find *graphically* the highest limit of income for which no tax is charged.

16. *P* and *Q* are two stations 34 Kms. apart. *A* starts from *P* at noon and walks towards *Q* at the rate of 4 Kms. per hour. At the end of $1\frac{1}{2}$ hours he takes rest for $1\frac{1}{2}$ hours and then resumes his journey at the rate of 5 Kms. per hour. *B* leaves *Q* 2 hours after *A* leaves *P* and travels at the rate of 5 Kms. per hour for 2 hours and takes rest for one hour and then resumes his journey at the rate of 4 Kms. per hour. When and where will *A* and *B* meet ? What was the distance between them at 3 P.M. and at what hour was the distance between them 8 Kms. ?

17. The following table shows the timings of two trains, one an express from Calcutta to Ranaghat, and the other a local from Naihati to Calcutta. Find by graphical methods when and where the trains

meet, assuming that all runs are at constant speed and that the local train waits one minute at each station between Naihati and Calcutta.

Distance from Calcutta in kilometres						
74	Ranaghat					17-56
38	Naihati	dep.		16-24		
36	Kakinara	"		16-29		
31	Shamnagar	"		16-36	↑	
27	Ichhapur	"		16-42		
25	Palta	"		16-45		
23	Barrackpur	"		16-49		
21	Tittagarh	"		16-53		
19	Khardah	"		16-57		
16	Sodepur	"	↓	17- 2		
14	Agarpara	"		17- 6		
12	Belghurria	"		17-11		
7	Dum-Dum	"		17-19		
	Calcutta	"		17-31		16-42

[B. C. S. 1922 (adapted)

CHAPTER XXIX

ELEMENTARY LAWS OF INDICES

188. Definition. The product of m factors each equal to a is represented by a^m. [Art. 16]

Thus, the meaning of a^m is clear when m is a *positive integer.*

189. The Index Law and the truths necessarily following from it.

To prove that $a^m \times a^n = a^{m+n}$, *where* m *and* n *are any two positive integers.*

Since $a^m = a.a.a. \cdots \cdots$ to m factors

and $a^n = a.a.a.a. \cdots \cdots$ to n factors

$\therefore\ a^m \times a^n = (a.a.a. \cdots \cdots$ to m factors)

$\times (a.a.a.a. \cdots \cdots$ to n factors)

$= a.a.a.a.a.a.a. \cdots \cdots$ to $(m+n)$ factors

$= a^{m+n}.$

This result is called the **Index Law.**

Cor. 1. $a^m \times a^n \times a^p = a^{m+n+p}$, *when* m, n *and* p *are positive integers*

For, $a^m \times a^n = a^{m+n}$; $\therefore\ a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{(m+n)+p} = a^{m+n+p}$

Hence, $a^m \times a^n \times a^p \times a^q \times \cdots = a^{m+n+p+q+\cdots}$

Thus, *the product of any number of powers of a given quantity is that power of the quantity whose index is equal to the sum of the indices of the factors.*

Cor. 2. $(a^m)^n = a^{mn}$, *when* m *and* n *are any two positive integers.**

For, $(a^m)^n = a^m \times a^m \times a^m \times \cdots$ to n factors

$= a^{m+m+m+\cdots \text{to } n \text{ terms}}$ [by Cor. 1]

and $\therefore\ = a^{mn}.$

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*This may also be proved as follows :

$(a^m)^n = a^m.\,a^m.\,a^m.\,a^m.........n$ factors

$= (a.\,a.\,a...$to m factors) $(a.\,a.\,a....$to m factors) $......n$ groups,

the group being repeated n times,

$= a.\,a.\,a.\,a.\,a.\,a.........mn$ factors

$= a^{mn}.$

Cor. 3. *$a^m \div a^n = a^{m-n}$, when m and n are positive integers and m is greater than n.*[*]

For, $a^{m-n} \times a^n = a^{(m-n)+n}$ [because $m-n$ is a positive integer]

$$= a^m ;$$

$$\therefore \quad a^m \div a^n = a^{m-n}.$$

190. Assuming the formula $a^m \times a^n = a^{m+n}$ to be true for *all* values of m and n, to find meanings for quantities with fractional or negative indices.

(i) To find the meaning of $a^{\frac{p}{q}}$, when p and q are any two positive integers.

Since, $a^m \times a^n = a^{m+n}$ for *all* values of m and n, putting $\frac{p}{q}$ for each of them, we have $a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}$.

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{3p}{q}}$; and so on.

Hence, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots\cdots$ to q factors

$$= a^{\frac{qp}{q}} = a^p.$$

Thus, $a^{\frac{p}{q}}$ is equal to the q^{th} root of a^p, and is, therefore, equivalent to $\sqrt[q]{a^p}$.

Cor. Hence, $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$; and so on.

Generally, $a^{\frac{1}{n}} = \sqrt[n]{a}$

Note. *From the Index Law it is also easy to see that $a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \cdots\cdots$ to p factors $= a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\cdots \text{to } p \text{ terms}} = a^{\frac{p}{q}}$. Thus, $a^{\frac{p}{q}}$ may as well be regarded as the p^{th} power of $a^{\frac{1}{q}}$, i.e., equivalent to $(\sqrt[q]{a})^p$. Thus, $a^{\frac{p}{q}}$ may be interpreted either as the q^{th} root of the p^{th} power of a, or as the p^{th} power of the q^{th} root of a.*

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[*] *This may also be proved as follows :*

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{a.\ a.\ a.\ a.\ \ldots\ldots \text{ to } m \text{ factors}}{a.\ a.\ a.\ \ldots\ldots\ldots \text{ to } n \text{ factors}},$$

If $m > n$, all the n factors of the denominator cancel with n factors of the numerator, leaving $m-n$ factors,

$$= a.\ a.\ \ldots\ldots (m-n) \text{ factors} = a^{m-n},$$

when $n > m$, the quotient is $\frac{1}{a^{n-m}}$.

(ii) To find the meaning of a^0.

Since, $a^m \times a^n = a^{m+n}$ is true for *all* values of m and n, putting $m = 0$ we have $a^0 \times a^n = a^{0+n} = a^n$;

$\therefore \quad a^0 = a^n \div a^n = 1.$

$$\left[\textit{Otherwise}, \ a^0 = a^{m-m} = \frac{a^m}{a^m} = 1. \right]$$

Thus, *any quantity raised to the power zero is equivalent to* 1.

(iii) To find the meaning of a^{-n}, where n is any positive integer.

Since, $a^m \times a^n = a^{m+n}$ is true for *all* values of m and n, putting $m = -n$, we have

$$a^{-n} \times a^n = a^{-n+n} = a^0 = 1 ;$$

$$\therefore \quad a^{-n} = \frac{1}{a^n}, \text{ and } a^n = \frac{1}{a^{-n}}.$$

Cor. Hence, $a^m \div a^n = a^{m-n}$ for *all* values of m and n.

For, $$a^m \div a^n = \frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}.$$

Example 1. Find the value of $8^{\frac{5}{3}}$.

$$8^{\frac{5}{3}} = (\sqrt[3]{8})^5 = 2^5 = 32.$$

Example 2. Find the value of $4^{-\frac{5}{2}}$.

$$4^{-\frac{5}{2}} = \frac{1}{4^{\frac{5}{2}}} = \frac{1}{(\sqrt{4})^5} = \frac{1}{2^5} = \frac{1}{32}.$$

Example 3. Multiply together $\sqrt{a^5}$, $a^{\frac{3}{4}}$, $\sqrt[4]{a^{-5}}$ and $\frac{1}{a^{-3}}$.

The required product $= a^{\frac{5}{2}} \times a^{\frac{3}{4}} \times a^{-\frac{5}{4}} \times a^3$

$$= a^{\frac{5}{2}+\frac{3}{4}-\frac{5}{4}+3}$$

$$= a^{\frac{5}{2}-\frac{1}{2}+3} = a^{2+3} = a^5.$$

EXERCISE 102

Express the following avoiding fractional or negative indices :

1. $a^{\frac{5}{7}}$. 2. $x^{-\frac{3}{2}}$. 3. $\frac{3}{x^{-\frac{1}{3}}}$. 4. $x^{-\frac{2}{5}} \times 3a^{-\frac{1}{2}}$.

5. $8m^{-2} \times m^{-\frac{3}{2}}$. 6. $x^{-\frac{4}{5}} \div 3a^{-\frac{5}{4}}$. 7. $x^{-\frac{3}{5}} \div 2x^{-\frac{1}{2}}$.

8 $\sqrt[5]{x^2} \div \sqrt[5]{x^{-a}}$. 9. $\sqrt[2m]{a^{-5}} \times \sqrt[m]{a^3}$. 10. $\sqrt[4a]{x^6} \div \sqrt[2a]{x^{-5}}$.

Express the following avoiding radical signs and negative indices :

11. $(\sqrt[3]{x})^7$. 12. $(\sqrt[4]{a})^{-6}$. 13. $\frac{1}{\sqrt[3]{x^{-2}}}$. 14. $\frac{1}{(\sqrt[5]{a})^{-2}}$.

15. $\sqrt[3]{x^{-4}}+(\sqrt[6]{x})^{-1}$. 16. $\sqrt[4]{a^{-3}}+(\sqrt[3]{a})^{-12}$.

Find the value of :

17. $4^{-\frac{3}{2}}$. 18. $8^{\frac{2}{3}}$. 19. $9^{\frac{3}{2}}$.

20. $16^{\frac{5}{4}}$. 21. $81^{-\frac{3}{4}}$. 22. $\frac{1}{6^{-2}}$.

23. $(125)^{-\frac{2}{3}}$. 24. $(\frac{1}{27})^{-\frac{4}{3}}$. 25. $(\frac{1}{216})^{-\frac{2}{3}}$.

26. Simplify $\frac{x^{m+2n}x^{3m-8n}}{x^{5m-6n}}$. [C. U. 1874]

191. **To prove that $(a^m)^n = a^{mn}$ is true for *all* values of m and n.**

(i) Let n be a *positive integer.* Then, whatever may be the value of m, we have

$$(a^m)^n = a^m \times a^m \times a^m \times \cdots\cdots\cdots \text{ to } n \text{ factors}$$
$$= a^{m+m+m+\cdots\cdots\cdots \text{to } n \text{ terms}}$$
$$= a^{mn}.$$

(ii) Let n be a *positive* fraction, equal to $\frac{p}{q}$, when p and q are positive integers Then, we have

$$\left(a^m\right)^n = \left(a^m\right)^{\frac{p}{q}} = \sqrt[q]{\left(a^m\right)^p} \quad [\text{ Art. } 190 \text{ (i) }]$$
$$= \sqrt[q]{a^{mp}} \quad [\text{ by (i) }]$$
$$= a^{\frac{mp}{q}} \quad [\text{ Art. } 190 \text{ (i) }]$$
$$= a^{mn}.$$

(iii) Let n be *any negative* quantity, equal to $-p$, where p is *positive.* Then, we have

$$(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} \quad [\text{ Art. } 190 \text{ (iii) }]$$
$$= \frac{1}{a^{mp}} \quad [\text{ by (i) and (ii) }]$$
$$= a^{-mp} \quad [\text{ Art. } 190 \text{ (iii) }]$$
$$= a^{m(-p)} = a^{mn}.$$

Thus, the proposition is established.

192. To prove that $a^n b^n = (ab)^n$ for *all* values of n.

(i) Let n be a *positive integer*. Then, we have

$$a^n b^n = (a.a.a \ldots\ldots \text{ to } n \text{ factors}) \times (b.b.b \ldots\ldots \text{ to } n \text{ factors})$$
$$= (ab.ab.ab \ldots\ldots \text{ to } n \text{ factors})$$
$$= (ab)^n$$

(ii) Let n be a *positive fraction*, equal to $\frac{p}{q}$, where p and q are positive integers. Then, putting x for $a^n b^n$, we have

$$x = a^{\frac{p}{q}} b^{\frac{p}{q}} ;$$

$$\therefore \quad x^q = \left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)^q = \left(a^{\frac{p}{q}}\right)^q \times \left(b^{\frac{p}{q}}\right)^q \qquad [\text{ by (i) }]$$
$$= a^p \times b^p \qquad [\text{ Art. 189 }]$$
$$= (ab)^p , \qquad [\text{ by (i) }]$$

$$\therefore \quad x = (ab)^{\frac{p}{q}} ; \text{ i.e., } a^n b^n = (ab)^n.$$

(iii) Let n be *any negative* quantity, equal to $-p$, where p is positive. Then, we have

$$a^n b^n = a^{-p} b^{-p}$$
$$= \frac{1}{a^p b^p} \qquad [\text{ Art. 190 (iii) }]$$
$$= \frac{1}{(ab)^p} \qquad [\text{ by (i) and (ii) }]$$
$$= (ab)^{-p} \qquad [\text{ Art. 190 (iii) }]$$
$$= (ab)^n.$$

Thus, the proposition is established.

Cor. 1. $\frac{a^n}{b^n} = a^n b^{-n} = a^n (b^{-1})^n = (ab^{-1})^n = \left(\frac{a}{b}\right)^n.$

Cor. 2. $a^n b^n c^n = (ab)^n c^n = (abc)^n$;

generally, $a^n b^n c^n d^n \ldots\ldots = (abcd \cdots)^n$.

193. Applications of the results proved in the last two articles.

Example 1. Simplify $\left(a^8 b^{\frac{5}{3}}\right)^{-\frac{3}{4}}$

$$\left(a^8 b^{\frac{5}{3}}\right)^{-\frac{3}{4}} = \left(a^8\right)^{-\frac{3}{4}} \times \left(b^{\frac{5}{3}}\right)^{-\frac{3}{4}}$$
$$= a^{8 \times \left(-\frac{3}{4}\right)} \times b^{\frac{5}{3} \times \left(-\frac{3}{4}\right)} = a^{-6} b^{-\frac{5}{4}}.$$

Example 2. Simplify $\sqrt{a^{-2}b} \times \sqrt[3]{ab^{-3}}$.

$$\sqrt{a^{-2}b} = (a^{-2}b)^{\frac{1}{2}} = (a^{-2})^{\frac{1}{2}} \times b^{\frac{1}{2}} = a^{-1}b^{\frac{1}{2}}\,;$$

and $$\sqrt[3]{ab^{-3}} = (ab^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}} \times (b^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}}b^{-1}.$$

Hence, the given expression

$$= a^{-1}b^{\frac{1}{2}} \times a^{\frac{1}{3}}b^{-1} = a^{-1+\frac{1}{3}} \times b^{\frac{1}{2}-1} = a^{-\frac{2}{3}}b^{-\frac{1}{2}}.$$

Example 3. Simplify $\sqrt{a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}} \div \sqrt[3]{a^4b^{-1}c^{\frac{5}{4}}}$.

$$\sqrt{a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}} = \left(a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}\right)^{\frac{1}{2}} = \left(a^3\right)^{\frac{1}{2}}\left(b^{-\frac{2}{3}}\right)^{\frac{1}{2}}\left(c^{-\frac{7}{6}}\right)^{\frac{1}{2}} = a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}}\,;$$

and $$\sqrt[3]{a^4b^{-1}c^{\frac{5}{4}}} = \left(a^4b^{-1}c^{\frac{5}{4}}\right)^{\frac{1}{3}} = \left(a^4\right)^{\frac{1}{3}}\left(b^{-1}\right)^{\frac{1}{3}}\left(c^{\frac{5}{4}}\right)^{\frac{1}{3}} = a^{\frac{4}{3}}b^{-\frac{1}{3}}c^{\frac{5}{12}}.$$

Hence, the given expression

$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}} \div a^{\frac{4}{3}}b^{-\frac{1}{3}}c^{\frac{5}{12}}$$

$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}} \times a^{-\frac{4}{3}}b^{\frac{1}{3}}c^{-\frac{5}{12}}$$

$$= a^{\frac{3}{2}-\frac{4}{3}}b^{-\frac{1}{3}+\frac{1}{3}}c^{-\frac{7}{12}-\frac{5}{12}}$$

$$= a^{\frac{1}{6}}b^0c^{-1} = a^{\frac{1}{6}}c^{-1}.$$

EXERCISE 103

Simplify .

1. $(a^{-\frac{3}{4}})^8$. 2. $(a^{-\frac{2}{3}}b^{\frac{6}{5}})^{\frac{3}{4}}$. 3. $(a^{-\frac{1}{2}}b^{-3})^{-2}$. 4. $(a^6b^{\frac{3}{4}})^{-\frac{4}{3}}$

5. $(\sqrt[3]{a^4b^3})^6$. 6. $(\sqrt[6]{x^9y^{-3}})^{-2}$. 7. $\sqrt[3]{x^2.\sqrt[4]{x^{-3}}}$.

8. $\sqrt{a^{-3}b^4} \times \sqrt[4]{a^2b^{-8}}$. 9. $\sqrt[4]{x^{-2}\sqrt{y^6}} \times \sqrt{x\sqrt[4]{y^3}}$.

10. $(8x^3+27a^{-3})^{\frac{2}{3}}$. 11. $(64x^3+27a^{-3})^{-\frac{2}{3}}$.

12. $\sqrt[3]{a^6b^{-2}c^{-4}} \times \sqrt[4]{a^{-6}b^4c^8}$. 13. $\sqrt{a^{-\frac{2}{3}}b^4c^{-\frac{1}{3}}} \div \sqrt[3]{a^2b^4c^{-1}}$

14. $\sqrt{ab^{-2}c^3} \div (\sqrt[3]{a^3b^2c^{-3}})^{-1}$. 15. $\left(\frac{a^{-1}b^2}{a^2b^{-4}}\right)^7 \div \left(\frac{a^3b^{-5}}{a^{-2}b^3}\right)^{-5}$.

194. Miscellaneous Examples.

Example 1. Divide $a+b+c+3a^{\frac{1}{3}}b^{\frac{2}{3}}+3a^{\frac{2}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

Let us proceed by arranging the dividend and the divisor according to descending powers of a:

$$\begin{array}{l|l|l}
a^{\frac{1}{3}}+\left(b^{\frac{1}{3}}+c^{\frac{1}{3}}\right) & a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+(b+c) & a^{\frac{2}{3}}+a^{\frac{1}{3}}\left(2b^{\frac{1}{3}}-c^{\frac{1}{3}}\right) \\
 & \underline{a+a^{\frac{2}{3}}\left(b^{\frac{1}{3}}+c^{\frac{1}{3}}\right)} & \qquad +\left(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}\right) \\
 & a^{\frac{2}{3}}\left(2b^{\frac{1}{3}}-c^{\frac{1}{3}}\right)+3a^{\frac{1}{3}}b^{\frac{2}{3}}+(b+c) & \\
 & \underline{a^{\frac{2}{3}}\left(2b^{\frac{1}{3}}-c^{\frac{1}{3}}\right)+a^{\frac{1}{3}}\left(2b^{\frac{2}{3}}+b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{2}{3}}\right)} & \\
 & \qquad a^{\frac{1}{3}}\left(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}\right)+(b+c) & \\
 & \qquad \underline{a^{\frac{1}{3}}\left(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}\right)+(b+c)} &
\end{array}$$

Thus, the required quotient $=a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}}+b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}$.

Note. *In multiplication as well as in division the arrangement of the expressions concerned according to ascending or descending powers of some common letter should never be overlooked. Such arrangements invariably give neatness to the required operations, if not always indispensable.*

Example 2. Divide $x+y^{\frac{1}{2}}+z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{6}}z^{\frac{1}{9}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{6}}+z^{\frac{1}{9}}$.

Putting a for $x^{\frac{1}{3}}$, b for $y^{\frac{1}{6}}$ and c for $z^{\frac{1}{9}}$, we have

$$\begin{aligned}
&x+y^{\frac{1}{2}}+z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{6}}z^{\frac{1}{9}} \\
&\qquad =a^3+b^3+c^3-3abc \\
&\qquad =(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\
&\qquad =\left(x^{\frac{1}{3}}+y^{\frac{1}{6}}+z^{\frac{1}{9}}\right)\left\{\left(x^{\frac{1}{3}}\right)^2+\left(y^{\frac{1}{6}}\right)^2+\left(z^{\frac{1}{9}}\right)^2-x^{\frac{1}{3}}y^{\frac{1}{6}}-y^{\frac{1}{6}}z^{\frac{1}{9}}-z^{\frac{1}{9}}x^{\frac{1}{3}}\right\} \\
&\qquad =\left(x^{\frac{1}{3}}+y^{\frac{1}{6}}+z^{\frac{1}{9}}\right)\left(x^{\frac{2}{3}}+y^{\frac{1}{3}}+z^{\frac{2}{9}}-x^{\frac{1}{3}}y^{\frac{1}{6}}-y^{\frac{1}{6}}z^{\frac{1}{9}}-z^{\frac{1}{9}}x^{\frac{1}{3}}\right).
\end{aligned}$$

Hence, the required quotient

$$\begin{aligned}
&=x^{\frac{2}{3}}+y^{\frac{1}{3}}+z^{\frac{2}{9}}-x^{\frac{1}{3}}y^{\frac{1}{6}}-y^{\frac{1}{6}}z^{\frac{1}{9}}-z^{\frac{1}{9}}x^{\frac{1}{3}} \\
&=x^{\frac{2}{3}}-x^{\frac{1}{3}}\left(y^{\frac{1}{6}}+z^{\frac{1}{9}}\right)+\left(y^{\frac{1}{3}}-y^{\frac{1}{6}}z^{\frac{1}{9}}+z^{\frac{2}{9}}\right).
\end{aligned}$$

Example 3. Divide

$$x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n} \text{ by } x^{2^{n-1}}-a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}$$

Let $p=x^{2^{n-2}}$ and $q=a^{2^{n-2}}$.

Then, $p^2=\left(x^{2^{n-2}}\right)^2=x^{2\times2^{n-2}}=x^{2^{n-1}}$,

and $p^4=(p^2)^2=\left(x^{2^{n-1}}\right)^2=x^{2\times2^{n-1}}=x^{2^n}$

Similarly, $q^2=a^{2^{n-1}}$ and $q^4=a^{2^n}$.

Hence,

$$\frac{x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n}}{x^{2^{n-1}}-a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}}$$

$$=\frac{p^4+p^2q^2+q^4}{p^2-pq+q^2}$$

$$=\frac{(p^2+q^2)^2-p^2q^2}{p^2-pq+q^2}$$

$$=\frac{(p^2+q^2+pq)(p^2+q^2-pq)}{p^2-pq+q^2}$$

$$=p^2+pq+q^2$$

$$=x^{2^{n-1}}+x^{2^{n-2}}a^{2^{n-2}}+a^{2^{n-1}}$$

Example 4. Find the H.C.F. of

$$a^2+2b^2+(a+2b)\sqrt{ab} \text{ and } a^2-b^2+(a-b)\sqrt{ab}.$$

The 1st expression $=a^2+a\sqrt{ab}+2b\sqrt{ab}+2b^2$

$$=a^2+a^{\frac32}b^{\frac12}+2a^{\frac12}b^{\frac32}+2b^2$$

$$=a^{\frac32}\left(a^{\frac12}+b^{\frac12}\right)+2b^{\frac32}\left(a^{\frac12}+b^{\frac12}\right)$$

$$=\left(a^{\frac12}+b^{\frac12}\right)\left(a^{\frac32}+2b^{\frac32}\right).$$

The 2nd expression $=a^2+a\sqrt{ab}-b\sqrt{ab}-b^2$

$$=a^2+a^{\frac32}b^{\frac12}-a^{\frac12}b^{\frac32}-b^2$$

$$=a^{\frac32}\left(a^{\frac12}+b^{\frac12}\right)-b^{\frac32}\left(a^{\frac12}+b^{\frac12}\right)$$

$$=\left(a^{\frac12}+b^{\frac12}\right)\left(a^{\frac32}-b^{\frac32}\right).$$

Hence, since $a^{\frac32}+2b^{\frac32}$ and $a^{\frac32}-b^{\frac32}$ have no common factor, the H.C.F. required

$$=a^{\frac12}+b^{\frac12}=\sqrt{a}+\sqrt{b}$$

Example 5. Simplify $\dfrac{x+(xy^2)^{\frac{1}{3}}-(x^2y)^{\frac{1}{3}}}{x+y}$.

$$\text{The numerator}=x+x^{\frac{1}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{1}{3}}$$

$$=x^{\frac{1}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}\right);$$

$$\text{and the denominator}=\left(x^{\frac{1}{3}}\right)^3+\left(y^{\frac{1}{3}}\right)^3$$

$$=\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)\left\{\left(x^{\frac{1}{3}}\right)^2-\left(x^{\frac{1}{3}}\right)\left(y^{\frac{1}{3}}\right)+\left(y^{\frac{1}{3}}\right)^2\right\}$$

$$=\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)\left(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}\right).$$

$$\text{Hence, the given expression}=\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}.$$

Example 6. Simplify $\left(\dfrac{x^l}{x^m}\right)^{l+m}\cdot\left(\dfrac{x^m}{x^n}\right)^{m+n}\cdot\left(\dfrac{x^n}{x^l}\right)^{n+l}$. [G. U. 1948, 1950]

$$\left(\frac{x^l}{x^m}\right)^{l+m}\cdot\left(\frac{x^m}{x^n}\right)^{m+n}\cdot\left(\frac{x^n}{x^l}\right)^{n+l}$$

$$=(x^{l-m})^{l+m}.(x^{m-n})^{m+n}.(x^{n-l})^{n+l}$$

$$=x^{l^2-m^2}.x^{m^2-n^2}.x^{n^2-l^2}=x^{l^2-m^2+m^2-n^2+n^2-l^2}=x^0=1$$

Example 7. Show that

$$\frac{1}{1+x^{m-n}+x^{m-p}}+\frac{1}{1+x^{n-m}+x^{n-p}}+\frac{1}{1+x^{p-m}+x^{p-n}}=1.$$

$$\text{The 1st term}=\frac{x^{-m}}{x^{-m}(1+x^{m-n}+x^{m-p})}=\frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}};$$

$$\text{the 2nd term}=\frac{x^{-n}}{x^{-n}(1+x^{n-m}+x^{n-p})}=\frac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}};$$

$$\text{and the 3rd term}=\frac{x^{-p}}{x^{-p}(1+x^{p-m}+x^{p-n})}=\frac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}.$$

Hence, the given expression

$$=\frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}+\frac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}}+\frac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}$$

$$=\frac{x^{-m}+x^{-n}+x^{-p}}{x^{-m}+x^{-n}+x^{-p}}=1.$$

N.B. In this type of sums each term is to be so reduced that the denominators are the same.

Example 8. If $a^b = b^a$, show that $\left(\frac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$; and if $a = 2b$, show that

Since $a^b = b^a$, $\therefore \quad a = b^{\frac{a}{b}}$ [extracting the bth root of both sides]

Hence, $\left(\frac{a}{b}\right)^{\frac{a}{b}} = \frac{a^{\frac{a}{b}}}{b^{\frac{a}{b}}} = \frac{a^{\frac{a}{b}}}{a} = a^{\frac{a}{b}-1}$.

If $a = 2b$, from the given relation, we have

$$(2b)^b = (b)^{2b} = (b^2)^b; \quad \therefore \quad 2b = b^2, \quad \therefore \quad b = 2.$$

Example 9. If $x = (a + \sqrt{a^2 + b^3})^{\frac{1}{3}} + (a - \sqrt{a^2 + b^3})^{\frac{1}{3}}$, show that $x^3 + 3bx - 2a = 0$.

Putting m for $a + \sqrt{a^2 + b^3}$, and n for $a - \sqrt{a^2 + b^3}$, we have

$$x^3 = \left(m^{\frac{1}{3}} + n^{\frac{1}{3}}\right)^3 = \left(m^{\frac{1}{3}}\right)^3 + \left(n^{\frac{1}{3}}\right)^3 + 3m^{\frac{1}{3}}.n^{\frac{1}{3}}\left(m^{\frac{1}{3}} + n^{\frac{1}{3}}\right)$$

$$= m + n + 3(mn)^{\frac{1}{3}}\left(m^{\frac{1}{3}} + n^{\frac{1}{3}}\right) = m + n + 3(mn)^{\frac{1}{3}}.x.$$

But $\quad m + n = 2a$,

and $(mn)^{\frac{1}{3}} = \{a^2 - (a^2 + b^3)\}^{\frac{1}{3}}$

$= (-b^3)^{\frac{1}{3}} = -b$;

$\therefore \quad x^3 = 2a - 3bx. \quad \therefore \quad x^3 + 3bx - 2a = 0.$

EXERCISE 104

Multiply :

1. $x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1$ by $x^{\frac{1}{3}} - 2x^{\frac{1}{6}} + 1$.
2. $a^{\frac{2}{3}} + 3a^{\frac{1}{3}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - 3b^{\frac{1}{3}}$.
3. $1 + ab^{-1} + a^2b^{-2}$ by $1 - ab^{-1} + a^2b^{-2}$
4. $x + 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$ by $x - 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$.
5. $x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$ by $x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.
6. $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}}$.
7. $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} - z^{\frac{1}{3}}x^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.
8. $a^m + 3b^n - 2c^p$ by $a^m - 3b^n + 2c^p$.

9. $a^{\frac{5}{2}}+8ab+4a^{\frac{3}{2}}b^{\frac{2}{3}}+2a^2b^{\frac{1}{3}}+32b^{\frac{5}{3}}+16a^{\frac{1}{2}}b^{\frac{4}{3}}$ by $a^{\frac{1}{2}}-2b^{\frac{1}{3}}$.

10. $a^{\frac{5}{8}}+a^{\frac{1}{4}}x^{-\frac{3}{8}}+x^{-\frac{5}{8}}+a^{\frac{3}{8}}x^{-\frac{1}{4}}+a^{\frac{1}{8}}x^{-\frac{1}{2}}+a^{\frac{1}{2}}x^{-\frac{1}{8}}$ by $a^{\frac{3}{8}}+a^{\frac{1}{8}}x^{-\frac{1}{4}}-x^{-\frac{3}{8}}-a^{\frac{1}{4}}x^{-\frac{1}{8}}$

Divide :

11. $x^{\frac{5}{2}}-4x^{\frac{3}{2}}-2x^{\frac{1}{2}}+6x-x^2$ by $x^{\frac{3}{2}}+2-4x^{\frac{1}{2}}$.

12. $8+12x^{-1}+2x^{-3}+2x^{-4}$ by $x^{-2}-2x^{-1}+4$.

13. $xy^{-1}+2x^{\frac{1}{2}}y^{-\frac{1}{2}}+3+2x^{-\frac{1}{2}}y^{\frac{1}{2}}+x^{-1}y$ by $x^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

14. $a^{\frac{5}{2}}-a^{\frac{3}{2}}b+ab^{\frac{3}{2}}-2a^{\frac{1}{2}}b^2+b^{\frac{5}{2}}$ by $a^{\frac{3}{2}}-ab^{\frac{1}{2}}+a^{\frac{1}{2}}b-b^{\frac{3}{2}}$.

15. $8x^{-n}-8x^n+5x^{3n}-3x^{-3n}$ by $5x^n-3x^{-n}$.

16. $8x^{\frac{3}{2}}+y^{-\frac{3}{2}}-z+6x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{3}}$ by $2x^{\frac{1}{2}}+y^{-\frac{1}{2}}-z^{\frac{1}{3}}$.

17. Show that $x^3+a^3+x^{\frac{3}{2}}a^{\frac{3}{2}}$ is divisible by $x^{\frac{3}{4}}+a^{\frac{3}{4}}+x^{\frac{3}{8}}a^{\frac{3}{8}}$.

18. Multiply $x^{2^{n-1}}+a^{2^{n-1}}$ by $x^{2^{n-1}}-a^{2^{n-1}}$.

19. Divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$. [C. U. 1879]

20. Simplify $\{(a^m)^{m-\frac{1}{m}}\}^{\frac{1}{m+1}}$.

21. Divide $2x^{-\frac{1}{4}}+3x^{\frac{3}{4}}-7x^{\frac{1}{4}}+x-2x^{\frac{1}{2}}$ by $x^{\frac{1}{4}}-2x^{-\frac{1}{4}}$.

22. Find the square of $x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-\frac{1}{4}}+y^{\frac{1}{2}}$.

23. Divide $x^{\frac{-3n}{2}}-a^{\frac{-3n}{2}}$ by $x^{\frac{n}{2}}-a^{\frac{n}{2}}$.

24. Find the square of $x^{\frac{1}{3}}-2x^{\frac{1}{2}}+x^{\frac{5}{6}}$.

25. Divide $ax^{-1}+a^{-1}x+2$ by $a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}-1$.

26. Simplify $\left(\dfrac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{a-b}\right)^{-1}$

Simplify the following :

27. $\dfrac{x^{\frac{1}{3}}+3y^{\frac{1}{3}}}{x^{\frac{1}{3}}-3y^{\frac{1}{3}}}+\dfrac{x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}{x^{\frac{2}{3}}+3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}$.

28. $$\frac{a^{\frac{3}{2}} - ax^{\frac{1}{2}} + a^{\frac{1}{2}}x - x^{\frac{3}{2}}}{a^{\frac{5}{2}} - a^2x^{\frac{1}{2}} + 3a^{\frac{3}{2}}x - 3ax^{\frac{3}{2}} + a^{\frac{1}{2}}x^2 - x^{\frac{5}{2}}}.$$

29. $$\frac{a^2+b^2-a^{-2}-b^{-2}}{a^2b^2-a^{-2}b^{-2}} + \frac{(a-a^{-1})(b-b^{-1})}{ab+a^{-1}b^{-1}}$$

30. $$\frac{x-y}{x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}} + \frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}}.$$

31. $(a+b+c)(a^{-1}+b^{-1}+c^{-1})-a^{-1}b^{-1}c^{-1}(b+c)(c+a)(a+b).$

32. $$\frac{a^{-1}(ab^{-1}-1)^2}{b^{-2}(1+a^{-1}b)} \times \frac{b^2(a^{-2}+b^{-2})}{a(ab^{-1}-a^{-1}b)} + \frac{1-a^{-1}b}{ab^{-1}+1}.$$

33. $$\frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x-a}\left(\frac{x}{x^{\frac{1}{3}}+a^{\frac{1}{3}}}+a^{\frac{1}{3}}\right)+\frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x+a}\left(\frac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}-a^{\frac{1}{3}}}-a^{\frac{1}{3}}\right).$$

34. $$\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}, \text{ when } x=\sqrt{\frac{a-b}{a+b}}.$$

35. Show that

$$\frac{x^{2^n}-y^{2^n}}{x-y}=(x+y)(x^2+y^2)(x^4+y^4)\cdots\cdots(x^{2^{n-1}}+y^{2^{n-1}}).$$

36. Write down, without actual division, the value of

$$\left(\frac{x^{-8}}{256}-\frac{y^{-12}}{625}\right)+\left(\frac{1}{4}x^{-2}+\frac{1}{5}y^{-3}\right).$$

Simplify :

37. $$\frac{\left(p+\frac{1}{q}\right)^m\left(p-\frac{1}{q}\right)^m}{\left(q+\frac{1}{p}\right)^m\left(q-\frac{1}{p}\right)^m}.$$ [B. U. 1889]

38. $$\frac{\left(p^2-\frac{1}{q^2}\right)^p\left(p-\frac{1}{q}\right)^{q-p}}{\left(q^2-\frac{1}{p^2}\right)^q\left(q+\frac{1}{p}\right)^{p-q}}.$$ [B. U. 1891]

39. $$\left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2}\times\left(\frac{x^m}{x^n}\right)^{m^2+mn+n^2}\times\left(\frac{x^n}{x^l}\right)^{n^2+nl+l^2}.$$ [C. U. 1904]

40. $$\frac{1}{1+x^{b-a}+x^{c-a}}+\frac{1}{1+x^{a-b}+x^{c-b}}+\frac{1}{1+x^{a-c}+x^{b-c}}.$$ [C. U. 1940]

41. $\sqrt[bc]{\dfrac{x^{\frac{b}{c}}}{x^{\frac{c}{b}}}}\cdot\sqrt[ca]{\dfrac{x^{\frac{c}{a}}}{x^{\frac{a}{c}}}}\cdot\sqrt[ab]{\dfrac{x^{\frac{b}{a}}}{x^{\frac{a}{b}}}}$. [C. U. 1938]

42. If $x=a^{\frac{1}{3}}-a^{-\frac{1}{3}}$, show that $x^3+3x=a-\dfrac{1}{a}$.

43. If $x=2+2^{\frac{2}{3}}+2^{\frac{1}{3}}$, show that $x^3-6x^2+6x-2=0$. [C. U. 1950]

44. Express the value of m in terms of n, when $a^{m^n}=(a^m)^n$. [P. U 1918]

45. If $x^{\frac{1}{m}}=y^{\frac{1}{n}}=z^{\frac{1}{p}}$ and $xyz=1$, prove that $m+n+p=0$.

195. Exponential Equations.

Definition : An equation in which the variable (or variables) occurs (or occur) as *indices* or *exponents*, is called **an exponential equation.**

Thus, $3^x=27$, $81^x=9^{x+4}$, etc. are called exponential equations ; likewise, $3^{x+y}=9$ and $8^x.16^y=128$ are a pair of simultaneous exponential equations.

The method of solution of exponential equations is based on the following axiom :

If $a^x=a^m$, whatever a may be, then will $x=m$.

Thus, to solve an exponential equation, we have

(i) *to reduce both the members of the equation to the same base*, and then,

(ii) *to equate their exponents.*

The following examples will illustrate the process :

Example 1. Solve $3.27^x=9^{x+4}$.

The left-hand side $=3.27^x=3.(3^3)^x=3.3^{3x}=3^{3x+1}$;
and the right-hand side $=9^{x+4}=(3^2)^{x+4}=3^{2(x+4)}=3^{2x+8}$.

$\therefore$ the equation reduces to $3^{3x+1}=3^{2x+8}$.

$\therefore\quad 3x+1=2x+8\ ;\qquad\therefore\quad x=7.$

Example 2. Solve the equation $a^{-x}(a^x+b^{-x})=\dfrac{a^2b^2+1}{a^2b^2}$.

We have $a^{-x}(a^x+b^{-x})=a^{-x}.a^x+a^{-x}.b^{-x}=1+(ab)^{-x}$.

$\therefore\ 1+(ab)^{-x}=1+\dfrac{1}{a^2b^2}=1+(ab)^{-2}$.

$\therefore\quad (ab)^{-x}=(ab)^{-2}\ ;\qquad\therefore\quad x=2.$

Example 3. Solve $a^x.a^{y+1}=a^7$ $\cdots$ (1)
$a^{2y}.a^{3x+5}=a^{20}.$ $\cdots$ (2)

From the 1st equation,

$$a^{x+y+1}=a^7\ ; \quad \therefore \quad x+y+1=7. \qquad \cdots \quad (3)$$

From the 2nd equation,

$$a^{2y+3x+5}=a^{20}\ ; \quad \therefore \quad 2y+3x+5=20. \qquad \cdots \quad (4)$$

From equations (3) and (4), we have, $x+y-6=0$,
and $3x+2y-15=0$.

$\therefore$ by cross-multiplication,

$$\frac{x}{-15+12}=\frac{y}{-18+15}=\frac{1}{2-3}, \quad \text{or,} \quad \frac{x}{-3}=\frac{y}{-3}=-1.$$

Hence, $x=3$ and $y=3$.

Example 4. Solve the equation $4.3^{x+1}=27+9^x$. [C. U. 1951]

$4.3^{x+1}=27+9^x$,

or, $4.3^x.3=27+3^{2x}$,

or, $12.3^x=27+(3^x)^2$,

or, $12p=27+p^2$, [putting $3^x=p$]

or, $p^2-12p+27=0$,

or, $(p-9)(p-3)=0$;

$\therefore \quad p=9$ or 3.

$\because \quad 3^x=9$ or $3=3^2$ or 3^1 ; $\therefore \quad x=2$ or 1.

EXERCISE 105

Solve the following equations :

1. $2^{x+7}=4^{x+2}$. 2. $(\sqrt{3})^{x+5}=(\sqrt[3]{3})^{2x+5}$. 3. $(\sqrt[5]{4})^{4x+7}=(\sqrt[11]{64})^{2x+7}$

4. $(\sqrt[3]{25})^{2x+1}=(\sqrt[5]{125})^{x+1}$. 5. $2^{x-4}=4a^{x-6}$, ($a \neq 0$).

6. $\left(\frac{a}{b}\right)^{cx-d}=\left(\frac{b}{a}\right)^{ax-b}$. 7. $2^{3x-5}.a^{x-2}=2^{x-3}.2a^{1-x}$.

8. $\frac{3^{3x-4}.a^{2x-5}}{3^{x+1}}=a^{2x-5}$. 9. $a^{x-2}(a^{2x+2}+a^{1-x})=a^{-3}(a^9+a^2)$.

10. $2^{x+1}-2^x-8=0$. 11. $3^{x+5}=3^{x+3}+\frac{8}{3}$.

12. $4^{x+2}=2^{2x+1}+14$. 13. $4^x-3.2^{x+2}+32=0$.

14. $4^x+\frac{1}{4^x}=16\frac{1}{16}$. 15. $x^y=y^x$ and $x=2y$.

16. $2^{x+1}.3^{y+2}=\frac{1}{6}$ and $2^{2x+1}.3^{3y+5}=\frac{1}{648}$.

17. $a^{x+1}.a^{y+2}=a^5$ and $a^{x+1}.a^{2(y+1)}=a^{11}$.

18. $a^{x-3}.a^{y+2}=a^2.a^x$ and $a^x.a^y=a^4$.

19. $a^{2x+1}.a^{3y+1}=a^8=a^{x+1}.a^{y+1}$. 20. $2^{x+y}=2^{2x-y}=\sqrt{8}$.

21. $2^x.3^y=18$ and $2^{2x}.3^y=36$. 22. $x^{2x+1}=2y^x$ and $x^{y-4}=1$.

23. $x^{y+na}=y^{x+nb}$ and $x^a=y^b$.

24. If $a^{n.m}=(a^m)^n$, find m in terms of n. [Pat. 1918]

25. If $a^x=b$, $b^y=c$ and $c^z=a$, prove that $xyz=1$.

26. If $a^x=m$, $a^y=n$ and $a^2=(m^yn^x)^z$, prove that $xyz=1$.

[Pat. 1919. '21]

27. If $x=3^{\frac{1}{3}}+3^{-\frac{1}{3}}$, prove that $3x^3-9x-10=0$.

28. If $a^2+2=3^{\frac{2}{3}}+3^{-\frac{2}{3}}$, prove that $3a^3+9a=8$.

29. If $xy^{p-1}=a$, $xy^{q-1}=b$, $xy^{r-1}=c$, show that

$$a^{q-r}.b^{r-p}.c^{p-q}=1.$$

[Pat. 1920]

Solve :

30. $$\left.\begin{aligned}2^{x+y+z}&=8^{x+z-y}\\5^{3y+2}&=25^{x+z}\\3^{2x+2z+y}&=9^{3x+y}\end{aligned}\right\}$$

31. $$\left.\begin{aligned}(\sqrt{a})^{x+y}&=(\sqrt[3]{a})^{y+z-1}\\(\sqrt[3]{b})^{x+z-2}&=(\sqrt[5]{b})^{y+z}\\(\sqrt[4]{c})^{y}&=(\sqrt[7]{c})^{x+y+z}\end{aligned}\right\}$$

32. $$\left.\begin{aligned}a^x&=(x+y+z)^y,\\a^y&=(x+y+z)^z,\\\text{and}\quad a^z&=(x+y+z)^x.\end{aligned}\right\}$$

CHAPTER XXX

ELEMENTARY SURDS

196. Definition. Any root of an arithmetical number which cannot be exactly found is called a **surd** or an **irrational quantity.** Thus, $\sqrt{2}$, $\sqrt{6}$, $\sqrt[3]{4}$ and $\sqrt[4]{5}$ are all surds.

Note. *Quantities which are not surds are called* ***rational quantities.*** *Hence, every root of an arithmetical number is either* ***rational*** *or* ***irrational.*** *Thus,* $\sqrt[3]{8}$, $\sqrt{25}$ *and* $\sqrt[4]{16}$ *are* ***rational quantities****, whilst* $\sqrt{2}$, $\sqrt[3]{5}$ *and* $\sqrt[4]{9}$ *are all* ***irrational quantities.***

An algebraical expression also, such as $\sqrt{x}$, is called a surd, although the value of x may be such that $\sqrt{x}$ is not in reality a surd For instance, if $x=4$, $\sqrt{x}=\sqrt{4}=2$, and is, therefore, not really a surd.

197. To express in the form of a surd the product of a rational quantity and a surd.

Example 1. $5\sqrt{3}=(5^2)^{\frac{1}{2}}\times 3^{\frac{1}{2}}=(5^2\times 3)^{\frac{1}{2}}$ [Art. 192]

$$=\sqrt{5^2\times 3}=\sqrt{75}.$$

Example 2. $2\sqrt[3]{9}=(2^3)^{\frac{1}{3}}\times 9^{\frac{1}{3}}=(2^3\times 9)^{\frac{1}{3}}$ [Art. 192]

$$=\sqrt[3]{2^3\times 9}=\sqrt[3]{72}.$$

EXERCISE 106

Express as a complete surd :

1. $3\sqrt{5}$. 2. $2\sqrt[3]{3}$. 3. $2\sqrt[4]{6}$. 4. $4\sqrt[4]{5}$.
5. $a\sqrt[n]{b}$. 6. $x^3\sqrt[a]{y}$. 7. $a^4\sqrt[5]{b^3}$.

198. A surd may sometimes be expressed as the product of a rational quantity and a surd.

Example 1. $\sqrt{32}=\sqrt{16\times 2}=(4^2\times 2)^{\frac{1}{2}}=(4^2)^{\frac{1}{2}}\times 2^{\frac{1}{2}}$ [Art. 192]

$$=4\times 2^{\frac{1}{2}}=4\sqrt{2}.$$

Example 2. $\sqrt[3]{40}=\sqrt[3]{8\times 5}=(2^3\times 5)^{\frac{1}{3}}=(2^3)^{\frac{1}{3}}\times 5^{\frac{1}{3}}$ [Art. 192]

$$=2\times 5^{\frac{1}{3}}=2\sqrt[3]{5}.$$

Example 3. $\sqrt[3]{-250.c^4}=\sqrt[3]{(-5c)^3.2c}$

$$=\{(-5c)^3\}^{\frac{1}{3}}\times(2c)^{\frac{1}{3}}=-5c\sqrt[3]{2c}.$$

EXERCISE 107

Simplify :

1. $\sqrt{18}$.
2. $\sqrt{80}$.
3. $\sqrt[3]{250}$.
4. $\sqrt[5]{128}$.
5. $\sqrt[4]{405}$.
6. $\sqrt[3]{1372}$.
7. $\sqrt[4]{1875}$.
8. $\sqrt[3]{a^6 b}$.
9. $\sqrt[n]{x^{4n}a}$.
10. $\sqrt[3]{-2560}$.
11. $\sqrt[3]{-192a^5 b^4}$.
12. $\sqrt[3]{500a^7 x^4}$.

199. **Similar Surds.** Two or more surds are said to be *similar* or *like* when they can be so reduced as to have the same irrational factor ; otherwise, they are said to be *unlike*. Thus, $\sqrt{45}$ and $\sqrt{80}$ are similar surds, for they are respectively equivalent to $3\sqrt{5}$ and $4\sqrt{5}$, and $\sqrt{12}$ and $\sqrt{20}$ are unlike. The sum of any number of similar surds may be found when they have been expressed in their simplest form.

Example 1. $\sqrt{147} + \sqrt{27} = \sqrt{49 \times 3} + \sqrt{9 \times 3} = 7\sqrt{3} + 3\sqrt{3} = 10\sqrt{3}$

Example 2.

$$\begin{aligned}&\sqrt[3]{625} - \sqrt[3]{135} + \sqrt[3]{40}\\ &= \sqrt[3]{125 \times 5} - \sqrt[3]{27 \times 5} + \sqrt[3]{8 \times 5}\\ &= \sqrt[3]{5^3 \times 5} - \sqrt[3]{3^3 \times 5} + \sqrt[3]{2^3 \times 5}\\ &= 5\sqrt[3]{5} - 3\sqrt[3]{5} + 2\sqrt[3]{5} = 4\sqrt[3]{5}.\end{aligned}$$

N. B. **Unlike surds cannot be added to give one quantity as the sum.**

EXERCISE 108

Simplify :

1. $\sqrt{12} + \sqrt{75}$.
2. $\sqrt{18} + \sqrt{32}$.
3. $\sqrt{20} + \sqrt{180}$.
4. $\sqrt{98} - \sqrt{50}$.
5. $\sqrt[3]{128} - \sqrt[3]{54}$.
6. $\sqrt[4]{80} + \sqrt[4]{405}$.
7. $\sqrt[4]{768} - \sqrt[4]{243}$.
8. $2\sqrt{27} - \sqrt{75} + \sqrt{12}$.
9. $2\sqrt{405} - 3\sqrt{125} + \sqrt{45}$.
10. $4\sqrt[3]{192} - 4\sqrt[3]{375} + 2\sqrt[3]{24}$.
11. $3\sqrt[3]{40} + 2\sqrt[3]{625} - 4\sqrt[3]{320}$.
12. $5\sqrt[3]{-54} - 2\sqrt[3]{-16} + 4\sqrt[3]{686}$
13. $\sqrt{45x^3} + \sqrt{80x^3} + \sqrt{5xy^2}$.
14. $x\sqrt[3]{x^3 a} + y\sqrt[3]{-8y^3 a} - z\sqrt[3]{-27z^3 a}$.
15. $2\sqrt[4]{32a^4 x} + 3\sqrt[4]{512a^4 x} - 4a\sqrt[4]{162x}$.

200. **Surds of the same order.** Surds are said to be *of the same order* or *equiradical* when they have all got the *same root-symbol* Thus, $\sqrt{5}$, $\sqrt{a^3}$ and $(a+x)^{\frac{5}{2}}$ are all surds of the same (*viz.*, the *second*) order.

A surd of the second order is often called a **quadratic surd** ; **whilst** one of the third order, such as, $\sqrt[3]{4}$ or $\sqrt[3]{a^2}$, is called a **cubic surd.**

Surds of different orders may be reduced to equivalent surds of the same order.

Example 1. Reduce $\sqrt{5}$ and $\sqrt[3]{4}$ to surds of the same order

The given surds are respectively of the 2nd and 3rd orders ; and the L.C.M. of 2 and 3 is 6. Hence, we can at once reduce them to surds of the 6th order, thus :

$$\sqrt{5}=5^{\frac{1}{2}}=5^{\frac{3}{6}}=\sqrt[6]{5^3}=\sqrt[6]{125}\,;$$

$$\sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{2}{6}}=\sqrt[6]{4^2}=\sqrt[6]{16}.$$

Thus, the required surds are $\sqrt[6]{125}$ and $\sqrt[6]{16}$.

Example 2. Reduce $\sqrt[6]{3}$ and $\sqrt[8]{2}$ to surds of the same order.

The L.C.M. of 6 and 8 is 24.

Thus, we have $\sqrt[6]{3}=3^{\frac{1}{6}}=3^{\frac{4}{24}}=\sqrt[24]{3^4}=\sqrt[24]{81}\,;$

and $\sqrt[8]{2}=2^{\frac{1}{8}}=2^{\frac{3}{24}}=\sqrt[24]{2^3}=\sqrt[24]{8}.$

Thus, the required surds are $\sqrt[24]{81}$ and $\sqrt[24]{8}$.

Example 3. Which is the greater : $\sqrt[3]{9}$ or $\sqrt[4]{20}$?

We have $\sqrt[3]{9}=9^{\frac{1}{3}}=9^{\frac{4}{12}}=\sqrt[12]{9^4}=\sqrt[11]{6561}\,;$

and $\sqrt[4]{20}=20^{\frac{1}{4}}=20^{\frac{3}{12}}=\sqrt[12]{20^3}=\sqrt[12]{8000}.$

Thus, the given surds are respectively equivalent to $\sqrt[12]{6561}$ and $\sqrt[12]{8000}$, and as the latter is greater than the former, therefore $\sqrt[4]{20}>\sqrt[3]{9}$.

EXERCISE 109

Reduce to surds of the same order :

1. $\sqrt{3}$ and $\sqrt[3]{2}$. 2. $\sqrt[3]{4}$ and $\sqrt[4]{5}$. 3. $\sqrt[5]{2}$ and $\sqrt[3]{3}$.
4. $\sqrt[4]{3}$ and $\sqrt[6]{5}$. 5. $\sqrt[6]{4}$ and $\sqrt[8]{6}$.

Which is greater :

6. $\sqrt{2}$ or $\sqrt[3]{3}$? 7. $\sqrt[3]{3}$ or $\sqrt[4]{4}$? 8. $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Arrange according to descending order of magnitude :

9. $\sqrt[4]{6}$, $\sqrt{2}$ and $\sqrt[3]{4}$. 10. $\sqrt[4]{3}$, $\sqrt[6]{10}$ and $\sqrt[12]{25}$.

201. Multiplication and Division of Surds.

A. (i) To multiply surds of the same order, multiply separately the rational factors and irrational factors.

(ii) If the surds are not of the same order, reduce them to surds of the same lowest order and multiply as in (i) above.

Example 1. $\sqrt[3]{6} \times \sqrt[3]{10} = 6^{\frac{1}{3}} \times 10^{\frac{1}{3}} = (6 \times 10)^{\frac{1}{3}} = \sqrt[3]{60}$.

Note. *In this example the given surds are of the same order.*

Example 2. $\sqrt[4]{5} \times \sqrt[6]{8} = 5^{\frac{1}{4}} \times 8^{\frac{1}{6}} = 5^{\frac{3}{12}} \times 8^{\frac{2}{12}}$

$= (5^3)^{\frac{1}{12}} \times (8^2)^{\frac{1}{12}}$ [Art. 191]

$= (5^3 \times 8^2)^{\frac{1}{12}}$ [Art. 192]

$= \sqrt[12]{125 \times 64} = \sqrt[12]{8000}$.

Note *In this example the given surds are of* ***different*** *orders.*

Example 3. $\sqrt[3]{2} \times \sqrt[5]{2} = 2^{\frac{1}{3}} \times 2^{\frac{1}{5}} = 2^{\frac{1}{3}+\frac{1}{5}} = 2^{\frac{8}{15}} = \sqrt[15]{2^8} = \sqrt[15]{256}$.

Note. *In this example the given surds have got the same quantity under the radical sign. They may as well be regarded as surds of different orders and treated like those in the last example.*

Example 4. $4\sqrt{18} \times \sqrt{75} = 4.3\sqrt{2} \times 5\sqrt{3} = 60\sqrt{2}.\sqrt{3} = 60\sqrt{6}$.

Note. *In this example the given surds have been reduced to simpler forms before multiplication.*

B. (i) To divide a surd by another surd of the same order, form a fraction with the dividend and the divisor as numerator and denominator respectively and cancel the common factors, if any.

(ii) If the surds are not of same order, reduce them to surds of the same lowest order and proceed as in B (i) above.

Example 5. $\sqrt[6]{4} \div \sqrt[4]{6} = 4^{\frac{1}{6}} \div 6^{\frac{1}{4}} = 4^{\frac{2}{12}} \div 6^{\frac{3}{12}}$

$= \dfrac{(4^2)^{\frac{1}{12}}}{(6^3)^{\frac{1}{12}}}$ [Art. 191]

$= \left(\dfrac{4^2}{6^3}\right)^{\frac{1}{12}}$ [Cor. 1, Art. 192]

$= \sqrt[12]{\dfrac{2}{27}}$.

Example 6. Express $\sqrt{5} \div 3\sqrt{3}$ as a fraction with a rational denominator.

We have $\sqrt{5} \div 3\sqrt{3} = \dfrac{\sqrt{5}}{3\sqrt{3}} = \dfrac{\sqrt{5} \times \sqrt{3}}{3\sqrt{3} \times \sqrt{3}} = \dfrac{\sqrt{15}}{3 \times 3} = \dfrac{\sqrt{15}}{9}$.

Note. ***For Arithmetical calculations it is convenient to reduce the quotient when one surd is divided by another to the form of a fraction with a rational denominator. Hence, even when the numerical value of a surd fraction is not required it is usual to express it in the above form.***

EXERCISE 110

Simplify :

1. $\sqrt{5} \times \sqrt{10}$.
2. $\sqrt{8} \times \sqrt{6}$.
3. $\sqrt{27} \times \sqrt{3}$.
4. $\sqrt{15} \times \sqrt{6}$.
5. $\sqrt{20} \times \sqrt{45}$.
6. $\sqrt[6]{5} \times \sqrt[9]{25}$.
7. $\sqrt[3]{6ax} \times \sqrt[3]{27a^2x^3}$.
8. $\sqrt[3]{2} \times \sqrt{6}$.
9. $\sqrt{2} \times \sqrt[3]{6}$.
10. $\sqrt[3]{4} \times \sqrt{8}$.
11. $\sqrt[3]{9} \times \sqrt{27}$.
12. $\sqrt[6]{2} \times \sqrt[9]{3}$.
13. $\sqrt[4]{3} \times \sqrt[6]{3}$.
14. $\sqrt[6]{2} \times \sqrt[9]{2}$.
15. $\sqrt[4]{4} \times \sqrt[6]{4}$.
16. $5\sqrt{8} \times 2\sqrt{6}$.
17. $8\sqrt{12} \times 3\sqrt{24}$.
18. $4\sqrt[3]{72} \times 5\sqrt[3]{576}$.
19. $7\sqrt[3]{8a^3x^2} \times 5\sqrt[3]{27b^3x^2}$.
20. $8\sqrt{10} + 4\sqrt{15}$.
21. $3\sqrt{12} + 6\sqrt{27}$.
22. $\sqrt[3]{36} + \sqrt[3]{48}$.
23. $\sqrt[9]{8} + \sqrt[6]{6}$.

Given $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{5} = 2{\cdot}236$, find to 3 places of decimals the numerical value of :

24. $\sqrt{2} + \sqrt{6}$.
25. $\sqrt{72} + \sqrt{40}$.
26. $\sqrt{275} + \sqrt{22}$.
27. $10\sqrt{108} + \sqrt{15}$.

202. Compound Surds. An expression consisting of two or more simple surds connected by the sign + or − is called a **compound surd.** Thus, $5\sqrt{2}$ and $4\sqrt{3}$ are simple surds, but $5\sqrt{2}+4\sqrt{3}$ and $5\sqrt{2}-4\sqrt{3}$ are compound surds.

Two or more compound surds are multiplied together in the same way as two or more compound algebraical expressions.

Example 1. Multiply $3\sqrt{x}+2\sqrt{3}$ by $\sqrt{x}-\sqrt{3}$.

$$(3\sqrt{x}+2\sqrt{3})(\sqrt{x}-\sqrt{3}) = 3\sqrt{x}.\sqrt{x}+2\sqrt{3}.\sqrt{x}-3\sqrt{x}.\sqrt{3}-2\sqrt{3}.\sqrt{3}$$
$$= 3x+2\sqrt{3x}-3\sqrt{3x}-6 = 3x-\sqrt{3x}-6.$$

Example 2. Multiply $7\sqrt{2}+\sqrt{3}$ by $7\sqrt{2}-\sqrt{3}$.

$$(7\sqrt{2}+\sqrt{3})(7\sqrt{2}-\sqrt{3}) = (7\sqrt{2})^2-(\sqrt{3})^2 = 49.2-3 = 98-3 = 95.$$

Example 3. Find the square of $\sqrt{3a+x}+\sqrt{3a-x}$.

$$(\sqrt{3a+x}+\sqrt{3a-x})^2 = (\sqrt{3a+x})^2+(\sqrt{3a-x})^2+2\sqrt{3a+x}.\sqrt{3a-x}$$
$$= (3a+x)+(3a-x)+2\sqrt{9a^2-x^2}$$
$$= 6a+2\sqrt{9a^2-x^2}.$$

EXERCISE 111

Multiply :

1. $\sqrt{a}+\sqrt{b}$ by $\sqrt{ab}$.
2. $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$.
3. $3\sqrt{a}-5$ by $2\sqrt{a}$.
4. $4\sqrt{x}+3\sqrt{y}$ by $4\sqrt{x}-3\sqrt{y}$.
5. $2\sqrt{x-5}+4$ by $3\sqrt{x-5}-6$
6. $3\sqrt{5}-4\sqrt{2}$ by $2\sqrt{5}+3\sqrt{2}$.

7. $\sqrt{2}+2\sqrt{3}+\sqrt{7}$ by $\sqrt{2}+2\sqrt{3}-\sqrt{7}$.

8. $3-\sqrt{5}+\sqrt{8}$ by $3-\sqrt{5}-\sqrt{8}$.

9. $\sqrt{11}+\sqrt{6}-\sqrt{3}$ by $\sqrt{11}-\sqrt{6}+\sqrt{3}$.

10. $\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{48}$ by $\sqrt[3]{2}+\sqrt[3]{3}$.

Find the square of :

11. $\sqrt{x+a}-\sqrt{x-a}$. 12. $2\sqrt{8}+5\sqrt{6}$. 13. $2\sqrt{5}+3\sqrt{7}$.

14. $\sqrt{a^2+2b^2}-\sqrt{a^2-2b^2}$. 15. $2\sqrt{x^2+y^2}+5\sqrt{x^2-y^2}$.

203. Rationalisation. If two surds be such that their product is *rational*, each of them is said to be rationalised when multiplied by the other. Thus, $2\sqrt{5}$ and $\sqrt{3}+\sqrt{2}$ are rationalised when respectively multiplied by $\sqrt{5}$ and $\sqrt{3}-\sqrt{2}$;

for $2\sqrt{5}\times\sqrt{5}=10$,

and $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=3-2=1$.

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be **conjugate** or **complementary** to each other. Thus, $\sqrt{3}+\sqrt{2}$ and $2\sqrt{5}-\sqrt{7}$ are respectively *conjugate* (or *complementary*) to $\sqrt{3}-\sqrt{2}$ and $2\sqrt{5}+\sqrt{7}$.

Evidently, therefore, *every binomial quadratic surd is rationalised when multiplied by the complementary surd.*

Hence, a fraction with a binomial quadratic surd for its denominator can be easily reduced to an equivalent fraction with a rational denominator.

Example 1. Given $\sqrt{2}=1{\cdot}414$, find to three places of decimals the value of $\dfrac{1+\sqrt{2}}{3-2\sqrt{2}}$.

$$\frac{1+\sqrt{2}}{3-2\sqrt{2}}=\frac{(1+\sqrt{2})(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})}=\frac{3+3\sqrt{2}+2\sqrt{2}+4}{9-8}$$
$$=7+5\sqrt{2}=7+5\times1{\cdot}414=7+7{\cdot}070=14{\cdot}070.$$

Example 2. Rationalise the denominator of

$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}.$$

The given expression

$$=\frac{(\sqrt{1+x^2}-\sqrt{1-x^2})^2}{(\sqrt{1+x^2}+\sqrt{1-x^2})(\sqrt{1+x^2}-\sqrt{1-x^2})}$$
$$=\frac{(1+x^2)+(1-x^2)-2\sqrt{1-x^4}}{(1+x^2)-(1-x^2)}$$
$$=\frac{2-2\sqrt{1-x^4}}{2x^2}=\frac{1-\sqrt{1-x^4}}{x^2}.$$

Example 3. Simplify $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

The denominator $=5\sqrt{3}-2\times 2\sqrt{3}-4\sqrt{2}+5\sqrt{2}=\sqrt{3}+\sqrt{2}$.

Hence, the given fraction

$$=\frac{3+\sqrt{6}}{\sqrt{3}+\sqrt{2}}=\frac{(3+\sqrt{6})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$=\frac{3\sqrt{3}+3\sqrt{2}-3\sqrt{2}-2\sqrt{3}}{3-2}=\sqrt{3}.$$

Example 4. Simplify $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

The 1st term

$$=\frac{3\sqrt{2}}{\sqrt{3}(1+\sqrt{2})}=\frac{\sqrt{6}}{\sqrt{2}+1}=\frac{\sqrt{6}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}=2\sqrt{3}-\sqrt{6}.$$

The 2nd term

$$=\frac{4\sqrt{3}}{\sqrt{2}(\sqrt{3}+1)}=\frac{2\sqrt{6}}{\sqrt{3}+1}=\frac{2\sqrt{6}(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$=\frac{2(3\sqrt{2}-\sqrt{6})}{2}=3\sqrt{2}-\sqrt{6}.$$

The 3rd term

$$=\frac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}=\frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}=3\sqrt{2}-2\sqrt{3}.$$

Hence, the given expression

$$=(2\sqrt{3}-\sqrt{6})-(3\sqrt{2}-\sqrt{6})+(3\sqrt{2}-2\sqrt{3})=0$$

EXERCISE 112

Reduce to an equivalent fraction with a rational denominator :

1. $\dfrac{5\sqrt{3}+\sqrt{7}}{4\sqrt{3}+2\sqrt{7}}$. 2. $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$. 3. $\dfrac{4+3\sqrt{2}}{3-2\sqrt{2}}$. 4. $\dfrac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

5. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$. 6. $\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$. 7. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$.

Given $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$, $\sqrt{5}=2{\cdot}236$, find to three places of decimals the value of :

8. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$. 9. $\dfrac{\sqrt{3}}{2-\sqrt{3}}$. 10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.

11. $\dfrac{3}{\sqrt{5}-\sqrt{2}}$. 12. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$. 13. $\dfrac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$.

Simplify :

14. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}$.

15. $\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$.

16. $\dfrac{\sqrt{2}(\sqrt{3}+1)(2-\sqrt{3})}{(\sqrt{2}-1)(3\sqrt{3}-5)(2+\sqrt{2})}$.

17. $\dfrac{4}{\sqrt{3}+\sqrt{5}-\sqrt{2}}$.

18. $(3+2\sqrt{2})^{-3}+(3-2\sqrt{2})^{-3}$.

19. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$.

20. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

Rationalise the denominator of :

21. $\dfrac{1}{\sqrt[3]{3}+\sqrt[3]{2}}$.

22. $\dfrac{1}{\sqrt[3]{4}-\sqrt[3]{3}}$.

204. The square root of a rational quantity cannot be partly rational and partly a quadratic surd.

If possible, let $\sqrt{n}=a+\sqrt{m}$.

Then, squaring both sides, we must have

$$n=a^2+m+2a\sqrt{m},$$

whence, $\sqrt{m}=\dfrac{n-a^2-m}{2a}$.

Thus, a surd is equal to a rational quantity, which is impossible.

205. If $a+\sqrt{b}=x+\sqrt{y}$, where a and x are rational, and $\sqrt{b}$ and $\sqrt{y}$ are irrational, then will $a=x$ and $b=y$.

For, if a be not equal to x, let $a=x+m$;

then, we have $x+m+\sqrt{b}=x+\sqrt{y}$;

$$\therefore\ m+\sqrt{b}=\sqrt{y}.$$

Thus, $\sqrt{y}$ is partly rational and partly a quadratic surd, which is impossible by the last article.

Therefore, $a=x$ and consequently $\sqrt{b}=\sqrt{y}$, or, $b=y$.

Note. *It should be distinctly borne in mind that the results proved above are true only when $\sqrt{b}$ and $\sqrt{y}$ are really irrational. For instance, from the relation $5+\sqrt{9}=3+\sqrt{25}$, we cannot conclude that $5=3$ and $9=25$.*

206. To find the square root of $a+\sqrt{b}$, where $\sqrt{b}$ is a surd.

Let $$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}.$$

Then, squaring both sides, we have
$$a+\sqrt{b}=x+y+2\sqrt{xy}.$$

Hence, by the last article,
$$\left.\begin{aligned} a&=x+y\\ \text{and}\quad \sqrt{b}&=2\sqrt{xy}\end{aligned}\right\}\quad \dots\quad \dots\quad (1)$$

Hence, $a^2-b=(x+y)^2-4xy=(x-y)^2$;

$\therefore$ $$\sqrt{a^2-b}=x-y.$$

Thus, we have $$\left.\begin{aligned} x+y&=a\\ \text{and}\quad x-y&=\sqrt{a^2-b}\end{aligned}\right\}.$$

Hence, by addition and subtraction,
$$2x=a+\sqrt{a^2-b},\quad\text{and}\quad 2y=a-\sqrt{a^2-b};$$
$$\therefore\quad x=\tfrac{1}{2}(a+\sqrt{a^2-b}),\ \text{and}\quad y=\tfrac{1}{2}(a-\sqrt{a^2-b}).$$

Thus, $$\sqrt{a+\sqrt{b}}=\sqrt{\tfrac{1}{2}(a+\sqrt{a^2-b})}+\sqrt{\tfrac{1}{2}(a-\sqrt{a^2-b})}.$$

Note. *From the values of x and y found above it is clear that unless $\sqrt{a^2-b}$ is rational the square root obtained is by far more complicated than the original expression. Thus, the process given above is of no great practical value except when a^2-b is a perfect square.*

Cor. From (1), we have $a-\sqrt{b}=x+y-2\sqrt{xy}=(\sqrt{x}-\sqrt{y})^2$;

$\therefore$ $$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

Thus, if $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then will $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

Example 1. Find the square root of $7+2\sqrt{10}$.

Let $$\sqrt{7+2\sqrt{10}}=\sqrt{x}+\sqrt{y}.$$

Then, squaring both sides,
$$7+2\sqrt{10}=x+y+2\sqrt{xy}.$$

Hence, $$\left.\begin{aligned} x+y&=7\\ \text{and}\quad xy&=10\end{aligned}\right\}$$

These relations are evidently satisfied by the numbers 5 and 2.

Hence, the required root $=\sqrt{5}+\sqrt{2}$.

Example 2. Find the square root of $19-8\sqrt{3}$.

Let $$\sqrt{19-8\sqrt{3}}=\sqrt{x}-\sqrt{y}.$$

Then, $$19-8\sqrt{3}=x+y-2\sqrt{xy}.$$

$$\left.\begin{array}{lll}\text{Hence,} & x+y=19 \quad \cdots & \cdots \quad (1)\\ \text{and} & 2\sqrt{xy}=8\sqrt{3},\text{ or, } xy=48 & \cdots \quad (2)\end{array}\right\}$$

Now, (1) and (2) are obviously satisfied by the numbers 16 and 3. Hence, the required root $=\sqrt{16}-\sqrt{3}=4-\sqrt{3}$.

Example 3. Find the square root of $16-5\sqrt{7}$.

Let $\sqrt{16-5\sqrt{7}}=\sqrt{x}-\sqrt{y}$.

Then, $16-5\sqrt{7}=x+y-2\sqrt{xy}$.

$$\left.\begin{array}{ll}\text{Therefore,} & x+y=16\\ \text{and} & 2\sqrt{xy}=5\sqrt{7}\end{array}\right\}$$

Hence,
$$\begin{aligned}(x-y)^2&=(x+y)^2-4xy=(16)^2-(5\sqrt{7})^2\\&=256-175=81;\end{aligned}$$

$$\therefore \quad x-y=9.$$

$$\text{Thus, we have } \left.\begin{array}{r}x+y=16\\ \text{and } x-y=\ \ 9\end{array}\right\}$$

Hence, $x=\frac{25}{2}$ and $y=\frac{7}{2}$.

Thus, the required root $=\sqrt{\frac{25}{2}}-\sqrt{\frac{7}{2}}$.

Example 4. Find the square root of $\sqrt{27}+\sqrt{15}$.

$$\sqrt{27}+\sqrt{15}=3\sqrt{3}+\sqrt{3}.\sqrt{5}=\sqrt{3}(3+\sqrt{5}).$$

Hence, $\sqrt{\sqrt{27}+\sqrt{15}}=\sqrt[4]{3}.\sqrt{3+\sqrt{5}}$.

Now, proceeding as in the last example, we find that

$$\sqrt{3+\sqrt{5}}=\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{5}{2}}.$$

Therefore, $\sqrt{\sqrt{27}+\sqrt{15}}=\sqrt[4]{3}.(\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{5}{2}})$.

Example 5. Find the value of

$$\frac{1+x}{1+\sqrt{1+x}}+\frac{1-x}{1-\sqrt{1-x}}, \text{ when } x=\frac{\sqrt{3}}{2}.$$

We have $1+x=1+\frac{\sqrt{3}}{2}=\frac{2+\sqrt{3}}{2}=\frac{4+2\sqrt{3}}{4}=\left(\frac{\sqrt{3}+1}{2}\right)^2$,

and $1-x=1-\frac{\sqrt{3}}{2}=\frac{2-\sqrt{3}}{2}=\frac{4-2\sqrt{3}}{4}=\left(\frac{\sqrt{3}-1}{2}\right)^2$.

Hence, the given expression

$$\begin{aligned}&=\frac{\frac{1}{2}(2+\sqrt{3})}{1+\frac{1}{2}(\sqrt{3}+1)}+\frac{\frac{1}{2}(2-\sqrt{3})}{1-\frac{1}{2}(\sqrt{3}-1)}=\frac{2+\sqrt{3}}{3+\sqrt{3}}+\frac{2-\sqrt{3}}{3-\sqrt{3}}\\&=\frac{(2+\sqrt{3})(3-\sqrt{3})+(2-\sqrt{3})(3+\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}\\&=\frac{(6+\sqrt{3}-3)+(6-\sqrt{3}-3)}{9-3}=\frac{6}{6}=1.\end{aligned}$$

Example 6. Find the value of

$$\frac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}, \text{ when } x=\frac{1}{2}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right).$$

$$\frac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}=\frac{2a\sqrt{1+x^2}(x-\sqrt{1+x^2})}{x^2-(1+x^2)}$$

$$=-2ax\sqrt{1+x^2}+2a(1+x^2).$$

Now, since $x=\frac{1}{2}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)$;

$$\therefore \quad x^2=\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}-2\right);$$

$$\therefore \quad \sqrt{1+x^2}=\sqrt{1+\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}-2\right)}$$

$$=\sqrt{\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}+2\right)}=\frac{1}{2}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right).$$

Hence, the required value

$$=-2a\cdot\frac{1}{4}\left(\frac{a}{b}-\frac{b}{a}\right)+2a\cdot\frac{1}{4}\left(\frac{a}{b}+\frac{b}{a}+2\right)$$

$$=2a\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{b}{a}\right)=a+b.$$

EXERCISE 113

Find the square root of :

1. $4-2\sqrt{3}$. 2. $7+4\sqrt{3}$. 3. $11-6\sqrt{2}$. 4. $8+2\sqrt{15}$.

5. $14-6\sqrt{5}$. 6. $28+10\sqrt{3}$. 7. $21-8\sqrt{5}$. 8. $17+12\sqrt{2}$.

9. $41+12\sqrt{5}$. 10. $37-20\sqrt{3}$. 11. $31+4\sqrt{21}$. 12. $73-12\sqrt{35}$.

13. $47+4\sqrt{33}$. 14. $4-\sqrt{7}$. 15. $6-\sqrt{35}$. 16. $\sqrt{18}-\sqrt{16}$.

17. $\sqrt{32}-\sqrt{24}$. 18. $\sqrt{27}+\sqrt{24}$. 19. $5\sqrt{5}+\sqrt{120}$.

20. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

21. Find the value of $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1+\sqrt{1-x}}$, when $x=\dfrac{\sqrt{3}}{2}$.

22. Find the value of $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$, when $x=\dfrac{2ab}{b^2+1}$.

Find the square root of :

23. $a^2+2x\sqrt{a^2-x^2}$.

24. $2a+2\sqrt{a^2-b^2}$.

25. $a+x+\sqrt{2ax+x^2}$.

26. $2x-1+2\sqrt{x^2-x-6}$.

27. $x+y+z+2\sqrt{xz+yz}$.

207. Equations involving Surds.

Example 1. Solve $\sqrt{x+12}=\sqrt{x}+2$.

Squaring both sides, we have $x+12=x+4+4\sqrt{x}$.

Hence, $4\sqrt{x}=8$,

or, $\sqrt{x}=2$. $\therefore$ $x=4$.

Example 2. Solve $2(x+2)=1+\sqrt{4x^2+9x+14}$. [C. U. 1877]

By transposition, we have $2x+3=\sqrt{4x^2+9x+14}$.

Squaring both sides, $4x^2+12x+9=4x^2+9x+14$,

or, $3x=5$. $\therefore$ $x=\frac{5}{3}$.

Example 3. Solve $\sqrt{x+6}+\sqrt{x-5}=11$.

By transposition, $\sqrt{x+6}=11-\sqrt{x-5}$.

Squaring both sides, $x+6=121-22\sqrt{x-5}+(x-5)$.

$\therefore$ $22\sqrt{x-5}=110$, [*by transposition*]

or, $\sqrt{x-5}=5$. $\therefore$ $x-5=25$. $\therefore$ $x=30$.

Example 4. Solve $\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3$.

[C. U. Entr. Paper, 1881]

By transposition, $\sqrt{x^2+11x+20}=3+\sqrt{x^2+5x-1}$.

Squaring both sides, $x^2+11x+20=9+(x^2+5x-1)+6\sqrt{x^2+5x-1}$,

or, $6x+12=6\sqrt{x^2+5x-1}$,

or, $x+2=\sqrt{x^2+5x-1}$.

$\therefore$ $x^2+4x+4=x^2+5x-1$, whence $x=5$.

Example 5. Solve $\dfrac{3x-1}{\sqrt{3x}+1}=1+\dfrac{\sqrt{3x}-1}{2}$.

Since, $3x-1=(\sqrt{3x}+1)(\sqrt{3x}-1)$,

$$\therefore \quad \frac{3x-1}{\sqrt{3x}+1}=\sqrt{3x}-1.$$

Hence, from the given equation, we have

$$\sqrt{3x}-1=1+\frac{\sqrt{3x}-1}{2},$$

or, $(\sqrt{3x}-1)(1-\tfrac{1}{2})=1$, [*by transposition*]

or, $\dfrac{\sqrt{3x}-1}{2}=1$, or, $\sqrt{3x}-1=2$,

or, $\sqrt{3x}=3$. $\therefore$ $3x=9$. $\therefore$ $x=3$.

Example 6. Solve $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b$.

Since $(\sqrt[3]{a+x}+\sqrt[3]{a-x})^3$

$$=(a+x)+(a-x)+3\sqrt[3]{a^2-x^2}\{\sqrt[3]{a+x}+\sqrt[3]{a-x}\}$$

$$=2a+3\sqrt[3]{a^2-x^2}.b,$$

therefore, cubing both sides of the equation, we have

$$2a+3\sqrt[3]{a^2-x^2}.b=b^3,$$

or, $3b\sqrt[3]{a^2-x^2}=b^3-2a$. $\therefore$ $a^2-x^2=\left(\dfrac{b^3-2a}{3b}\right)^3$.

$$\therefore\quad x^2=a^2-\left(\frac{b^3-2a}{3b}\right)^3.$$

$$\therefore\quad x=\sqrt{a^2-\left(\frac{b^3-2a}{3b}\right)^3}.$$

Example 7. Solve $\dfrac{x-8}{\sqrt{x+1}-3}+\dfrac{x-26}{\sqrt{x-1}+5}=\dfrac{4x-5}{\sqrt{4x-1}+2}$.

$$\frac{x-8}{\sqrt{x+1}-3}=\frac{(x-8)(\sqrt{x+1}+3)}{(x+1)-9}=\sqrt{x+1}+3\,;$$

$$\frac{x-26}{\sqrt{x-1}+5}=\frac{(x-26)(\sqrt{x-1}-5)}{(x-1)-25}=\sqrt{x-1}-5\,;$$

$$\frac{4x-5}{\sqrt{4x-1}+2}=\frac{(4x-5)(\sqrt{4x-1}-2)}{(4x-1)-4}=\sqrt{4x-1}-2.$$

Hence, from the given equation, we have

$$(\sqrt{x+1}+3)+(\sqrt{x-1}-5)=\sqrt{4x-1}-2,$$

or, $\sqrt{x+1}+\sqrt{x-1}=\sqrt{4x-1}$.

$\therefore$ $(x+1)+(x-1)+2\sqrt{x^2-1}=4x-1$,

or, $2\sqrt{x^2-1}=2x-1$,

or, $4(x^2-1)=4x^2-4x+1$,

or, $4x=5$. $\therefore$ $x=\tfrac{5}{4}$.

Example 8. Solve $\sqrt{2x^2+9}+\sqrt{2x^2-9}=9+3\sqrt{7}$.

We have, for *all* values of x, $(2x^2+9)-(2x^2-9)=18$, and hence, this relation is also true for the particular value which x has in the given equation.

Therefore, the required value of x will also satisfy the equation

$$\frac{(2x^2+9)-(2x^2-9)}{\sqrt{2x^2+9}+\sqrt{2x^2-9}}=\frac{18}{9+3\sqrt{7}},$$

or, $$\sqrt{2x^2+9}-\sqrt{2x^2-9}=\frac{18(9-3\sqrt{7})}{81-63}=9-3\sqrt{7}.$$

Adding together the given equation and this, we have

$$2\sqrt{2x^2+9}=18, \qquad \text{or,} \qquad \sqrt{2x^2+9}=9.$$

$$\therefore \quad 2x^2+9=81. \qquad \therefore \quad x^2=36. \qquad \therefore \quad x=6.$$

Example 9. Solve $4x^2+6x+\sqrt{2x^2+3x+4}=13$. [W. B. S. F. 1953]

$$4x^2+6x+\sqrt{2x^2+3x+4}=13,$$

or, $4x^2+6x+8+\sqrt{2x^2+3x+4}=21$ [adding 8 to both sides]

or, $2(2x^2+3x+4)+\sqrt{2x^2+3x+4}=21$

or, $2p^2+p=21$ [putting p for $\sqrt{2x^2+3x+4}$]

or, $2p^2+p-21=0$, or, $(p-3)(2p+7)=0$;

$\therefore$ $p=3$, rejecting the negative value $-\frac{7}{2}$.

$\therefore$ $\sqrt{2x^2+3x+4}=3$, or, $2x^2+3x+4=9$,

or, $2x^2+3x-5=0$, or, $(x-1)(2x+5)=0$. $\therefore$ $x=1, -\frac{5}{2}$.

EXERCISE 114

Solve the following equations :

1. $\sqrt{x+7}=1+\sqrt{x}$.
2. $\sqrt{3x+16}=\sqrt{3x}+2$.
3. $\sqrt{x+9}=1+\sqrt{x}$.
4. $\sqrt{3x}-4=\sqrt{3x+4}$.
5. $\sqrt{5x+10}=\sqrt{5x}+2$.
6. $\sqrt{x-16}+\sqrt{x}=8$.
7. $\sqrt{2x+9}+\sqrt{2x}=9$.
8. $\sqrt{x+11}-\sqrt{x}=1$.
9. $\sqrt{8x+33}-3=2\sqrt{2x}$.
10. $x+\sqrt{2ax+x^2}=a$.
11. $x+a+\sqrt{2ax+x^2}=b$.
12. $\sqrt{x-4}+3=\sqrt{x+11}$.
13. $\sqrt{x-5}=6-\sqrt{x+7}$.
14. $\sqrt{x+9}-\sqrt{x+2}=1$.

15. $\sqrt{3x+1}-\sqrt{3x-11}=2$.

16. $\sqrt{5x+6}+\sqrt{5x-14}=10$.

17. $\sqrt{7x+4}+\sqrt{7x-12}=8$.

18. $\sqrt{x^2-3x+5}-\sqrt{x^2-x+1}=1$.

19. $\dfrac{x-1}{\sqrt{x}+1}=4+\dfrac{\sqrt{x}-1}{2}$.

20. $\dfrac{ax-1}{\sqrt{ax}+1}=4+\dfrac{\sqrt{ax}-1}{2}$.

[C. U. Entr. Paper, 1885]

21. $\dfrac{ax-b^2}{\sqrt{ax}+b}=c+\dfrac{\sqrt{ax}-b}{c}$.

22. $\dfrac{200+120\sqrt{5x}}{9x-5}=(3\sqrt{x}-\sqrt{5})^2$.

23. $\sqrt{4a+x}-\sqrt{a+x}=2\sqrt{x-2a}$.

24. $\sqrt{x}+\sqrt{a+x}=\dfrac{3a}{\sqrt{a+x}}$.

25. $\sqrt{x}+\sqrt{x+13}=\dfrac{91}{\sqrt{x+13}}$.

26. $\sqrt{x+a}+\sqrt{x-a}=\dfrac{b}{\sqrt{x+a}}$.

27. $\dfrac{3\sqrt{x}-4}{\sqrt{x}+2}=\dfrac{15+3\sqrt{x}}{\sqrt{x}+40}$.

28. $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1$.

29. $\sqrt{x}+\sqrt{8-\sqrt{x^2+8x}}=2\sqrt{2}$.

30. $\sqrt{1-x}+\sqrt{1-x+\sqrt{1+x}}=\sqrt{1+x}$.

31. $\dfrac{1}{a}\sqrt{a+x}+\dfrac{1}{x}\sqrt{a+x}=\dfrac{1}{b}\sqrt{x}$.

32. $\sqrt[5]{x+8}=\sqrt[10]{x^2+64x+36}$.

33. $(1+x)^{\frac{1}{3}}+(1-x)^{\frac{1}{3}}=2^{\frac{1}{3}}$. [C. U. Entr. Paper, 1885]

34. $(a+x)^{\frac{2}{3}}+(a-x)^{\frac{2}{3}}=3(a^2-x^2)^{\frac{1}{3}}$.

35. $x^2+18=8x+6\sqrt{x^2-8x+9}$.

36. $\left(\dfrac{x}{a}+\dfrac{a}{b}\right)^{\frac{1}{2}}+9\left(\dfrac{x}{a}-\dfrac{a}{b}\right)^{\frac{1}{2}}=6\left(\dfrac{x^2}{a^2}-\dfrac{a^2}{b^2}\right)^{\frac{1}{4}}$.

37. $\dfrac{x-47}{\sqrt{x+2}-7}+\dfrac{x-19}{\sqrt{x-3}-4}=\dfrac{4x-124}{\sqrt{4x-3}-11}$.

38. $\dfrac{2x-49}{\sqrt{2x+15}-8}+\dfrac{18x+22}{\sqrt{18x+31}+3}=\dfrac{8x+191}{2\sqrt{2x+54}-5}$.

39. $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a$.

40. $\sqrt{x^2+9}+\sqrt{x^2-9}=4+\sqrt{34}$.

41. $\sqrt{3x^2+16}-\sqrt{3x^2-16}=8-4\sqrt{2}$.

CHAPTER XXXI

EVOLUTION : SQUARE AND CUBE ROOTS

208. Evolution. The process of finding the roots of quantities is called **Evolution.**

Thus, Evolution is the inverse of Involution. [Art. 127]

209. The ordinary method of finding the square root of a compound algebraical expression. From our previous knowledge of formulæ the following results are obvious :

$$(a+b)^2 = a^2+(2a+b)b\,;$$

$$(a+b+c)^2 = a^2+(2a+b)b+(2a+2b+c)c\,;$$

$$(a+b+c+d)^2 = a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d\,;$$

and so on.

Clearly, therefore, we must have

$(ax^2+bx+c)^2 = a^2x^4+(2ax^2+bx)bx+(2ax^2+2bx+c)c$, and this latter when arranged according to the descending powers of x,

$$= a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2.$$

Now, if it is proposed to find the square root of the above expression, let us see what means we have of discovering successively the several terms of the root.

The first term of the root, *viz.*, ax^2, is evidently the square root of the first term of the given expression which is a^2x^4 ;

if we subtract a^2x^4 from the given expression, the remainder is $\{(2ax^2+bx)bx+(2ax^2+2bx+c)c\}$, in which the term containing the highest power of x, $= 2ax^2 \times bx$, *i.e.*, = twice the first term of the root *into* the second term ; this enables us to get the second term after having obtained the first ;

if now from the above remainder we subtract $(2ax^2+bx)bx$, the second remainder is $(2ax^2+2bx+c)c$, in which the term containing the highest power of x, $= 2ax^2 \times c$, *i.e.*, = twice the first term of the root *into* the third ; this shows how to get the third term after having obtained the first and second.

Thus, we are furnished with a clue for successively discovering the terms of the expression ax^2+bx+c when its square is given.

The operation may be performed as follows :

$$\begin{array}{rll}
 & a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2 & \big(\, ax^2+bx+c \\
 & \underline{a^2x^4} & \\
2ax^2+bx & \big)\, 2abx^3+(b^2+2ac)x^2+2bcx+c^2 & \\
 & \quad \underline{2abx^3+b^2x^2} & \\
2ax^2+2bx+c & \big)\, 2acx^2+2bcx+c^2 & \\
 & \quad \underline{2acx^2+2bcx+c^2} &
\end{array}$$

(1) Find the square root of a^2x^4, the first term of the proposed expression, and set it down as the first term of the required root;

(2) subtract a^2x^4 from the given expression, and bring down the remainder $2abx^3+(b^2+2ac)x^2+2bcx+c^2$;

(3) set down $2ax^2$, *i.e.*, twice the first term of the root, on the left of the above remainder as the first term of a divisor;

(4) divide the first term of the remainder by $2ax^2$, and set down the quotient, bx, as the second term of the root and also as the second term of the divisor;

(5) multiply the divisor thus obtained by the second term of the root and subtract the product from the first remainder;

(6) bring down the second remainder $2acx^2+2bcx+c^2$ and put $2ax^2+2bx$ (*i.e.*, twice the sum of the two terms of the root already obtained) on the left of this remainder for the first two terms of a divisor;

(7) divide the first term of the new remainder by the first term of the new divisor and set down the quotient, c, as the third term of the root and also as the third term of the divisor;

(8) multiply the complete divisor thus obtained by the third term of the root and subtract the product from the second remainder.

After this nothing remains, and we obtain ax^2+bx+c for the required root.

Note. *The expression considered above stands arranged according to descending powers of x. Similarly, every expression of which the square root is sought must be arranged according to descending or ascending order of the powers of the same letter.*

Example 1. Extract the square root of

$$x^6+8x^4-2x^3+16x^2-8x+1.$$

$$\begin{array}{r|l|l}
 & x^6+8x^4-2x^3+16x^2-8x+1 & x^3+4x-1 \\
 & x^6 & \\ \cline{2-2}
2x^3+4x & 8x^4-2x^3+16x^2-8x+1 & \\
 & 8x^4 \qquad\quad +16x^2 & \\ \cline{2-2}
2x^3+8x-1 & -2x^3 \qquad\qquad -8x+1 & \\
 & -2x^3 \qquad\qquad -8x+1 & \\ \cline{2-2}
\end{array}$$

Thus, the required root $=x^3+4x-1$.

Example 2. Extract the square root of

$$x^4+2(y+z)x^3+(3y^2+2yz+3z^2)x^2+2(y^3+y^2z+yz^2+z^3)x$$
$$+y^4+2y^2z^2+z^4. \quad \text{[C. U. 1888]}$$

The given expression $= x^4 + 2(y+z)x^3 + (3y^2 + 2yz + 3z^2)x^2$
$+ 2(y+z)(y^2+z^2)x + (y^2+z^2)^2$,

which stands arranged according to descending powers of x; so we can at once proceed thus :

$$\begin{array}{r|l}
 & \quad\quad\quad\quad\quad\quad\quad\quad + 2(y+z)(y^2+z^2)x + (y^2+z^2)^2 \; \Big(\; x^2 + (y+z)x \\
 & x^4 + 2(y+z)x^3 + (3y^2+2yz+3z^2)x^2 \quad\quad\quad\quad\quad\quad\quad\quad + (y^2+z^2) \\
 & x^4 \\
\hline
2x^2 + (y+z)x & 2(y+z)x^3 + (3y^2+2yz+3z^2)x^2 \\
 & 2(y+z)x^3 + (\;y^2+2yz+\;z^2)x^2 \\
\hline
2x^2 + 2(y+z)x & 2(y^2+z^2)x^2 + 2(y+z)(y^2+z^2)x + (y^2+z^2)^2 \\
+ (y^2+z^2) & 2(y^2+z^2)x^2 + 2(y+z)(y^2+z^2)x + (y^2+z^2)^2 \\
\hline
\end{array}$$

Thus, the required root $= x^2 + xy + xz + y^2 + z^2$.

Example 3. Find the square root of $\frac{x^4}{4} + 4x^2 + \frac{ax^2}{3} + \frac{a^2}{9} - 2x^3 - \frac{4ax}{3}$.

[C. U. 1889

Arrange the expression according to descending powers of x and then proceed thus :

$$\begin{array}{r|l}
 & \frac{x^4}{4} - 2x^3 + 4x^2 + \frac{ax^2}{3} - \frac{4ax}{3} + \frac{a^2}{9} \; \Big(\; \frac{x^2}{2} - 2x + \frac{a}{3} \\
 & \frac{x^4}{4} \\
\hline
x^2 - 2x & -2x^3 + 4x^2 \\
 & -2x^3 + 4x^2 \\
\hline
x^2 - 4x + \frac{a}{3} & \frac{ax^2}{3} - \frac{4ax}{3} + \frac{a^2}{9} \\
 & \frac{ax^2}{3} - \frac{4ax}{3} + \frac{a^2}{9} \\
\hline
\end{array}$$

Thus, the required root $= \frac{x^2}{2} - 2x + \frac{a}{3}$.

Example 4. Extract the square root of $\frac{x^4}{4y^4} + \frac{4y^4}{x^4} + \frac{x^2}{y^2} + \frac{4y^2}{x^2} + 3$.

The expression when arranged according to descending powers of x stands thus :

$$\frac{x^4}{4y^4} + \frac{x^2}{y^2} + 3 + \frac{4y^2}{x^2} + \frac{4y^4}{x^4},$$

or now the indices of the powers of x in the successive terms are respectively 4, 2, 0, -2 and -4, which numbers evidently are in descending order of magnitude. Hence, we proceed as follows :

$$\begin{array}{l}
\dfrac{x^4}{4y^4}+\dfrac{x^2}{y^2}+3+\dfrac{4y^2}{x^2}+\dfrac{4y^4}{x^4}\,\Big(\,\dfrac{x^2}{2y^2}+1+\dfrac{2y^2}{x^2}\\
\dfrac{x^4}{4y^4}\\
\hline
\dfrac{x^2}{y^2}+1\,\Big)\;\dfrac{x^2}{y^2}+3\\
\qquad\qquad\;\dfrac{x^2}{y^2}+1\\
\hline
\dfrac{x^2}{y^2}+2+\dfrac{2y^2}{x^2}\,\Big)\;2+\dfrac{4y^2}{x^2}+\dfrac{4y^4}{x^4}\\
\qquad\qquad\qquad\quad\;\, 2+\dfrac{4y^2}{x^2}+\dfrac{4y^4}{x^4}\\
\hline
\end{array}$$

Thus, the required root $=\dfrac{x^2}{2y^2}+1+\dfrac{2y^2}{x^2}$.

Example 5. Extract the square root of

$$x^{\frac{8}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+a^{\frac{8}{5}}. \quad \text{[C. U. 1880]}$$

Let us proceed by arranging the expression according to descending powers of x, thus :

$$\begin{array}{l}
a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+x^{\frac{8}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{\frac{8}{5}}\,\big(\,a^{-\frac{3}{5}}x^{\frac{7}{5}}-x^{\frac{4}{5}}-a^{\frac{4}{5}}\\
a^{-\frac{6}{5}}x^{\frac{14}{5}}\\
\hline
2a^{-\frac{3}{5}}x^{\frac{7}{5}}-x^{\frac{4}{5}}\,\big)\;-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+x^{\frac{8}{5}}\\
\qquad\qquad\qquad\quad\; -2a^{-\frac{3}{5}}x^{\frac{11}{5}}+x^{\frac{8}{5}}\\
\hline
2a^{-\frac{3}{5}}x^{\frac{7}{5}}-2x^{\frac{4}{5}}-a^{\frac{4}{5}}\,\big)\;-2a^{\frac{1}{5}}x^{\frac{7}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{\frac{8}{5}}\\
\qquad\qquad\qquad\qquad\qquad\;\; -2a^{\frac{1}{5}}x^{\frac{7}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{\frac{8}{5}}\\
\hline
\end{array}$$

Thus, the required root $=a^{-\frac{3}{5}}x^{\frac{7}{5}}-x^{\frac{4}{5}}-a^{\frac{4}{5}}$.

EXERCISE 115

Find the square root of :

1. $4x^2z^2+12xyz+9y^2$.
2. $x^4-4x^3+10x^2-12x+9$.
3. $x^6-2x^4+2x^3+x^2-2x+1$.
4. $4x^4-12x^3+25x^2-24x+16$.
5. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$. [C. U. 1870]
6. $9x^4-2x^3y+\frac{163}{9}x^2y^2-2xy^3+9y^4$. [C. U. 1874]

7. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$. 8. $\frac{1051x^2}{25}-\frac{6x}{5}-\frac{14x^3}{5}+49x^4+9$.

9. $x^4+\frac{4}{x^2}-2+4x-x^3+\frac{x^2}{4}$. 10. $\frac{a^2}{x^2}+\frac{x^2}{a^2}+\frac{a^4}{4}+\frac{a^3}{x}-2-ax$.

11. $\frac{a^2}{4b^2}-\frac{a}{b}+\frac{4b^2}{a^2}-1+\frac{4b}{a}$. 12. $\frac{9a^2}{x^2}-\frac{6a}{5x}+\frac{101}{25}-\frac{4x}{15a}+\frac{4x^2}{9a^2}$.

13. $4x^4-8x^3y^2+4xy^6+y^8$. 14. $\frac{49x^2}{y^2}+\frac{y^2}{49x^2}-\frac{42x}{y}+\frac{6y}{7x}+7$.

15. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\frac{3}{4}$. 16. $25\frac{3}{7}-\frac{20x}{7y}+\frac{9y^2}{16x^2}-\frac{15y}{2x}+\frac{4x^2}{49y^2}$.

17. $x^2-2x^{\frac{3}{2}}+3x-2x^{\frac{1}{2}}+1$. 18. $x^{\frac{5}{3}}-4x^{\frac{4}{3}}+2x+4x^{\frac{2}{3}}+x^{\frac{1}{3}}$.

19. $a^2x^{-2}+2ax^{-1}+a^{-2}x^2+3+2a^{-1}x$.

20. $x^{\frac{3}{2}}+xy^{-\frac{1}{2}}-2x^{\frac{5}{4}}y^{-\frac{1}{4}}-2x^{\frac{1}{2}}y^{\frac{1}{4}}+2x^{\frac{3}{4}}y^{\frac{1}{2}}+y$.

21. $\frac{9x^3}{4}-5x^{\frac{5}{2}}y^{\frac{1}{2}}+\frac{179x^2y}{45}-\frac{4x^{\frac{3}{2}}y^{\frac{3}{2}}}{3}+\frac{4xy^2}{25}$. 22. $a^{2m}-4a^{m+n}+4a^{2n}$.

23. $9a^{2m}+6a^{3m+1}+25c^{2m-4}-30a^mc^{m-2}+a^{4m+2}-10a^{2m+1}c^{m-2}$.

210. Extraction of square roots by the application of the formula $a^2\pm2ab+b^2=(a\pm b)^2$ and Miscellaneous examples.

Example 1. Find the square root of $4-4c+2b+c^2-bc+\frac{b^2}{4}$.

[C. U. 1876]

The given expression, arranged according to descending powers of b,

$$=\frac{b^2}{4}-b(c-2)+(c^2-4c+4)$$

$$=\left(\frac{b}{2}\right)^2-2\left\{\frac{b}{2}(c-2)\right\}+(c-2)^2$$

$$=\left\{\frac{b}{2}-(c-2)\right\}^2=\left(\frac{b}{2}-c+2\right)^2.$$

Therefore, the required root $=\frac{b}{2}-c+2$.

Example 2. Extract the square root of $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$.

The given expression

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}+2\right)+12$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4=\left(x^2-2+\frac{1}{x^2}\right)^2.$$

Therefore, the required root $=x^2-2+\frac{1}{x^2}$.

Example 3. Extract the square root of

$$\frac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2}+4\frac{a}{a+b}\times\frac{b}{a-b}. \qquad \text{[C. U. 1886]}$$

The given expression

$$\frac{(a^2+b^2)^2}{(a^2-b^2)^2}+\frac{4ab}{a^2-b^2}=\frac{(a^2+b^2)^2+4ab(a^2-b^2)}{(a^2-b^2)^2},$$

of which the numerator $=\{(a^2-b^2)^2+4a^2b^2\}+4ab(a^2-b^2)$

$$=(a^2-b^2)^2+4ab(a^2-b^2)+4a^2b^2$$
$$=\{(a^2-b^2)+2ab\}^2.$$

$\therefore$ the given expression $=\dfrac{(a^2+2ab-b^2)^2}{(a^2-b^2)^2}.$

Therefore, the reqd. root $=\dfrac{a^2+2ab-b^2}{a^2-b^2}.$

Example 4. Extract the square root of $(ab+ac+bc)^2-4abc(a+c)$.

[C. U. 1888]

The given expression

$$=\{b(a+c)+ac\}^2-4abc(a+c)$$
$$=b^2(a+c)^2+a^2c^2-2abc(a+c)$$
$$=\{b(a+c)-ac\}^2=(ab-ac+bc)^2.$$

Therefore, the required root $=ab-ac+bc$.

Example 5. Extract the square root of

$$a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)+2a^2c^2+2b^2d^2.$$

Arranging the given expression according to descending powers of a, we have

$$a^4-2a^2(b^2+d^2-c^2)+\{b^4+c^4+d^4-2c^2(b^2+d^2)+2b^2d^2\},$$

and the expression within the braces arranged according to descending powers of b,

$$=b^4-2b^2(c^2-d^2)+(c^4+d^4-2c^2d^2)$$
$$=b^4-2b^2(c^2-d^2)+(c^2-d^2)^2$$
$$=\{b^2-(c^2-d^2)\}^2.$$

Hence, the given expression

$$=a^4-2a^2(b^2-c^2+d^2)+(b^2-c^2+d^2)^2$$
$$=\{a^2-(b^2-c^2+d^2)\}^2$$
$$=(a^2-b^2+c^2-d^2)^2.$$

Therefore, the required root $=a^2-b^2+c^2-d^2$.

Example 6. Find the square root of

$$4\{(a^2-b^2)cd+ab(c^2-d^2)\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2.$$

The given expression

$$\begin{aligned}
&=4\{(a^2-b^2)^2c^2d^2+2abcd(a^2-b^2)(c^2-d^2)+a^2b^2(c^2-d^2)^2\}\\
&\quad+\{(a^2-b^2)^2(c^2-d^2)^2-8abcd(a^2-b^2)(c^2-d^2)+16a^2b^2c^2d^2\}\\
&=\{4(a^2-b^2)^2c^2d^2+4a^2b^2(c^2-d^2)^2\}+\{(a^2-b^2)^2(c^2-d^2)^2\\
&\qquad\qquad+16a^2b^2c^2d^2\}\\
&=(a^2-b^2)^2\{(c^2-d^2)^2+4c^2d^2\}+4a^2b^2\{(c^2-d^2)^2+4c^2d^2\}\\
&=\{(a^2-b^2)^2+4a^2b^2\}\{(c^2-d^2)^2+4c^2d^2\}\\
&=(a^4+2a^2b^2+b^4)(c^4+2c^2d^2+d^4)\\
&=(a^2+b^2)^2(c^2+d^2)^2.
\end{aligned}$$

Therefore, the required root $=(a^2+b^2)(c^2+d^2)$.

Example 7. Express $(x-2a)(x-5a)(x-8a)(x-11a)+81a^4$ as a perfect square. [C. U. 1945]

The given expression

$$\begin{aligned}
&=\{(x-2a)(x-11a)\}\{(x-5a)(x-8a)\}+81a^4\\
&=(x^2-13ax+22a^2)(x^2-13ax+40a^2)+81a^4\\
&=(p+22a^2)(p+40a^2)+81a^4 \quad [\text{ supposing } p=x^2-13ax\ ]\\
&=p^2+62pa^2+880a^4+81a^4\\
&=p^2+62pa^2+961a^4\\
&=(p+31a^2)^2\\
&=(x^2-13ax+31a^2)^2.
\end{aligned}$$

Example 8. For what value of x, will the expression $9x^4+12x^3+46x^2+30x+45$ be a perfect square ?

$$\begin{array}{r|l}
 & 9x^4+12x^3+46x^2+30x+45 \ \big(\ 3x^2+2x+7\\
 & \underline{9x^4}\\
6x^2+2x & 12x^3+46x^2\\
 & \underline{12x^3+\ \ 4x^2}\\
6x^2+4x+7 & 42x^2+30x+45\\
 & \underline{42x^2+28x+49}\\
 & \qquad\qquad 2x-\ \ 4
\end{array}$$

Had there been no remainder the expression would have been a perfect square. If $2x-4$ would be equal to 0, there would have been no remainder.

$$2x-4=0, \text{ only when } x=2.$$

$\therefore$ the expression will be a perfect square if $x=2$.

Example 9. What least number should be added to $x^4+8x^3+26x^2+40x+15$, to make it a perfect square?

$$\begin{array}{r|l}
 & x^4+8x^3+26x^2+40x+15 \;\big(\, x^2+4x+5 \\
 & \underline{x^4} \\
2x^2+4x & 8x^3+26x^2 \\
 & \underline{8x^3+16x^2} \\
2x^2+8x+5 & 10x^2+40x+15 \\
 & \underline{10x^2+40x+25} \\
 & \qquad\qquad -10
\end{array}$$

The expression would have been a perfect square, if there was no remainder. But the remainder is -10. If 10 is added to it, the remainder is 0. Therefore, 10 is the required number.

EXERCISE 116

Find the square root of:

1. $25x^2y^2-40xy+16$.
2. $49a^2x^4-42ab^2x^2+9b^4$.
3. $49a^6b^6+126a^7b^7+81a^8b^8$.
4. $\frac{1}{4}x^8y^4-\frac{1}{5}x^7y^7+\frac{1}{25}x^6y^{10}$.
5. $\frac{25a^2b^2}{4}+\frac{c^4}{9}-\frac{5abc^2}{3}$.
6. $a^2+b^2+c^2+2ab+2ac+2bc$.
7. $a^2+b^2+c^2-2ab+2ac-2bc$.
8. $4a^2+b^2+9c^2+6bc-12ac-4ab$.
9. $a^4+4b^4+9c^4+4a^2b^2-6a^2c^2-12b^2c^2$.
10. $4a^4+9b^4+25c^4-12a^2b^2+20a^2c^2-30b^2c^2$.
11. $x^2+\frac{a^2}{9}-bx+\frac{b^2}{4}-\frac{ab}{3}+\frac{2ax}{3}$.
12. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$.
13. $x^4+\frac{1}{x^4}+2\left(x^2+\frac{1}{x^2}\right)+3$.
14. $\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{2a}{b}+\frac{2b}{a}+3$.
15. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}+2\frac{1}{2}$.
16. $\frac{9x^2}{a^2}+\frac{a^2}{9x^2}-6\frac{x}{a}-\frac{2a}{3x}+3$.

17. $x^2+\frac{1}{x^2}+4\left(x+\frac{1}{x}\right)+6.$ 18. $-2+a^{2\sqrt{2}}+a^{-2\sqrt{2}}.$

19. $a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd.$

20. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4).$

21. $a^4+b^4+c^4+d^4-2a^2(b^2+d^2)-2b^2(c^2-d^2)+2c^2(a^2-d^2).$

22. $a^4+2a^3-a+\frac{1}{4}.$

23. $2a^2(b+c)^2+2b^2(c+a)^2+2c^2(a+b)^2+4abc(a+b+c).$

24. Prove that $(x-1)(x-3)(x-5)(x-7)+16$ is a perfect square. [C. U. 1941]

25. For what value of x will the expression $x^4+6x^3+11x^2+3x+31$ be a perfect square ? [C. U. 1927]

26. For what value of a will the expression $4x^4-12x^3+25x^2-24x+a$ become a perfect square ? [A. U. 1948]

27. Find the least number to be added to $x^4-6x^3+13x^2-12x+1$, so that the sum may be a perfect square. [C. U. 1915]

28. Under what condition x^2+px+q is a perfect square ? [G. U. 1951]

211. The ordinary method of finding the cube root of a compound algebraical expression.

Evidently, we have $(ax^2+bx+c)^3$

$$\begin{aligned}&=(ax^2+bx)^3+3(ax^2+bx)^2c+3(ax^2+bx)c^2+c^3\\&=a^3x^6+3(a^2x^4)(bx)+3(ax^2)(bx)^2+(bx)^3\\&\qquad+3(ax^2+bx)^2c+3(ax^2+bx)c^2+c^3.\end{aligned}$$

Hence, if we are asked to find the cube root of the above expression, we see that we have the following means of discovering successively, the several terms of the root :

The first term of the root, *viz.*, ax^2, is evidently the cube root of the first term of the given expression, which is a^3x^6.

If we subtract a^3x^6 from the given expression, the term containing the highest power of x in the remainder is $3(a^2x^4)(bx)$, *i.e.*, equal to three times the square of the first term of the root *into* the second term ; the second term is, therefore, discovered.

If from the above remainder we now subtract $\{3(a^2x^4)+3(ax^2)(bx)+(bx)^2\}(bx)$, the second remainder is $3(ax^2+bx)^2c+3(ax^2+bx)c^2+c^3$; the term containing the highest power of x in this remainder is $3a^2x^4c$, *i.e.*, equal to three times the square of the first term of the root *into* the third.

Hence, the third term is discovered.

If from the second remainder we now subtract $\{3(ax^2+bx)^2+3(ax^2+bx)c+c^2\}c$, nothing is left and we obtain the required root $=ax^2+bx+c$.

Let us illustrate the process by an example.

Example. Find the cube root of

$$x^6-6x^5y+24x^4y^2-56x^3y^3+96x^2y^4-96xy^5+64y^6.$$

The given expression stands arranged according to descending powers of x; we need not, therefore, change the order of the terms.

The first term of the cube root $=$ the cube root of the first term of the expression $=$ cube root of $x^6=x^2$.

The second term of the root, *viz.*, $-2xy$ as shown on the next page, is obtained by dividing $-6x^5y$ by $3x^4$ (*i.e.*, three times the square of the first term).

Then the divisor, $3x^4-6x^3y+4x^2y^2$, is formed as shown on the next page.

The product of this divisor by $(-2xy)$, *viz.*, $-6x^5y+12x^4y^2-8x^3y^3$, is now subtracted from the expression which stands above it and the remainder is put down below the line.

Now, take three times the square of the part of the root already obtained and put down the result, $3x^4-12x^3y+12x^2y^2$, as part of a divisor.

The third term of the root, *viz.*, $4y^2$, is obtained by dividing $12x^4y^2$, the first term of the remainder, by $3x^4$, the first term of the divisor.

The complete divisor is then formed as shown on the next page, and the product of this divisor by the third term of the root is subtracted from the expression which stands above it.

As no remainder is now left, we find the required root

$$=x^2-2xy+4y^2.$$

$$x^6-6x^5y+24x^4y^2-56x^3y^3+96x^2y^4-96xy^5+64y^6 \; \Big(\; x^2-2xy+4y^2$$

$$x^6$$

$$-6x^5y+24x^4y^2-56x^3y^3+96x^2y^4-96xy^5+64y^6$$

$$3\times(x^2)^2=3x^4$$

$$3\times x^2\times(-2xy)= \qquad -\;6x^3y$$

$$(-2xy)^2= \qquad\qquad +\;4x^2y^2$$

$$3x^4-\;6x^3y+\;4x^2y^2$$

$$-6x^5y+12x^4y^2-\;8x^3y^3$$

$$12x^4y^2-48x^3y^3+96x^2y^4-96xy^5+64y^6$$

$$3\times(x^2-2xy)^2=3x^4-12x^3y+12x^2y^2$$

$$\times(x^2-2xy)\times(4y^2)= \qquad +12x^2y^2-24xy^3$$

$$(4y^2)^2= \qquad\qquad +16y^4$$

$$3x^4-12x^3y+24x^2y^2-24xy^3+16y^4$$

$$12x^4y^2-48x^3y^3+96x^2y^4-96xy^5+64y^6$$

EXERCISE 117

Find the cube root of :

1. $x^3+27x^2+243x+729$.
2. $27x^3-216x^2+576x-512$.
3. $64a^3-144a^2b+108ab^2-27b^3$.
4. $33x^4-36x+x^6-63x^3+8-9x^5+66x^2$.
5. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.
6. $1-9x^2+33x^4-63x^6+66x^8-36x^{10}+8x^{12}$.
7. $c^6-63c^3x^3+8x^6-9c^5x+66c^2x^4-36cx^5+33c^4x^2$.

CHAPTER XXXII

RATIO AND PROPORTION

Ratio

212. Definitions. The *ratio* of one quantity to another of the same kind is defined to be the **abstract number** (integral or fractional) which expresses what multiple, part or parts, the former is of the latter. Thus,

since 2 hours is a portion of time which is three times as large as 40 minutes, the ratio of 2 hours to 40 minutes $=3$;

since a length of 25 centimetres is a fourth part of 1 metre, the ratio of 25 cm. to 1 metre $=\frac{1}{4}$;

since the sum of £1. 4*s*. is obtained by dividing 18*s*. into 3 equal parts and taking 4 of such parts, the ratio of £1. 4*s*. to 18*s*. $=\frac{4}{3}$;

and so on.

Hence, it is clear that the ratio of one *concrete* quantity to another (of the same kind) is a fraction, of which the numerator and denominator are respectively the *measures* of those quantities (*referred to one and the same unit*) ; and the ratio of one *abstract* quantity to another is a fraction, of which the numerator and denominator are respectively the quantities themselves.

The ratio of any number a to any other number b is usually expressed by the notation $a:b$; thus, $a:b$ is the same as $\frac{a}{b}$. The quantities a and b are respectively called the **antecedent** and the **consequent** (or the ***first term*** and the ***second term***) of the ratio $a:b$.

A ratio is called *a ratio of greater inequality*, of *less inequality*, or of *equality*, according as it is *greater* than, *less* than, or *equal* to 1.

Note. *Since a ratio is only a fraction, there is no difficulty in seeing that the value of a ratio remains unaltered if its terms be multiplied or divided by the same number. Thus, the ratios* 3 : 4, 6 : 8, 15 : 20 *and* $3n : 4n$ *are equal to one another. Hence, also two or more ratios can be easily compared with one another by reducing them to ratios with the lowest common consequent by multiplying both the terms by the same number ; for instance, the ratios* 2 : 3, 4 : 5 *and* 7 : 10 *being respectively equivalent to* 20 : 30, 24 : 30 *and* 21 : 30, *we see at once that the second of them is the greatest and the first the least.*

213. A ratio of less inequality is increased and a ratio of greater inequality is diminished, by adding the same number to both its terms.

Let $\frac{a}{b}$ be any given ratio, and let $\frac{a+x}{b+x}$ be the new ratio formed by adding x to both its terms.

Then, $$\frac{a+x}{b+x}-\frac{a}{b}=\frac{x(b-a)}{b(b+x)},$$

and, therefore, it is positive or negative according as a is less or greater than b.

Hence, if $a < b$, $\frac{a+x}{b+x} > \frac{a}{b}$; and if $a > b$, $\frac{a+x}{b+x} < \frac{a}{b}$;

which proves the proposition.

Note. *Similarly, it can be proved that a ratio of less inequality is diminished and a ratio of greater inequality is increased by subtracting from both its terms any number which is less than each of those terms. This is left as an exercise for the student.*

214. Composition of Ratios. The ratio of the product of the antecedents of any number of ratios to the product of their consequents is called the ratio **compounded** of the given ratios.

Thus, the ratio compounded of three ratios

$$3 : 4, \qquad 8 : 9, \qquad 2x : 3y$$

is $$3\times8\times2x : 4\times9\times3y, \quad \text{or,} \quad 4x : 9y.$$

When the ratio $a : b$ is compounded with itself, the resulting ratio $a^2 : b^2$ is called the **duplicate** ratio of $a : b$. Similarly, $a^3 : b^3$ is called the **triplicate** ratio of $a : b$; $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the **sub-duplicate** ratio of $a : b$; and $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is called the **sub-triplicate** ratio of $a : b$.

215. Approximate values of Ratios. If x is *very small* compared with a, to show that the ratio $(a+x)^2 : a^2$ is approximately the same as $a+2x : a$.

We have $\dfrac{(a+x)^2}{a^2} = \dfrac{a^2+2ax+x^2}{a^2} = 1 + \dfrac{2x}{a} + \dfrac{x^2}{a^2}$,

and $\therefore$ *approximately*, $= 1 + \dfrac{2x}{a}$,

since $\dfrac{x^2}{a^2}\left(\text{which} = \dfrac{x}{a} \times \dfrac{x}{a}\right)$ is very small compared with $\dfrac{2x}{a}$, and smaller still than 1.

Thus, approximately we have

$$\frac{(a+x)^2}{a^2} = 1 + \frac{2x}{a} = \frac{a+2x}{a}. \quad \cdots \quad (1)$$

Cor. From (1), we have $\sqrt{\dfrac{a+2x}{a}} = \dfrac{a+x}{a}$. Hence, if x is very small compared with a, we have

$$\sqrt{a+x} : \sqrt{a} = a + \tfrac{1}{2}x : a.$$

Note. *By a similar mode of reasoning it can be shown that when x is very small compared with a, $(a+x)^3 : a^3 = a+3x : a$; $(a+x)^4 : a^4 = a+4x : a$; $(a+x)^{\frac{1}{3}} : a^{\frac{1}{3}} = a + \frac{1}{3}x : a$; and so on.*

216. Incommensurable Quantities. If two quantities be such that their ratio cannot be exactly expressed by the ratio of two integers, they are said to be **incommensurable quantities.** Thus, $\sqrt{3}$ and 2 are incommensurable quantities, since no two integers can be found whose ratio is *exactly* equal to $\sqrt{3} : 2$.

Although the ratio of two incommensurable quantities cannot be *exactly* expressed by the ratio of two integers, we can always find two integers, however, whose ratio differs from such a ratio by as small a quantity as we please.

For instance, $\dfrac{\sqrt{3}}{2} = \dfrac{1\text{·}73205\ldots\ldots}{2} = \text{·}86602\ldots\ldots$

and therefore, $\dfrac{\sqrt{3}}{2} > \dfrac{86602}{100000}$ and $< \dfrac{86603}{100000}$;

thus, $\sqrt{3} : 2$ differs from either $86602 : 100000$ or $86603 : 100000$ by even less than a hundred-thousandth part of unity. A further approximation might evidently be arrived at by calculating the value of $\sqrt{3}$ to more places of decimals.

Note. ***Any number which cannot be exactly expressed as the ratio of two whole numbers is also sometimes called incommensurable. From this point of view every surd is an incommensurable quantity.***

EXAMPLES

Example 1. Two numbers are in the ratio of 2 to 3, and if 9 be added to each, they are in the ratio of 3 to 4. Find the numbers.

Since the numbers are in the ratio of 2 to 3, evidently we can represent them by $2x$ and $3x$ respectively.

Hence, by the second condition, we have

$$\frac{2x+9}{3x+9}=\frac{3}{4}.$$

Hence, $8x+36=9x+27$, whence $x=9$.

Therefore, the numbers are 18 and 27.

Example 2. What is the ratio of x to y, if

$$10x+3y : 5x+2y=9 : 5\ ?$$

We have $\dfrac{9}{5}=\dfrac{10x+3y}{5x+2y}=\dfrac{10\cdot\frac{x}{y}+3}{5\cdot\frac{x}{y}+2}.$

Hence, $45\cdot\dfrac{x}{y}+18=50\cdot\dfrac{x}{y}+15.$

$\therefore\ 5.\dfrac{x}{y}=3. \qquad \therefore\ \dfrac{x}{y}=\dfrac{3}{5}.$

Example 3. Which is the greater (x and y being positive)

$$x^3+y^3 : x^2+y^2, \quad \text{or,} \quad x^2+y^2 : x+y\ ?$$

We have $\dfrac{x^3+y^3}{x^2+y^2}-\dfrac{x^2+y^2}{x+y}=\dfrac{xy^3+x^3y-2x^2y^2}{(x^2+y^2)(x+y)}=\dfrac{xy(x-y)^2}{(x^2+y^2)(x+y)},$

which evidently is a positive quantity, since $(x-y)^2$ is positive whether x is greater or less than y.

Hence, $x^3+y^3 : x^2+y^2 > x^2+y^2 : x+y.$

Example 4. What number must be added to each term of the ratio $4 : 7$, that it may become equal to $7 : 8$?

Suppose the required number is x.

Therefore, from the given conditions,

$$\frac{4+x}{7+x}=\frac{7}{8},$$

or, $32+8x=49+7x,$

or, $x=17.$

$\therefore$ 17 is the required number.

Example 5. Two armies number 11000 and 7000 men respectively; before they fight, each is reinforced by 1000 men; in favour of which army is the increase? [C. U. 1879]

The new strength of the 1st army : its original strength

$$=12000 : 11000=12 : 11,$$

whilst, the new strength of the 2nd army : its original strength

$$=8000 : 7000=8 : 7.$$

Now, since $12 : 11=84 : 77,$

and $8 : 7=88 : 77,$

it is clear that $8 : 7 > 12 : 11.$

Thus, *compared* with the original strength, the new strength of the second army is greater than that of the first.

Hence, the increase is in favour of the second army.

EXERCISE 118

Which is the greater:

1. $4 : 5$ or $7 : 8$? 2. $7 : 10$ or $11 : 14$? 3. $9 : 5$ or $13 : 8$?

4. $22 : 27$ or $32 : 45$? 5. $28 : 39$ or $49 : 65$?

Find the ratio compounded of:

6. $a : b$, $b : c$ and $c : d$. 7. $3 : 5$, $7 : 9$ and $15 : 28$.

8. $a+x : a-x$, $a^2+x^2 : (a+x)^2$ and $(a^2-x^2)^2 : a^4-x^4$.

9. $16 : 5$, the triplicate ratio of $5 : 4$ and the sub-duplicate ratio of $9 : 4$.

10. $25 : 18$, the sub-duplicate ratio of $81 : 49$, the triplicate ratio of $2 : 3$ and the duplicate ratio of $7 : 5$.

11. If $2x+5y : 3x+5y=9 : 10$, find $x : y$.

12. If $x : y=3 : 4$, find the value of $5x+9y : 16x+5y$.

13. Two numbers are in the ratio of $7 : 8$, and their sum is 135. Find the numbers.

14. Find two numbers which are in the ratio of $5 : 3$, and whose difference is 34.

15. Two numbers are in the ratio of $4 : 5$, and if 7 be added to each, the sums are in the ratio of $5 : 6$. Find the numbers.

16. Two numbers are in the ratio of $7 : 9$, and if 10 be subtracted from each, the remainders are in the ratio of $8 : 11$. Find the numbers.

17. For what value of x will the ratio $23+x : 19+x$ be equal to 2?

18. What number must be added to each term of the ratio 25 : 37, that it may become equal to 5 : 6 ?

19. What number must be added to each term of the ratio 29 : 38, that it may become equal to 4 : 7 ?

20. What quantity must be added to each of the terms of the ratio $a : b$, that it may become equal to $c : d$?

21. Show that if $a > x$, the ratio $a^2 - x^2 : a^2 + x^2$ is greater than the ratio $a - x : a + x$.

22. Show that the ratio $a^2 + b^2 : a + b$ is less than the ratio $a^3 - b^3 : a - b$.

Find approximately the values of :

23. $(226)^3 : (225)^3$. **24.** $\surd(3546) : \surd(3542)$.

25. A, B, C are three school boys getting monthly allowances of Rs. 15, Rs. 20 and Rs. 25 respectively ; out of these amounts they respectively spend Rs. $8\frac{3}{4}$, Rs. $11\frac{1}{4}$ and Rs. $15\frac{5}{8}$ per month. Which of them is the most frugal ?

Proportion

217. Definitions. Four quantities are said to be *proportionals* when the ratio of the first to the second is equal to the ratio of the third to the fourth. Thus, a, b, c, d are proportionals, if $a : b = c : d$. This is often expressed as $a : b : : c : d$ and is read 'a is to b as c is to d'.

The terms a and d are called the **extremes** and the terms b and c, the **means**. The term d is also called the **fourth proportional** to a, b, c.

Three or more quantities are said to be in continued proportion when the first is to the second as the second is to the third, as the third is to the fourth ; and so on. Thus, a, b, c, d are in continued proportion, when $a : b = b : c = c : d$.

If three quantites, a, b, c are in continued proportion ($a : b : : b : c$), then b is called the **mean proportional** between a and c, and c is called the **third proportional** to a and b.

218. If $a : b : : c : d$, then will $ad = bc$.

Since, $$\frac{a}{b} = \frac{c}{d},$$

multiplying both sides by bd, we have $ad = bc$.

Thus, *if four quantities are proportionals, the product of the extremes is equal to the product of the means.*

[Conversely, if $ad = bc$, then $a : b : : c : d$. This is obvious by dividing both sides of the equality by bd.]

Cor. If $a:b::b:c$, then $ac=b^2$; *i.e., if three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.*

Note. *From the result above established we can at once find a third proportional to, or a mean proportional between two given quantities as well as a fourth proportional to three given quantities.*

Example 1. Find a third proportional to 6, 8.

Let x be the required third proportional, then

$$\frac{6}{8}=\frac{8}{x},$$

or, $6x=64$; $\therefore\ x=\frac{64}{6}=10\frac{2}{3}$.

Example 2. Find a fourth proportional to 4, 6, 8.

Let x be the required fourth proportional, then

$$\frac{4}{6}=\frac{8}{x},$$

or, $4x=48$; $\therefore\ x=12$.

Example 3. Find the number that must be added to each of 8, 11, 16, 21, so that the sums are proportional.

Let x be the required number, then

$$\frac{8+x}{11+x}=\frac{16+x}{21+x},$$

or, $(8+x)(21+x)=(16+x)(11+x)$,

or, $x^2+29x+168=x^2+27x+176$,

or, $2x=8$; $\therefore\ x=4$.

Note. *In example 3, if, 'what number must be deducted' was the question, the equation should have been framed with $-x$ in place of x.*

EXERCISE 119

Find a third proportional to :

1. 9, 6.
2. 8, 12.
3. 6, 15.
4. 16, 24.

Find a fourth proportional to :

5. 6, 8, 15.
6. 14, 24, 35.
7. ·0014, 1·4, ·02.

Find a mean proportional between :

8. 4, 9.
9. 7, 28.
10. 6, 54.

11. What number must be added to each of 4, 7, 8, 12, so that the sums will become proportional ?

12. What number must be added to each of a, b, m, n, so that the sums will become proportional?

13. What number must be deducted from each of 8, 10, 17, 22, so that the differences will become proportional?

14. What number must be deducted from each of a, b, m, n, so that the differences will become proportional?

219. If $a : b :: b : c$, then $a : c :: a^2 : b^2$.

For, $\frac{a}{b} = \frac{b}{c}$, $\therefore \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}$; or, $\frac{a}{c} = \frac{a^2}{b^2}$.

Thus, *if three quantities are in continued proportion, the first is to the third in the duplicate ratio of the first is to the second.*

Note. *Similarly, if $a : b = b : c = c : d$, it can be easily proved that $a : d = a^3 : b^3$, which is left as an exercise for the student.*

220. If $a : b :: c : d$, then $b : a :: d : c$.

For, $\frac{a}{b} = \frac{c}{d}$,

$\therefore \quad 1 + \frac{a}{b} = 1 + \frac{c}{d}$, whence, $\frac{b}{a} = \frac{d}{c}$.

Thus, *if four quantities be proportionals, they are also proportionals when taken inversely.*

This operation is called **Invertendo.**

221. If $a : b :: c : d$, then $a : c :: b : d$.

For, $\frac{a}{b} = \frac{c}{d}$,

$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$, or, $\frac{a}{c} = \frac{b}{d}$.

Thus, *if four quantities be proportionals, they are proportionals when taken alternately.*

This operation is called **Alternando.**

222. If $a : b :: c : d$, then $a+b : b :: c+d : d$.

For, $\frac{a}{b} = \frac{c}{d}$,

$\therefore \quad \frac{a}{b} + 1 = \frac{c}{d} + 1$, or, $\frac{a+b}{b} = \frac{c+d}{d}$.

Thus, *when four quantities are proportionals, the first together with the second is to the second as the third together with the fourth is to the fourth.*

This operation is called **Componendo**.

223. **If $a : b :: c : d$, then $a-b : b :: c-d : d$.**

For, $\frac{a}{b}=\frac{c}{d}$,

$\therefore$ $\frac{a}{b}-1=\frac{c}{d}-1$, or, $\frac{a-b}{b}=\frac{c-d}{d}$.

Thus, *when four quantities are proportionals, the excess of the first over the second is to the second as the excess of the third over the fourth is to the fourth.*

This operation is called **Dividendo**.

Cor. **If $a : b :: c : d$, then $a : a-b :: c : c-d$.**

For, $\frac{a-b}{b}=\frac{c-d}{d}$, $\therefore$ inversely, $\frac{b}{a-b}=\frac{d}{c-d}$.

Hence, $\frac{b}{a-b}\times\frac{a}{b}=\frac{d}{c-d}\times\frac{c}{d}$, or, $\frac{a}{a-b}=\frac{c}{c-d}$.

Thus, *when four quantities are proportionals, the first is to the excess of the first over the second as the third is to the excess of the third over the fourth.*

This operation is called **Convertendo**.

223A. **If $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$.**

Since $\frac{a}{b}=\frac{c}{d}$, $\therefore$ $\frac{a}{c}=\frac{b}{d}$ [Alternando]

$\therefore$ $\frac{a+c}{c}=\frac{b+d}{d}$ [Componendo]

$\therefore$ $\frac{a+c}{b+d}=\frac{c}{d}$ [Alternando]

$\therefore$ $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$.

Similarly it can be proved that

$$\frac{a}{b}=\frac{c}{d}=\frac{a-c}{b-d},$$

and $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\ldots\ldots$, each of the ratios $=\frac{a+c+e+g+\cdots}{b+d+f+h+\cdots}$.

This operation is called Addendo.

224. If $a : b :: c : d$, then $a+b : a-b :: c+d : c-d$.

From Art. 222, $\dfrac{a+b}{b}=\dfrac{c+d}{d}$. ... (1)

From Art. 223, $\dfrac{a-b}{b}=\dfrac{c-d}{d}$. ... (2)

Hence, dividing (1) by (2), $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

Thus, *when four quantities are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

This result is often spoken of as **Componendo and Dividendo.**

Note. *The result proved in this article is of great use in solving a certain class of equations. This will be illustrated in some of the following examples.*

Example 1. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$.

By componendo and dividendo, we have

$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}}=\frac{b+1}{b-1}.$$

Hence, squaring, $\dfrac{a+x}{a-x}=\left(\dfrac{b+1}{b-1}\right)^2=\dfrac{b^2+2b+1}{b^2-2b+1}$.

Again, applying componendo and dividendo,

$$\frac{2a}{2x}=\frac{2(b^2+1)}{4b}, \quad \text{or,} \quad \frac{a}{x}=\frac{b^2+1}{2b}.$$

$$\therefore \quad x(b^2+1)=2ab. \qquad \therefore \quad x=\frac{2ab}{b^2+1}.$$

Example 2. Solve $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1$.

We have $\sqrt{\dfrac{1+bx}{1-bx}}=\dfrac{1+ax}{1-ax}$. $\therefore$ $\dfrac{1+bx}{1-bx}=\dfrac{1+2ax+a^2x^2}{1-2ax+a^2x^2}$

Hence, by componendo and dividendo,

$$\frac{1}{bx}=\frac{1+a^2x^2}{2ax};$$

$$\therefore \quad b(1+a^2x^2)=2a, \quad \text{or,} \quad a^2x^2=\frac{2a}{b}-1.$$

$$\therefore \quad x=\frac{1}{a}\sqrt{\frac{2a}{b}-1}.$$

Example 3. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, when $x=\frac{4ab}{a+b}$.

[A. U. 1892]

From the given relation, we have

$$\frac{x}{2a}=\frac{2b}{a+b}, \quad \text{and} \quad \frac{x}{2b}=\frac{2a}{a+b}.$$

Hence, by componendo and dividendo,

$$\frac{x+2a}{x-2a}=\frac{a+3b}{b-a}, \quad \text{and} \quad \frac{x+2b}{x-2b}=\frac{3a+b}{a-b}.$$

Hence, the given expression

$$=\frac{-(a+3b)}{a-b}+\frac{3a+b}{a-b}=\frac{2(a-b)}{a-b}=2.$$

Note. *For a different solution of this example see Art. 171, Ex. 2.*

Example 4. If $(a+b+c+d)(a-b-c+d)$
$=(a-b+c-d)(a+b-c-d)$, show that $a : b :: c : d$.

From the given relation, we have

$$\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}.$$

Hence, by componendo and dividendo,

$$\frac{a+b}{c+d}=\frac{a-b}{c-d};$$

$$\therefore \quad \frac{a+b}{a-b}=\frac{c+d}{c-d} \quad [\text{ Alternando }];$$

whence by a second application of componendo and dividendo,

$$\frac{a}{b}=\frac{c}{d}.$$

Example 5. If $x=\frac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$, show that

$$x^3-3mx^2+3x-m=0.$$

From the given relation, by componendo and dividendo, we have

$$\frac{x+1}{x-1}=\frac{\sqrt[3]{m+1}}{\sqrt[3]{m-1}}. \qquad \therefore \quad \frac{m+1}{m-1}=\frac{(x+1)^3}{(x-1)^3}=\frac{x^3+3x^2+3x+1}{x^3-3x^2+3x-1}$$

Hence, by a second application of componendo and dividendo,

we have $$\frac{m}{1}=\frac{x^3+3x}{3x^2+1};$$

$$\therefore \quad m(3x^2+1)=x^3+3x,$$

whence, $x^3-3mx^2+3x-m=0.$

EXERCISE 120

Solve the following equations :

1. $\left.\begin{aligned}\frac{x+y}{x-y}&=5\\ 2x+3y&=36\end{aligned}\right\}$ 2. $\left.\begin{aligned}\frac{3x-5y}{3x+5y}&=\frac{1}{4}\\ 4x-9y&=19\end{aligned}\right\}$ 3. $\left.\begin{aligned}\frac{5x-7y}{5x+7y}&=\frac{1}{7}\\ 3x-5y&=18\end{aligned}\right\}$

4. $16\left(\dfrac{a-x}{a+x}\right)^3=\dfrac{a+x}{a-x}.$ [C. U. 1886] 5. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=4.$ 6. $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}=\dfrac{1}{3}$

7. $\dfrac{\sqrt{36x+1}+\sqrt{36x}}{\sqrt{36x+1}-\sqrt{36x}}=9.$ 8. $\dfrac{1+x+x^2}{1-x+x^2}=\dfrac{62}{63}.\dfrac{1+x}{1-x}.$

9. $\dfrac{\sqrt{5}+\sqrt{5-x}}{\sqrt{5}-\sqrt{5-x}}=5.$ 10. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{b}{x}.$

11. $\dfrac{a^{\frac{1}{2}}-\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}{a^{\frac{1}{2}}+\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}=b.$

Prove that $a : b :: c : d$—

12. If $(a+3b+2c+6d)(a-3b-2c+6d)$
$=(a-3b+2c-6d)(a+3b-2c-6d).$

13. If $(2a+b+4c+2d)(2a-b-4c+2d)$
$=(2a-b+4c-2d)(2a+b-4c-2d).$

14. If $x=\dfrac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$, show that $3bx^2-4ax+3b=0.$

15. If $x=\dfrac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$, find the value of $\dfrac{x+\sqrt{8}}{x-\sqrt{8}}+\dfrac{x+\sqrt{12}}{x-\sqrt{12}}.$

225. An Important Theorem. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, then each of these ratios $=\left(\dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$, where p, q, r, n are any quantities whatever.

Supposing each of the given ratios $=k$, we have $a=bk,\ c=dk,\ e=fk.$

Hence, $\left.\begin{aligned}pa^n&=p(bk)^n=pb^n.k^n\\ qc^n&=q(dk)^n=qd^n.k^n\\ re^n&=r(fk)^n=rf^n.k^n\end{aligned}\right\}\ \therefore\ pa^n+qc^n+re^n=(pb^n+qd^n+rf^n)k^n;$

whence, $k^n=\dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}$; and $\therefore\ k=\left(\dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$

which proves the proposition.

Cor. As a particular case, if p, q, r, n be each equal to 1, we have each of the given ratios $=\dfrac{a+c+e}{b+d+f}$.

Similarly, giving different sets of values to p, q, r, n several particular cases may be at once deduced.

Note. *What is proved above for three equal ratios is obviously true for any number of equal ratios*, namely, if $\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\dfrac{a_3}{b_3}=\cdots=\dfrac{a_p}{b_p}$, then each is equal to $\left(\dfrac{k_1a_1{}^n+k_2a_2{}^n+k_3a_3{}^n+\cdots+k_pa_p{}^n}{k_1b_1{}^n+k_2b_2{}^n+k_3b_3{}^n+\cdots+k_pb_p{}^n}\right)^{\frac{1}{n}}$, *the same reasoning being applicable to all cases. It is always a very good exercise for the student, however, to work out independently every fresh example of this class, applying the mode of demonstration illustrated above. Hence, an exercise is added below with a recommendation to the student that he should find the result in each case without using the formula established in this article.*

EXERCISE 121

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each of these ratios is equal to :

1. $\dfrac{a-c+e}{b-d+f}$. 2. $\dfrac{a+3c-5e}{b+3d-5f}$. 3. $\dfrac{5a-7c-13e}{5b-7d-13f}$. 4. $\dfrac{ka+lc+me}{kb+ld+mf}$.
[C. U. 1875]

5. $\left(\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2}\right)^{\frac{1}{2}}$. 6. $\left(\dfrac{a^3-2c^3+3e^3}{b^3-2d^3+3f^3}\right)^{\frac{1}{3}}$. 7. $\dfrac{\sqrt[3]{a^3+c^3+e^3}}{\sqrt[3]{b^3+d^3+f^3}}$.
[C. U. 1882]

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\dfrac{g}{h}$, prove that each of these ratios is equal to :

8. $\left(\dfrac{a^{-1}+c^{-1}+e^{-1}+g^{-1}}{b^{-1}+d^{-1}+f^{-1}+h^{-1}}\right)^{-1}$. 9. $\sqrt[4]{\dfrac{a^4-2c^4+3e^4-4g^4}{b^4-2d^4+3f^4-4h^4}}$.

10. $\sqrt{\left(\dfrac{3a^{-2}-7c^{-2}-8e^{-2}+15g^{-2}}{3b^{-2}-7d^{-2}-8f^{-2}+15h^{-2}}\right)^{-1}}$.

226. Miscellaneous Examples.

Example 1. If $x:y::m^2:n^2$, and

$$m:n::\sqrt{p^2+x^2}:\sqrt{p^2-y^2},\text{ then } p^2:xy::x+y:x-y.$$

We have $\dfrac{x}{y}=\dfrac{m^2}{n^2}=\dfrac{p^2+x^2}{p^2-y^2}$.

$\therefore \quad x(p^2-y^2)=y(p^2+x^2)$ [Art. 218] or, $$p^2(x-y)=xy(x+y);$$

$$\therefore\ \frac{p^2}{xy}=\frac{x+y}{x-y}; \qquad \text{[Art. 218, Converse]}$$

i.e., $$p^2 : xy :: x+y : x-y.$$

Example 2. If $a : b :: c : d$, show that

$$ma+nc : mb+nd :: (a^2+c^2)^{\frac{1}{2}} : (b^2+d^2)^{\frac{1}{2}}. \qquad \text{[C. U. 1880]}$$

Since, $$\frac{a}{b}=\frac{c}{d}; \qquad \therefore\ \frac{ma}{mb}=\frac{nc}{nd},$$

and, therefore, each of them $=\dfrac{ma+nc}{mb+nd}$. [Art. 225]

Again, since $$\frac{a}{b}=\frac{c}{d}; \qquad \therefore\ \frac{a^2}{b^2}=\frac{c^2}{d^2},$$

and, therefore, each of them $=\dfrac{a^2+c^2}{b^2+d^2}$. [Art. 225]

Thus, we have $$\frac{ma+nc}{mb+nd}=\frac{ma}{mb}=\frac{a}{b}, \qquad \cdots \quad (1)$$

and $$\frac{a^2+c^2}{b^2+d^2}=\frac{a^2}{b^2}. \qquad \cdots \quad (2)$$

Hence, from (1) and (2),

$$\frac{ma+nc}{mb+nd}=\frac{(a^2+c^2)^{\frac{1}{2}}}{(b^2+d^2)^{\frac{1}{2}}},$$ which was to be proved.

Example 3. If $\dfrac{x}{(b-c)(b+c-2a)}=\dfrac{y}{(c-a)(c+a-2b)}=\dfrac{z}{(a-b)(a+b-2c)}$, find the value of $x+y+z$. [C. U. 1889]

Let each of the given ratios $=k$.

Then, $$x=k(b-c)(b+c-2a)=k\{(b^2-c^2)-2a(b-c)\},$$
$$y=k(c-a)(c+a-2b)=k\{(c^2-a^2)-2b(c-a)\},$$
$$z=k(a-b)(a+b-2c)=k\{(a^2-b^2)-2c(a-b)\}$$

Hence, $$x+y+z=k[\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\} \\ -2\{a(b-c)+b(c-a)+c(a-b)\}] \\ =0.$$

Example 4. If $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}$ show that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

Let each of the given ratios $=k$.

Then, we have $$(ay-bx)c=kc^2,$$
$$(cx-az)b=kb^2$$
$$(bz-cy)a=ka^2$$

Hence, by addition,

$$k(a^2+b^2+c^2)=0 \qquad \therefore \quad k=0$$

Hence, $\quad ay-bx=0 \quad \therefore \; ay=bx. \quad \therefore \; \dfrac{x}{a}=\dfrac{y}{b}, \qquad \cdots \; (1)$

also, $\quad cx-az=0. \quad \therefore \; cx=az. \quad \therefore \; \dfrac{x}{a}=\dfrac{z}{c}. \qquad \cdots \; (2)$

Hence, from (1) and (2), $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}.$

Example 5. If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, then will

$$(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2.$$

From the given relations, we have

(i) $b^2=ac$; (ii) $c^2=bd$; (iii) $bc=ad$. [Art. 218]

Now, $(b-c)^2+(c-a)^2+(d-b)^2$

$$=(b^2+c^2-2bc)+(c^2+a^2-2ac)+(d^2+b^2-2bd)$$
$$=2(b^2-ac)+2(c^2-bd)+a^2+d^2-2bc$$
$$=a^2+d^2-2bc \qquad \text{[from (i) and (ii)]}$$
$$=a^2+d^2-2ad \qquad \text{[from (iii)]}$$
$$=(a-d)^2.$$

Example 6. If $a:b::c:d$, show that

$$4(a+b)(c+d)=bd\left\{\frac{a+b}{b}+\frac{c+d}{d}\right\}^2. \qquad \text{[C. U. 1874]}$$

Since, $\dfrac{a}{b}=\dfrac{c}{d}$, $\quad \therefore \; \dfrac{a+b}{b}=\dfrac{c+d}{d}$; [componendo]

clearly, therefore, $\dfrac{a+b}{b}+\dfrac{c+d}{d}=\dfrac{2(a+b)}{b}=\dfrac{2(c+d)}{d}.$

Hence, $\quad \left\{\dfrac{a+b}{b}+\dfrac{c+d}{d}\right\}^2=\dfrac{2(a+b)}{b}\times\dfrac{2(c+d)}{d}=\dfrac{4(a+b)(c+d)}{bd}.$

$\therefore \quad bd\left\{\dfrac{a+b}{b}+\dfrac{c+d}{d}\right\}^2=4(a+b)(c+d).$

Example 7. If $a:b::p:q$, show that

$$a^2+b^2:\frac{a^3}{a+b}::p^2+q^2:\frac{p^3}{p+q}.$$

From the given relations, we have

$$\frac{b}{a}=\frac{q}{p}, \quad \text{and} \quad \therefore \; \frac{b^2}{a^2}=\frac{q^2}{p^2}.$$

Hence, (i) $\dfrac{a+b}{a}=\dfrac{p+q}{p}$, and (ii) $\dfrac{a^2+b^2}{a^2}=\dfrac{p^2+q^2}{p^2}$.

Multiplying together (i) and (ii), we have

$$\frac{(a^2+b^2)(a+b)}{a^3}=\frac{(p^2+q^2)(p+q)}{p^3},$$

or,
$$\frac{a^2+b^2}{\left(\dfrac{a^3}{a+b}\right)}=\frac{p^2+q^2}{\left(\dfrac{p^3}{p+q}\right)};$$

i.e.,
$$a^2+b^2:\frac{a^3}{a+b}::p^2+q^2:\frac{p^3}{p+q}.$$

Example 8. If $m:n::p:q$, prove that

$$\frac{(m-n)(m-p)}{m}=(m+q)-(n+p). \qquad [\text{C. U. } 1859]$$

We have $\dfrac{m}{n}=\dfrac{p}{q}$; $\quad\therefore\ \dfrac{m-n}{n}=\dfrac{p-q}{q}$;

alternately, $\dfrac{m}{p}=\dfrac{n}{q}$; $\quad\therefore\ \dfrac{m-p}{p}=\dfrac{n-q}{q}$.

Hence, $\dfrac{(m-n)(m-p)}{np}=\dfrac{(p-q)(n-q)}{q^2}$,

or, $\dfrac{(m-n)(m-p)}{mq}=\dfrac{(p-q)(n-q)}{q^2}$, $\qquad[\because\ np=mq]$

$$\begin{aligned}\therefore\quad \frac{(m-n)(m-p)}{m}&=\frac{pn-q(n+p)+q^2}{q}\\&=\frac{mq+q^2-q(n+p)}{q} \qquad [\because\ pn=mq]\\&=(m+q)-(n+p).\end{aligned}$$

Example 9. If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, show that

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2. \qquad [\text{C. U. } 1887]$$

Let each of the given ratios $=k$.

Then,
$$\left.\begin{aligned}k^2b^2&=a^2\\k^2c^2&=b^2\\k^2d^2&=c^2\end{aligned}\right\}\qquad \begin{aligned}&\therefore\ k^2(b^2+c^2+d^2)=a^2+b^2+c^2;\\&\therefore\ k^2=\frac{a^2+b^2+c^2}{b^2+c^2+d^2}; \qquad \cdots\ (1)\end{aligned}$$

also,
$$\begin{aligned}kb&=a;\\kc&=b;\\kd&=c;\end{aligned}\quad \left.\begin{aligned}\therefore\ kb^2&=ab\\\therefore\ kc^2&=bc\\\therefore\ kd^2&=cd\end{aligned}\right\}\qquad \begin{aligned}&\therefore\ k(b^2+c^2+d^2)=ab+bc+cd;\\&\therefore\ k=\frac{ab+bc+cd}{b^2+c^2+d^2}. \qquad \cdots\ (2)\end{aligned}$$

Hence, equating the value of k^2 from (1) and (2), we have

$$\frac{a^2+b^2+c^2}{b^2+c^2+d^2}=\frac{(ab+bc+cd)^2}{(b^2+c^2+d^2)^2};$$

$$\therefore\quad (a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2.$$

Example 10. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that

$$\sqrt{(a+c+e)(b+d+f)}=(ab)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}+(ef)^{\frac{1}{2}}.$$

Let each of the given ratios $=k$.

$$\text{Then, }\left.\begin{array}{l}a=bk\\ c=dk\\ e=fk\end{array}\right\}\qquad \begin{array}{l}\therefore\ a+c+e=k(b+d+f);\\ \therefore\ (a+c+e)(b+d+f)=k(b+d+f)^2;\end{array}$$

$$\therefore\quad \sqrt{(a+c+e)(b+d+f)}=(b+d+f)\sqrt{k}. \qquad \cdots \quad (1)$$

$$\text{Also, we have}\quad \begin{array}{ll}ab=b^2k; & \therefore\ (ab)^{\frac{1}{2}}=b\sqrt{k}\\ cd=d^2k; & \therefore\ (cd)^{\frac{1}{2}}=d\sqrt{k}\\ ef=f^2k; & \therefore\ (ef)^{\frac{1}{2}}=f\sqrt{k}\end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}$$

$$\therefore\quad (ab)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}+(ef)^{\frac{1}{2}}=(b+d+f)\sqrt{k}. \qquad \cdots \quad (2)$$

Hence, from (1) and (2),

$$\sqrt{(a+c+e)(b+d+f)}=(ab)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}+(ef)^{\frac{1}{2}}.$$

EXERCISE 122

If a be the greatest of the four quantities a, b, c, d and if $a:b::c:d$, show that :

1. b and c are each $> d$. 2. $a-b > c-d$. 3. $a+d > b+c$.

If $a:b::c:d$, show that :

4. $ma+nb : b :: mc+nd : d$.
5. $ma+nb : mc+nd :: pa-qb : pc-qd$.
6. $a : b :: a+c : b+d$. 7. $a^2 : b^2 :: a^2+c^2 : b^2+d^2$.
8. $a^2+c^2 : b^2+d^2 :: ac : bd$. [C. U. 1877]
9. $(a-c)^2 : (b-d)^2 = a^2 : b^2$.
10. $(a+c)^3 : (b+d)^3 = a(a-c)^2 : b(b-d)^2$. [C. U. 1888]
11. $a^2+b^2 : a^2-b^2 = ac+bd : ac-bd$.
12. $a(a+c) : c^2 :: b(b+d) : d^2$. 13. $c : d = \sqrt{a^2+c^2} : \sqrt{b^2+d^2}$.
14. $a+b : c+d = \sqrt{a^2+b^2} : \sqrt{c^2+d^2}$.
15. $a+b : c+d :: \sqrt{3a^2+5b^2} : \sqrt{3c^2+5d^2}$.
16. $a^2+ab+b^2 : a^2-ab+b^2 :: c^2+cd+d^2 : c^2-cd+d^2$.
17. $a^2+ac+c^2 : a^2-ac+c^2 :: b^2+bd+d^2 : b^2-bd+d^2$.

If $a : b = c : d = e : f$, show that

18. $\dfrac{ma+nb}{mc+nd}=\dfrac{b^2c}{d^2a}$. [C. U. 1876]

19. $ac : bd :: 2a^2+3c^2+5e^2 : 2b^2+3d^2+5f^2$.

20. $a^2+c^2+e^2 : b^2+d^2+f^2 :: ce : df$. [C. U. 1878]

21. $pa+qc+re : pb+qd+rf :: \sqrt[3]{ace} : \sqrt[3]{bdf}$.

22. $a^2 : b^2 :: ac+ce+ae : bd+df+bf$.

23. $a^3+c^3+e^3 : b^3+d^3+f^3 :: ace : bdf$.

24. $\sqrt{a^3c^3+c^3e^3+a^3e^3} : \sqrt{b^3d^3+d^3f^3+b^3f^3} :: ace : bdf$.

25. $\dfrac{a^2+c^2+e^2}{ab+cd+ef}=\dfrac{ab+cd+ef}{b^2+d^2+f^2}$.

26. If a, b, c, d, e be in continued proportion, show that

$$a : e :: a^4 : b^4.$$

27. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, find the value of

$(b-c)x+(c-a)y+(a-b)z$. [C. U. 1878]

28. If $a : b :: c : d$, prove that

$$a^2+c^2 : b^2+d^2 :: \sqrt{a^4+c^4} : \sqrt{b^4+d^4}.$$

29. If a, b, c, d be in continued proportion, prove that

(i) $a+b : c+d = a^2+b^2+c^2 : b^2+c^2+d^2$. [C. U. 1939]

(ii) $(a^2-b^2)(c^2-d^2)=(b^2-c^2)^2$. [C. U. 1943]

(iii) $(a+b)(c+d)=(b+c)^2$. [D. B. 1924]

30. If $a : b = c : d = e : f$, show that

$$27(a+b)(c+d)(e+f)=bdf\left(\frac{a+b}{b}+\frac{c+d}{d}+\frac{e+f}{f}\right)^3.$$

31. If $a : b :: c : d$, show that $ad+bc : 2bd :: a^2+c^2 : ab+cd$.

32. If $a : b :: c : d$, show that

$$a^2+b^2 : ab+ad-bc :: c^2+d^2 : cd-ad+bc.$$

If $a : b :: b : c$, show that

33. $a^2+ab+b^2 : b^2+bc+c^2 = a : c$.

34. $a-2b+c=\dfrac{(a-b)^2}{a}=\dfrac{(b-c)^2}{c}$.

35. $a^2b^2c^2\left(\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}\right)=a^3+b^3+c^3$.

If $a : b = b : c = c : d$, show that

36. $(b+c)(b+d)=(c+a)(c+d)$. 37. $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$.

38. $\left(\dfrac{a-b}{c}+\dfrac{a-c}{b}\right)^2-\left(\dfrac{d-b}{c}+\dfrac{d-c}{b}\right)^2=(a-d)^2\left(\dfrac{1}{c^2}-\dfrac{1}{b^2}\right)$.

39. $a : b = \frac{1}{b} + \frac{1}{c} + \frac{1}{d} : \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

40. $a : d :: a^3 + b^3 + c^3 : b^3 + c^3 + d^3$.

41. If $a : b :: c : d$, show that $a^2 + ab : c^2 + cd :: b^2 - 2ab : d^2 - 2cd$.

42. If $a : b = c : d = e : f$, show that

$$(a^2+b^2)(ce+df)^2 = (c^2+d^2)(ae+bf)^2 = (e^2+f^2)(ac+bd)^2.$$

CHAPTER XXXIII

ELIMINATION, MISCELLANEOUS THEOREMS AND ARTIFICES

I. Elimination

227. If there be *two* equations involving *one* unknown quantity, they will generally not be satisfied by the same value of it. For instance, the same value of x will *not* satisfy the equations $x+3=7$ and $x+4=9$. But this cannot be strictly said of the two equations $x+a=7$ and $x+b=9$, where a and b have no fixed numerical values; the appropriate remark in this case would be "the two equations *will be* satisfied by the same value of x *if* $7-a=9-b$, or, $b-a=2$". Thus, if *one* unknown quantity occurs in *two* equations which *also* involve other *algebraical symbols*, there always exists a particular relation between these other symbols for which, and for which alone, *both* the given equations are satisfied by the *same* value of the unknown quantity. The process of finding this relation is called the **Elimination** of the unknown quantity from the given equations, and the relation obtained is called the **Eliminant** of those equations.

Similarly there may be a question of eliminating two unknown quantities from three given equations. For instance, the three equations $x+y=a$, $x+2y=b$, $x+3y=c$, *cannot* be *all* satisfied by the *same* values of x and y *unless* the quantities a, b, c are connected with one another in a certain way, and this connection may be necessary to investigate.

A few simple cases of elimination will now be presented to the student, calculated to give him a tolerably clear idea of the subject, as also to familiarise him with some of the various ways of dealing with such questions.

Example 1. Eliminate x from the equations

$$a_1x+b_1=0,\ a_2x+b_2=0.$$

From the first equation, we have $x=-\frac{b_1}{a_1}$, and from the second equation, $x=-\frac{b_2}{a_2}$.

Evidently, therefore, both the equations will be satisfied by the same value of x if $\frac{b_1}{a_1}=\frac{b_2}{a_2}$, or, $a_1b_2=a_2b_1$.

Thus, $a_1b_2=a_2b_1$ is the required eliminant.

Example 2. Eliminate x from the equations

$$a_1x^2+b_1x+c_1=0,\ a_2x^2+b_2x+c_2=0.$$

Let α be the value of x which satisfies both the equations. Then, we must have

$$\left.\begin{array}{l} a_1\alpha^2+b_1\alpha+c_1=0 \\ a_2\alpha^2+b_2\alpha+c_2=0 \end{array}\right\}$$

Hence, by cross multiplication,

$$\frac{\alpha^2}{b_1c_2-b_2c_1}=\frac{\alpha}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}.$$

$$\therefore \quad \frac{\alpha^2}{b_1c_2-b_2c_1}\times\frac{1}{a_1b_2-a_2b_1}=\left(\frac{\alpha}{c_1a_2-c_2a_1}\right)^2,$$

whence, $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^2$, which is the required eliminant.

Example 3. Eliminate x and y from the equations

$$\left.\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \\ a_3x+b_3y+c_3=0 \end{array}\right\}$$

From the first two equations, by cross multiplication, we have

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1};$$

$$\therefore \quad x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \quad y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

If the third equation also be satisfied by these values of x and y, we must evidently have

$$a_3.\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}+b_3.\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}+c_3=0,$$

or, $$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0,$$

which is the required eliminant.

Example 4. Eliminate x, y, z from the equations

$$\frac{ax}{by+cz}=\frac{by}{cz+ax}=\frac{z}{x+y}=\frac{1}{2}.$$

We have $$\frac{ax}{by+cz}=\frac{1}{2},$$

$$\therefore \quad 2ax=by+cz, \text{ or, } 2ax-by-cz=0. \qquad \cdots \quad (1)$$

Also, $$\frac{by}{cz+ax}=\frac{1}{2},$$

$$\therefore \quad 2by=cz+ax, \text{ or, } ax-2by+cz=0. \qquad \cdots \quad (2)$$

Hence, from (1) and (2), by cross multiplication, we have

$$\frac{x}{-bc-2bc}=\frac{y}{-ca-2ca}=\frac{z}{-4ab+ab},$$

$$\text{or,} \quad \frac{x}{-3bc}=\frac{y}{-3ca}=\frac{z}{-3ab}, \qquad \text{or,} \quad \frac{x}{bc}=\frac{y}{ca}=\frac{z}{ab}.$$

Supposing each of these ratios $=k$, we have

$$x=k.bc, \quad y=k.ca, \quad z=k.ab.$$

Substituting these values of x, y, z in the third equation which is $2z=x+y$, we have

$$2k.ab=k(bc+ca), \text{ or, } 2ab=bc+ac,$$

$$\therefore \quad \frac{2}{c}=\frac{1}{a}+\frac{1}{b},$$

which is the required eliminant.

Note. *It may be noticed in this example that the three given equations* $2ax-by-cz=0$, $ax-2by+cz=0$ *and* $2z=x+y$ *virtually involve two unknown quantities, instead of three; for they are respectively equivalent to* $2a\left(\frac{x}{z}\right)-b\left(\frac{y}{z}\right)-c=0$, $a\left(\frac{x}{z}\right)-2b\left(\frac{y}{z}\right)+c=0$ *and* $2=\left(\frac{x}{z}\right)+\left(\frac{y}{z}\right)$, *in which the only unknown quantities are* $\frac{x}{z}$ *and* $\frac{y}{z}$.

It is owing to this disguised character (so to speak) of the three given equations that we have been able to eliminate from them the three unknown quantities x, y, z; *otherwise a fourth equation would have been required for the purpose.*

N.B. The number of equations required to eliminate any number of unknown quantities is one more than the number of unknown quantities to be eliminated.

Example 5. Eliminate x from the equations

$$x^3+\frac{3}{x}=4(a^3+b^3), \quad 3x+\frac{1}{x^3}=4(a^3-b^3).$$

Adding together the equations, we have

$$x^3+3x+\frac{3}{x}+\frac{1}{x^3}=8a^3,$$

$$\text{or,}\quad \left(x+\frac{1}{x}\right)^3=(2a)^3,$$

$$\therefore\quad x+\frac{1}{x}=2a.\quad \cdots \qquad \cdots \ (1)$$

Subtracting the second equation from the first, we have

$$x^3-3x+\frac{3}{x}-\frac{1}{x^3}=8b^3,$$

$$\text{or,}\quad \left(x-\frac{1}{x}\right)^3=(2b)^3.$$

$$\therefore\quad x-\frac{1}{x}=2b.\quad \cdots \qquad \cdots \ (2)$$

From (1) and (2), by addition,

$$2x=2(a+b),\quad \text{or,}\quad x=a+b\,;$$

and by subtraction, $\frac{2}{x}=2(a-b)$, or, $\frac{1}{x}=a-b$.

Hence, $(a+b)(a-b)=x\times\frac{1}{x}=1$.

Thus, $a^2-b^2=1$ is the required eliminant.

Example 6. Eliminate x, y, z from the equations

$$\left.\begin{array}{rll} x+y+z=a & \cdots \qquad \cdots & (1)\\ 2(yz+zx+xy)=b^2 & \cdots \qquad \cdots & (2)\\ x^3+y^3+z^3=c^3 & \cdots \qquad \cdots & (3)\\ 3xyz=d^3 & \cdots \qquad \cdots & (4) \end{array}\right\}$$

Since, $x^2+y^2+z^2=(x+y+z)^2-2(yz+zx+xy)$,

$\therefore$ from (1) and (2), $x^2+y^2+z^2=a^2-b^2 \quad \cdots \quad (5)$

Now, since $x^3+y^3+z^3-3xyz$

$$=(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$

$$=(x+y+z)\{(x^2+y^2+z^2)-(yz+zx+xy)\},$$

from (3), (4), (1), (5) and (2), we must have

$$c^3-d^3=a\{(a^2-b^2)-\tfrac{1}{2}b^2\}=a^3-\tfrac{3}{2}ab^2,$$

$$\text{or,}\quad 2a^3-3ab^2-2c^3+2d^3=0,$$

which is the required eliminant.

Example 7. Eliminate x, y, z from the equations

(i) $x^2(y+z)=a^2$; (ii) $y^2(x+z)=b^2$;

(iii) $z^2(x+y)=c^2$; (iv) $xyz=abc$.

Multiplying the first three equations together, we have

$$x^2y^2z^2(y+z)(z+x)(x+y)=a^2b^2c^2. \quad \cdots \quad (A)$$

Hence, from (A) and (iv), $(y+z)(z+x)(x+y)=1$. $\cdots$ (a)

But
$$\begin{aligned}(y+z)(z+x)(x+y)&=(y+z)\{x^2+x(y+z)+yz\}\\&=x^2(y+z)+x(y^2+z^2+2yz)+yz(y+z)\\&=x^2(y+z)+y^2(x+z)+z^2(x+y)+2xyz,\end{aligned}$$

and $\therefore$ from the given equations, $(y+z)(z+x)(x+y)=a^2+b^2+c^2+2abc$.

Hence, from (a), we have $a^2+b^2+c^2+2abc=1$, as the required eliminant.

EXERCISE 123

Eliminate x from the equations :

1. $\left.\begin{aligned}a^2x^2-b^2&=0\\cx-d&=0\end{aligned}\right\}$ 2. $\left.\begin{aligned}ax^2-b&=0\\cx^3-d&=0\end{aligned}\right\}$

3. $\left.\begin{aligned}mx^3-n&=0\\px^4-q&=0\end{aligned}\right\}$ 4. $\left.\begin{aligned}ax^2+bx+c&=0\\x+d&=0\end{aligned}\right\}$

5. $\left.\begin{aligned}lx^2+mx+n&=0\\ax+b&=0\end{aligned}\right\}$ 6. $\left.\begin{aligned}ax^2+bx+c&=0\\lx^2+mx+n&=0\end{aligned}\right\}$

7. $\left.\begin{aligned}x+\frac{1}{x}&=a+b\\x-\frac{1}{x}&=a-b\end{aligned}\right\}$ 8. $\left.\begin{aligned}2x+\frac{3}{x}&=5p+7q\\2x-\frac{3}{x}&=5p-7q\end{aligned}\right\}$

9. $\left.\begin{aligned}a_1x^3+b_1x+c_1&=0\\a_2x^3+b_2x+c_2&=0\end{aligned}\right\}$ 10. $\left.\begin{aligned}a_1x^3+b_1x^2+c_1&=0\\a_2x^3+b_2x^2+c_2&=0\end{aligned}\right\}$

11. $\left.\begin{aligned}a_1x^4+b_1x^3+c_1&=0\\a_2x^4+b_2x^3+c_2&=0\end{aligned}\right\}$ 12. $\left.\begin{aligned}ax^3+bx+c&=0 \cdots (1)\\x^2+mx+n&=0 \cdots (2)\end{aligned}\right\}$

[Multiply (2) by ax and subtract (1) from the resulting equation ; we thus get $amx^2+(an-b)x-c=0$. Now eliminate x from this equation and (2).]

13. $\left.\begin{aligned}ax^2+bx+c&=0\\x^3+2x^2+3&=0\end{aligned}\right\}$

Eliminate x and y from the equations :

14. $\left.\begin{aligned} ax+by&=m\\ bx-ay&=n\\ x^2+y^2&=1 \end{aligned}\right\}$ 15. $\left.\begin{aligned} ax+b&=cy\\ a_1y+b_1&=c_1x\\ x^2+y^2&=1 \end{aligned}\right\}$ 16. $\left.\begin{aligned} ax+by&=0\\ lx^2+mxy+ny^2&=0 \end{aligned}\right\}$

Eliminate x, y, z from the equations :

17. $\dfrac{x}{y+z}=a,\ \dfrac{y}{z+x}=b,\ \dfrac{z}{x+y}=c.$

18. $\dfrac{y-z}{y+z}=a,\ \dfrac{z-x}{z+x}=b,\ \dfrac{x-y}{x+y}=c.$

19. $\dfrac{y}{z}+\dfrac{z}{y}=a,\ \dfrac{z}{x}+\dfrac{x}{z}=b,\ \dfrac{x}{y}+\dfrac{y}{x}=c.$ [Example 6, Art. 171, may be consulted with profit.]

20. $x^2(y-z)=a,\ y^2(z-x)=b,\ z^2(x-y)=c,\ xyz=d.$

21. Eliminate a, b, c from the equations :

$$bz+cy=a,\ az+cx=b,\ ay+bx=c.$$

II. Miscellaneous Theorems

228. Theorem. *If the sum of the squares of any number of real quantities be zero, then each of the quantities is zero.*

Let $A^2+B^2+C^2+D^2+\cdots\cdots=0$, where A, B, C, D,...... are real quantities.

To prove that $A=0$, $B=0$, $C=0$, $D=0$, ...

Proof. If the sum of any number of quantities be zero, evidently they must be partly positive and partly negative *unless each of them is zero.*

Hence, A, B, C, D, etc. being real, their squares A^2, B^2, C^2, D^2, etc. are *all* positive. Hence, the sum of $A^2+B^2+C^2+D^2+\cdots\cdots$ cannot be zero unless each of A^2, B^2, C^2, etc. is zero ;

$$\therefore\quad A^2=0,\ B^2=0,\ C^2=0, \text{ etc.,}$$

$$i.e.,\quad A=0,\ B=0,\ C=0, \text{ etc.}$$

Example 1. If $a^2+b^2+c^2-bc-ca-ab=0$, prove that $a=b=c$, a, b, c being real.

We have, $a^2+b^2+c^2-bc-ca-ab=\frac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\}=0.$

Hence, $b-c=0$, $c-a=0$, and $a-b=0$, *i.e.*, $a=b=c$.

Example 2. If x, y, a and b be real, solve $(x-a)^2+(y-b)^2=0$

Since, x, y, a and b are real, $(x-a)$ and $(y-b)$ are both real.

$\therefore$ from the given equation, we have

$$x-a=0,\ i.e.,\ x=a,\ \text{and}\ \ y-b=0,\ i.e.,\ y=b.$$

Example 3. Show that if $(x^2+y^2+z^2)(a^2+b^2+c^2)$

$$=(ax+by+cz)^2, \text{ then } \frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

From the given relation, we have

$$a^2(y^2+z^2)+b^2(x^2+z^2)+c^2(x^2+y^2)=2abxy+2acxz+2bcyz.$$

Hence, by transposition, $(a^2y^2+b^2x^2-2abxy)$

$$+(a^2z^2+c^2x^2-2acxz)+(b^2z^2+c^2y^2-2bcyz)=0.$$

or, $(ay-bx)^2+(az-cx)^2+(bz-cy)^2=0.$

$$\text{Hence, } \left.\begin{array}{lll} ay-bx=0\,; & \therefore & \frac{x}{a}=\frac{y}{b} \\ az-cx=0\,; & \therefore & \frac{x}{a}=\frac{z}{c} \\ bz-cy=0\,; & \therefore & \frac{y}{b}=\frac{z}{c} \end{array}\right\}$$

Thus, we have $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

EXERCISE 124

[*N.B. Letters stand for real quantities in the following examples.*]

1. If $(x+a)^2+(y+b)^2=4(xa+yb)$, prove that $x=a$, $y=b$.
2. If $(x+a)^2+(y+b)^2+(z+c)^2=4(xa+yb+zc)$, prove that $x=a$, $y=b$ and $z=c$.
3. If $a^2+b^2+c^2+bc+ca+ab=0$, prove that $a=b=c=0$.
4. Solve $(x^2+y^2)(a^2+b^2)-(ax+by)^2+(y-b)^2=0$.
5. Solve $x^2+y^2+2=(1+x)(1+y)$.
6. Solve $x^2+2y^2+a^2=2y(x+a)$.
7. Solve $2(x+y-1)=x^2+y^2+z^2$.
8. Solve $1+ax+by=\sqrt{\{(1+x^2+y^2)(1+a^2+b^2)\}}$.

229. Inequalities. If a and b be two real quantities, a is said to be $>b$, when $a-b$ is positive.

Thus, $7>5$, since $7-5=+2$;

$-3>-8$, since $(-3)-(-8)=+5$;

$a^2+1>2a$, since $a^2+1-2a=(a-1)^2=$ a positive quantity.

An Inequality $a>b$ is, therefore, established if $a-b$ can be proved to be positive.

Theorem. *If x and y be real and unequal, then $x^2+y^2 > 2xy$.*

$$(x^2+y^2)-(2xy)=x^2-2xy+y^2$$
$$=(x-y)^2=\text{a positive quantity ;}$$
$$\therefore \quad x^2+y^2 > 2xy.$$

Note. *If $x=y$, $(x^2+y^2)-(2xy)=(x-y)^2=0$,*
i.e., $x^2+y^2=2xy$.

Hence, x^2+y^2 is never less than $2xy$.

Most of the results in Inequalities may be obtained by the application of the above theorem.

Example 1. If x, y and z be real and unequal quantities, show that $$x^2+y^2+z^2 > yz+zx+xy.$$

We have $x^2+y^2 > 2xy,$
$y^2+z^2 > 2yz,$
and $z^2+x^2 > 2zx.$

Adding, $2(x^2+y^2+z^2) > 2(xy+yz+zx),$

or, $x^2+y^2+z^2 > yz+zx+xy.$

Otherwise : $x^2+y^2+z^2-(yz+zx+xy)$
$=\frac{1}{2}[(y-z)^2+(z-x)^2+(x-y)^2]=$ a positive quantity ;

$\therefore \quad x^2+y^2+z^2 > yz+zx+xy.$

Example 2. If a, b, c be positive, real and unequal quantities, prove that

(i) $(b+c)(c+a)(a+b) > 8abc,$

and (ii) $a^2(b+c)+b^2(c+a)+c^2(a+b) > 6abc.$

(i) We have $b+c=(\sqrt{b})^2+(\sqrt{c})^2 > 2\sqrt{b}\sqrt{c}.$

Similarly, $c+a>2\sqrt{c}\sqrt{a},$
and $a+b>2\sqrt{a}\sqrt{b}.$

Multiplying, $(b+c)(c+a)(a+b)>(2\sqrt{b}\sqrt{c})(2\sqrt{c}\sqrt{a})(2\sqrt{a}\sqrt{b}),$
i.e., $>8abc.$

(ii) Also, $(b+c)(c+a)(a+b)$
$=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc>8abc;$

$\therefore \quad a^2(b+c)+b^2(c+a)+c^2(a+b) > 6abc.$

EXERCISE 125

[*N.B. Letters stand for real, positive and unequal quantities in the following examples.*]

Prove that :

1. $a^2-ab+b^2>ab$. 2. $a^3+b^3>ab(a+b)$. 3. $x+\frac{1}{x}>2$.

4. $\frac{a+b}{2}>\frac{2ab}{a+b}$. 5. $a+b+c>\frac{2bc}{b+c}+\frac{2ca}{c+a}+\frac{2ab}{a+b}$.

6. $(a+b+c)(bc+ca+ab)>9abc$. 7. $a^3+b^3+c^3>3abc$.

8. $(a+b+c)^3-a^3-b^3-c^3>24abc$.

9. $(b^2-bc+c^2)(c^2-ca+a^2)(a^2-ab+b^2)>a^2b^2c^2$.

10. $a(b+c)^2+b(c+a)^2+c(a+b)^2>12abc$.

230. Theorem. If the fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, etc. *be unequal*, then $\frac{a+c+e+\cdots\cdots\cdots}{b+d+f+\cdots\cdots\cdots}$ *is greater than the least and less than the greatest of them, the denominators b, d, f,............* **being positive.**

Let $\frac{a}{b}$ be the *smallest* of the fractions.

Hence, $\frac{c}{d}>\frac{a}{b}$; $\frac{e}{f}>\frac{a}{b}$; and so on.

Let $\frac{a}{b}=k$; $\therefore$ $\frac{c}{d}>k$, $\frac{e}{f}>k$; and so on.

Hence, $a=bk$, $c>dk$, $e>fk$, etc.

Adding, $a+c+e+\cdots\cdots>bk+dk+fk+\cdots$

i.e., $>(b+d+f+\cdots\cdots)k$;

$\therefore$ $\frac{a+c+e+\cdots\cdots}{b+d+f+\cdots\cdots}>k$,

i.e., $>$ the least of the fractions.

Similarly, $\frac{a+c+e+\cdots\cdots\cdots}{b+d+f+\cdots\cdots\cdots}$ can be proved to be less than the greatest of all fractions.

231. Maximum and Minimum Values of Expressions.

Example 1. Find the maximum values of $5-2x-x^2$ (*i.e.*, find the algebraically greatest value of $5-2x-x^2$ for various values of x).

The given expression $=5-2x-x^2=6-(1+2x+x^2)=6-(x+1)^2$
$=6+\{-(x+1)^2\}$.

Since, $(x+1)^2$ cannot be negative,
$\{-(x+1)^2\}$ can *never be positive.*

Hence, whatever real values x may have, the given expression can *never* be *greater* than 6.

Evidently, the given expression $=6$, when $x+1=0$,
i.e., when $x=-1$.

Hence, we notice that the expression can be equal to 6 but can never be greater than 6.

$\therefore$ the maximum value of the expression $=6$.

Example 2. Find the minimum value of $4x^2+12x+18$ (*i.e.*, find the algebraically smallest possible value of $4x^2+12x+18$ for various values of x).

The given expression $=(2x+3)^2+9$.

Since, $(2x+3)^2$ cannot be negative, the given expression can *never* be *less* than 9 but can be equal to 9, when $2x+3=0$, *i.e.*, when $x=-1\frac{1}{2}$.

$\therefore$ the smallest value required $=9$.

EXERCISE 126

Find the maximum value of :

1. $6x-x^2-1$. 2. $5+8x-8x^2$. 3. $5+4x-4x^2$.

4. $3+5x-2x^2$. 5. $17+8x-x^2$.

Find the minimum value of :

6. $x^2+\frac{1}{x^2}+4$. 7. $2x^2-7x+6$. 8. $4x^2-9x+5$.

9. $3x^2-5x+4$. 10. $2x^2-13x+22$.

11. Divide 32 into two parts so that their product has the maximum value.

III. Miscellaneous Artifices

232. We shall now work out some examples which require for their solution either the application of some principle with which the student is not already acquainted or some special artifice.

Example 1. Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the difference of two square quantities. [C. U. 1887]

The given expression

$$=\{(x+3a)(x+9a)\}\{(x+5a)(x+7a)\}$$
$$=\{x^2+12ax+27a^2\}\{x^2+12ax+35a^2\}$$
$$=\{(x^2+12ax+31a^2)-4a^2\}\{(x^2+12ax+31a^2)+4a^2\}$$
$$=(x^2+12ax+31a^2)^2-16a^4.$$

Example 2. A man receives $\frac{x}{y}$ths of Rs. 10 and afterwards $\frac{y}{x}$ths of Rs. 10. He then gives away Rs. 20. Show that he cannot lose by the transaction. [C. U. 1881]

The man receives altogether $\left(\frac{x}{y}+\frac{y}{x}\right)\cdot 10$ rupees and gives away 20 rupees.

Clearly, therefore, he loses

$$\text{if } \left(\frac{x}{y}+\frac{y}{x}\right)\cdot 10 < 20,$$

i.e., if $\frac{x}{y}+\frac{y}{x}<2$,

i.e., if $x^2+y^2<2xy$,

i.e., if $x^2+y^2-2xy<0$,

i.e., if $(x-y)^2$ be a negative quantity.

But whichever of x and y may be the greater, $(x-y)^2$ can *never* be negative.

Hence, the man *cannot* lose.

Note. *It may be observed that there is always a gain in this transaction except when $x=y$.*

Example 3. If $\frac{a}{b+c}+\frac{c}{a+b}=\frac{2b}{c+a}$, prove that

$$a^2+c^2=2b^2, \text{ or, } a+b+c=0.$$

From the given relation, we have

$$\frac{a}{b+c}-\frac{b}{c+a}=\frac{b}{c+a}-\frac{c}{a+b},$$

$$\text{or, } \frac{c(a-b)+a^2-b^2}{(b+c)(c+a)}=\frac{a(b-c)+b^2-c^2}{(c+a)(a+b)},$$

or, $\dfrac{(a-b)(c+\overline{a+b})}{b+c}=\dfrac{(b-c)(a+b+c)}{a+b}$,

or, $(a^2-b^2)(a+b+c)=(b^2-c^2)(a+b+c)$,

or, $(a+b+c)\{(a^2-b^2)-(b^2-c^2)\}=0$,

or, $(a+b+c)(a^2+c^2-2b^2)=0$.

Therefore, *either*, $a+b+c=0$, *or*, $a^2+c^2-2b^2=0$,

and $\therefore$ $a^2+c^2=2b^2$.

Note. *It may be observed in this connection that whenever any relation of quality is reduced to the form* $xp=xp_1$ [or, $x(p-p_1)=0$], *it is obviously satisfied ither* (i) *when* $x=0$, *or*, (ii) *when* $p=p_1$, *and that of these two alternative results we annot accept one as the only conclusion to which we are led unless it is known that e other is impossible.*

In the present example, we have got $(a^2-b^2)(a+b+c)=(b^2-c^2)(a+b+c)$ *as ne of the steps in the solution, and it is not difficult to see from this that it would be mistake to remove the common factor* $a+b+c$ *from both sides and set down* a^2-b^2 b^2-c^2 *as the next step ; for the above relation may be true not on account of* a^2-b^2 *ing equal to* b^2-c^2, *but on account of* $a+b+c$ *being equal to zero. We might move* $a+b+c$ *from both sides of the equation, however, if we know that owing to rtain restrictions on the values of the letters* a, b, c, *the expression* $a+b+c$ *could t possibly vanish.*

Hence, the only legitimate conclusion from the relation $xp=xp_1$ [*or*, $x(p-p_1)=0$) '*either*, $x=0$, *or*, $p=p_1$' *but not simply* '$p=p_1$' *except when* x *is known to be t equal to zero.*

Example 4. Show that if $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c+a}{b}=1$, and $a-b+c$ is ot $=0$, then $\dfrac{1}{a}=\dfrac{1}{b}+\dfrac{1}{c}$. [C. U. 1875]

From the given relation, we have

$$1-\frac{b-c}{a}=\frac{a-b}{c}+\frac{c+a}{b}, \quad \text{or,} \quad \frac{a-b+c}{a}=\frac{b(a-b)+c(c+a)}{bc}$$

$$=\frac{b(a-b+c)+c(c+a-b)}{bc}=\frac{(a-b+c)(b+c)}{bc}.$$

Hence, *either*, $a-b+c=0$,

or, $\dfrac{1}{a}=\dfrac{b+c}{bc}$. [See Note, last example.]

But by hypothesis, $a-b+c$ is *not* zero.

Therefore, we must have $\dfrac{1}{a}=\dfrac{b+c}{bc}=\dfrac{1}{b}+\dfrac{1}{c}$.

Example 5. If $a+b+c=0$, show that

$$2(a^4+b^4+c^4)=(a^2+b^2+c^2)^2.$$

From the given relation, we have

$$a+b=-c, \qquad \therefore \quad a^2+2ab+b^2=c^2;$$

$$\therefore \quad a^2+b^2-c^2=-2ab;$$

$$\therefore \quad (a^2+b^2-c^2)^2=4a^2b^2,$$

or $\quad a^4+b^4+c^4+2a^2b^2-2a^2c^2-2b^2c^2=4a^2b^2;$

$\therefore \quad a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2).$

Hence, $\quad 2(a^4+b^4+c^4)=a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)$

$$=(a^2+b^2+c^2)^2.$$

Example 6. If $a+b+c=0$, show that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0.$$

From the given relation, we have

$$a+b=-c; \qquad \therefore \quad a^2+2ab+b^2=c^2;$$

$\therefore \quad a^2+b^2-c^2=-2ab.$

Similarly, $b^2+c^2-a^2=-2bc,$ and $c^2+a^2-b^2=-2ca.$

Hence, the proposed expression

$$=\frac{1}{-2bc}+\frac{1}{-2ca}+\frac{1}{-2ab}=\frac{a+b+c}{-2abc}=\frac{0}{-2abc}=0.$$

Example 7. If $a+b+c=0$, show that

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}=1.$$

We have $\quad 2a^2+bc=a^2+a.a+bc$

$$=a^2-a(b+c)+bc \qquad [\because \; a=-(b+c)]$$

$$=(a-b)(a-c).$$

Similarly, $\quad 2b^2+ca=b^2-b(a+c)+ca=(b-c)(b-a),$

and $\quad 2c^2+ab=c^2-c(a+b)+ab=(c-a)(c-b).$

Hence, the proposed expression

$$=\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$$

$$=\frac{a^2}{(a-b)(a-c)}+\frac{-b^2}{(b-c)(a-b)}+\frac{c^2}{(a-c)(b-c)}$$

$$=\frac{a^2(b-c)-b^2(a-c)+c^2(a-b)}{(a-b)(a-c)(b-c)}$$

$$=\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{(a-b)(a-c)(b-c)}=\frac{(a-b)(a-c)(b-c)}{(a-b)(a-c)(b-c)}=1.$$

[Art. 129]

Example 8. Prove that $\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2$

$$=4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right), \text{ if } xyz=1.$$

$$\begin{aligned}
\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2&=\left(x^2+2+\frac{1}{x^2}\right)+\left(y^2+2+\frac{1}{y^2}\right)\\
&=4+\left(x^2+y^2\right)+\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\\
&=4+xy\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{xy}\left(\frac{y}{x}+\frac{x}{y}\right)\\
&=4+\left(\frac{x}{y}+\frac{y}{x}\right)\left(xy+\frac{1}{xy}\right)\\
&=4+\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{1}{z}+z\right). \quad [\because xyz=1]
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ the given exp. }&=4+\left(\frac{x}{y}+\frac{y}{x}\right)\left(z+\frac{1}{z}\right)+\left(z+\frac{1}{z}\right)^2\\
&=4+\left(z+\frac{1}{z}\right)\left(\frac{x}{y}+\frac{y}{x}+z+\frac{1}{z}\right)\\
&=4+\left(z+\frac{1}{z}\right)\left\{\left(\frac{x}{y}+\frac{1}{z}\right)+\left(\frac{y}{x}+z\right)\right\}\\
&=4+\left(z+\frac{1}{z}\right)\left\{\left(\frac{x}{y}+xy\right)+\left(\frac{y}{x}+\frac{1}{xy}\right)\right\} \quad [\because xyz=1]\\
&=4+\left(z+\frac{1}{z}\right)\left\{x\left(\frac{1}{y}+y\right)+\frac{1}{x}\left(y+\frac{1}{y}\right)\right\}\\
&=4+\left(z+\frac{1}{z}\right)\left(y+\frac{1}{y}\right)\left(x+\frac{1}{x}\right).
\end{aligned}$$

Example 9. If $xy+yz+zx=1$, show that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

Since, $xy+yz+zx=1$, we have

$$\left.\begin{aligned}
xy+yz&=1-zx, &\text{or, } y(x+z)&=1-zx \cdots \text{(i)}\\
yz+zx&=1-xy, &\text{or, } z(y+x)&=1-xy \cdots \text{(ii)}\\
zx+xy&=1-yz, &\text{or, } x(z+y)&=1-yz \cdots \text{(iii)}
\end{aligned}\right\}$$

Now, the given expression

$$\frac{x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)}{(1-x^2)(1-y^2)(1-z^2)}$$

of which the numerator

$$=x\{1-(y^2+z^2)+y^2z^2\}+y\{1-(z^2+x^2)+z^2x^2\}+z\{1-(x^2+y^2)+x^2y^2\}$$

$$=(x+y+z)-y^2(z+x)-z^2(x+y)-x^2(y+z)+xyz(yz+zx+xy)$$

$$=(x+y+z)-y\{y(z+x)\}-z\{z(x+y)\}-x\{x(y+z)\}+xyz.1$$

$$=(x+y+z)-y(1-x)-z(1-xy)-x(1-yz)+xyz$$

[by (i), (ii) and (iii)]

$$=(x+y+z)-(y+z+x)+3xyz+xyz=4xyz.$$

Hence, the given expression $=\dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$.

Example 10. If $x-a$ be the H. C. F. of $a_1x^2+b_1x+c_1$ and $a_2x^2+b_2x+c_2$, prove that

(i) $a=\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$;

and (ii) $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^2$.

Since, $x-a$ must be a factor of each of the expressions $a_1x^2+b_1x+c_1$ and $a_2x^2+b_2x+c_2$, we have by the factor theorem (Art. 155),

$$a_1a^2+b_1a+c_1=0,$$

and $$a_2a^2+b_2a+c_2=0.$$

Hence, by cross multiplication,

$$\frac{a^2}{b_1c_2-b_2c_1}=\frac{a}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1};$$

$$\therefore\quad a=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Also, $\dfrac{a^2}{b_1c_2-b_2c_1}\times\dfrac{1}{a_1b_2-a_2b_1}=\left(\dfrac{a}{c_1a_2-c_2a_1}\right)^2$,

whence, $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^2$.

Example 11. By performing the operation for extracting the square root, find the value of x, which will make $x^4+6x^3+11x^2+3x+31$ a perfect square.

$$\begin{array}{r|l|l}
 & x^4+6x^3+11x^2+3x+31 & x^2+3x+1 \\
 & x^4 & \\
\hline
2x^2+3x & 6x^3+11x^2 & \\
 & 6x^3+\ \ 9x^2 & \\
\hline
2x^2+6x+1 & 2x^2+3x+31 & \\
 & 2x^2+6x+\ \ 1 & \\
\hline
 & -3x+30 &
\end{array}$$

Now, in order that the given expression may be a perfect square the remainder $(-3x+30)$ must be $=0$, and, therefore, $3x=30$, or, $x=10$

Hence, when $x=10$, the given expression is a perfect square.

Example 12. If $x(b-c)+y(c-a)+z(a-b)=0$,

then will $\dfrac{bz-cy}{b-c}=\dfrac{cx-az}{c-a}=\dfrac{ay-bx}{a-b}$.

$$\left.\begin{array}{ll}\text{We have} & x(b-c)+y(c-a)+z(a-b)=0 \\ \text{and identically also,} & a(b-c)+b(c-a)+c(a-b)=0\end{array}\right\}$$

Hence, by cross multiplication,

$$\frac{b-c}{cy-bz}=\frac{c-a}{az-cx}=\frac{a-b}{bx-ay}, \quad \text{whence,} \quad \frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b},$$

Example 13. Solve

$$\left.\begin{array}{lr} x+y+z=a+b+c & \cdots (1) \\ \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=3 \quad \cdots & \cdots (2) \\ \dfrac{x}{a^2}+\dfrac{y}{b^2}+\dfrac{z}{c^2}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}. & \cdots (3)\end{array}\right\}$$

From (1), $(x-a)+(y-b)+(z-c)=0$.

From (2), $\dfrac{1}{a}(x-a)+\dfrac{1}{b}(y-b)+\dfrac{1}{c}(z-c)=0$.

Hence, by cross multiplication,

$$\frac{x-a}{\frac{1}{c}-\frac{1}{b}}=\frac{y-b}{\frac{1}{a}-\frac{1}{c}}=\frac{z-c}{\frac{1}{b}-\frac{1}{a}};$$

and supposing each of these fractions $=k$, we have

$$x-a=k\cdot\frac{b-c}{bc}; \quad y-b=k\cdot\frac{c-a}{ca}; \quad z-c=k\cdot\frac{a-b}{ab}. \quad \cdots (a)$$

Now, from (3), $\dfrac{1}{a^2}(x-a)+\dfrac{1}{b^2}(y-b)+\dfrac{1}{c^2}(z-c)=0$.

Substituting in this equation the values of $x-a$, $y-b$, $z-c$, found above, we have

$$k\left\{\frac{b-c}{bc}\cdot\frac{1}{a^2}+\frac{c-a}{ca}\cdot\frac{1}{b^2}+\frac{a-b}{ab}\cdot\frac{1}{c^2}\right\}=0,$$

or, $k\cdot\dfrac{bc(b-c)+ca(c-a)+ab(a-b)}{a^2b^2c^2}=0$,

or, $k\cdot\dfrac{(b-c)(a-b)(a-c)}{a^2b^2c^2}=0$; [Art. 129]

$$\therefore \quad k=0,$$

since, a, b, c being impliedly unequal, none of the factors $b-c$, $a-b$, $a-c$ is zero.

Hence, from (a),

$$\left.\begin{array}{lll} x-a=0, & \text{or,} & x=a \\ y-b=0, & \text{or,} & y=b \\ z-c=0, & \text{or,} & z=c \end{array}\right\}$$

Example 14. If $x=cy+bz$, $y=az+cx$ and $z=bx+ay$ show that $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

From the given relations, we have

$$\left.\begin{array}{ll} x-cy-bz=0 & \cdots\quad\cdots\quad\cdots(1) \\ cx-y+az=0 & \cdots\quad\cdots\quad\cdots(2) \\ bx+ay-z=0 & \cdots\quad\cdots\quad\cdots(3) \end{array}\right\}$$

From (1) and (2), by cross multiplication,

$$\frac{x}{-ac-b}=\frac{y}{-bc-a}=\frac{z}{-1+c^2}.$$

or, $$\frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}. \quad \cdots \quad \cdots(4)$$

Similarly, from (2) and (3), $\dfrac{x}{1-a^2}=\dfrac{y}{ab+c}=\dfrac{z}{ac+b}$; $\quad\cdots(5)$

and from (1) and (3), $\dfrac{x}{ab+c}=\dfrac{y}{1-b^2}=\dfrac{z}{bc+a}$. $\quad\cdots(6)$

Now, from (4) and (5),

$$\left.\begin{array}{r} \dfrac{x}{ac+b}=\dfrac{z}{1-c^2}, \\ \text{and}\quad \dfrac{x}{1-a^2}=\dfrac{z}{ac+b}, \end{array}\right\} \quad \text{whence,}\quad \frac{x^2}{1-a^2}=\frac{z^2}{1-c^2}.$$

Again, from (5) and (6),

$$\left.\begin{array}{r} \dfrac{x}{1-a^2}=\dfrac{y}{ab+c}, \\ \text{and}\quad \dfrac{x}{ab+c}=\dfrac{y}{1-b^2}, \end{array}\right\} \quad \text{whence,}\quad \frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}.$$

Hence, $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

Example 15. Show that if $ax+by+cz=0$, and

$\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=0$, then will

$$ax^2+by^2+cz^2+(a+b+c)(xy+yz+zx)=0.$$

From the given relations, we have

$$\left.\begin{aligned} ax+by+cz&=0 \\ \text{and}\qquad ayz+bzx+cxy&=0 \end{aligned}\right\}$$

Hence, by cross multiplication,

$$\frac{a}{x(y^2-z^2)}=\frac{b}{y(z^2-x^2)}=\frac{c}{z(x^2-y^2)},$$

and $\therefore$ each of these ratios

$$=\frac{ax^2+by^2+cz^2}{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)},$$

$$\text{and also,}=\frac{a+b+c}{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}. \qquad \text{[Art. 225]}$$

Thus, we have
$$\frac{ax^2+by^2+cz^2}{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}$$
$$=\frac{a+b+c}{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}.$$

Hence,
$$\frac{ax^2+by^2+cz^2}{a+b+c}=\frac{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}{x^2(z-y)+y^2(x-z)+z^2(y-x)}$$
$$=\frac{(y-z)(x-z)(x-y)(xy+yz+zx)}{-(y-z)(x-z)(x-y)}$$

[See Arts. 140 and 129

$$=-(xy+yz+zx);$$

whence, $ax^2+by^2+cz^2+(a+b+c)(xy+yz+zx)=0.$

Example 16. If $\dfrac{x}{a}=\dfrac{y}{b}$, show that

$$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}=\frac{(x+y)^3+(a+b)^3}{(x+y)^2+(a+b)^2}.$$

Let each of the given ratios $=k$. Then, we have $x=ak$ and $y=bk$.

Hence,
$$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}=\frac{a^3(k^3+1)}{a^2(k^2+1)}+\frac{b^3(k^3+1)}{b^2(k^2+1)}$$
$$=\frac{a(k^3+1)}{k^2+1}+\frac{b(k^3+1)}{k^2+1}=\frac{(k^3+1)(a+b)}{k^2+1}$$
$$=\frac{(k^3+1)(a+b)^3}{(k^2+1)(a+b)^2}=\frac{k^3(a+b)^3+(a+b)^3}{k^2(a+b)^2+(a+b)^2}$$
$$=\frac{(ka+kb)^3+(a+b)^3}{(ka+kb)^2+(a+b)^2}=\frac{(x+y)^3+(a+b)^3}{(x+y)^2+(a+b)^2}.$$

Example 17. Show that $(bcd+cda+dab+abc)^2$
$$-abcd(a+b+c+d)^2=(bc-ad)(ca-bd)(ab-cd).$$

We have
$$\begin{aligned}(bcd+cda+dab+abc)^2&=\{cd(a+b)+ab(c+d)\}^2\\&=c^2d^2(a+b)^2+2abcd(a+b)(c+d)+a^2b^2(c+d)^2;\end{aligned}$$
and
$$(a+b+c+d)^2=(a+b)^2+2(a+b)(c+d)+(c+d)^2.$$

Hence, the given expression
$$\begin{aligned}&=c^2d^2(a+b)^2+a^2b^2(c+d)^2-abcd(a+b)^2-abcd(c+d)^2\\&=ab(c+d)^2(ab-cd)-cd(a+b)^2(ab-cd)\\&=(ab-cd)\{ab(c+d)^2-cd(a+b)^2\}\\&=(ab-cd)\{ab(c^2+d^2)-cd(a^2+b^2)\}\\&=(ab-cd)\{ac(bc-ad)-bd(bc-ad)\}\\&=(ab-cd)(bc-ad)(ac-bd).\end{aligned}$$

Example 18. Show that the following expression is an exact square: $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)$.

Putting a for x^2-yz, b for y^2-zx and c for z^2-xy, we have the given expression
$$\begin{aligned}&=a^3+b^3+c^3-3abc\\&=(a+b+c)(a^2+b^2+c^2-bc-ca-ab) &&[\text{ Art. 128 }]\\&=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}. &&\cdots\ (1)\end{aligned}$$

Now,
$$\begin{aligned}a-b&=(x^2-yz)-(y^2-zx)\\&=(x^2-y^2)+z(x-y)=(x-y)(x+y+z).\end{aligned}$$

Similarly, $b-c=(y-z)(x+y+z)$,

and $c-a=(z-x)(x+y+z)$;

whence,
$$\begin{aligned}&(a-b)^2+(b-c)^2+(c-a)^2\\&=(x+y+z)^2\{(x-y)^2+(y-z)^2+(z-x)^2\}\\&=2(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy). &&\cdots\ (2)\end{aligned}$$

Also,
$$a+b+c=x^2+y^2+z^2-yz-zx-xy. \qquad\cdots\ (3)$$

Therefore, from (1), (2) and (3), the given expression
$$\begin{aligned}&=\tfrac{1}{2}(x^2+y^2+z^2-yz-zx-xy)\\&\qquad\times\{2(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy)\}\\&=\{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)\}^2\\&=(x^3+y^3+z^3-3xyz)^2.\end{aligned}$$

Example 19. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, show that
$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}},$$
where n is any positive integer.

From the given relation, we have

$$\frac{bc+a(b+c)}{abc}-\frac{1}{a+b+c}=0\,;$$

$$\therefore\quad \{a(b+c)+bc\}\{a+(b+c)\}-abc=0.$$

Now, the left-hand expression

$$=a^2(b+c)+a(b+c)^2+bc(b+c)$$

$$=(b+c)\{a^2+a(b+c)+bc\}=(b+c)(a+b)(a+c)\,;$$

$$\therefore\quad (b+c)(a+b)(a+c)=0.$$

Hence, either, $b+c=0$, or, $a+b=0$, or, $a+c=0$.

Taking $b+c=0$, we have $c=-b$.

$$\text{Hence, } \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1}=\left(\frac{1}{a}\right)^{2n+1} \qquad \left[\because\ \frac{1}{b}+\frac{1}{c}=0.\right]$$

$$=\frac{1}{a^{2n+1}}=\frac{1}{a^{2n+1}+b^{2n+1}-b^{2n+1}}$$

$$=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$$

[$\because\ c^{2n+1}=(-b)^{2n+1}=-b^{2n+1}$; see foot-note, page 125]

Example 20. Having given $x=by+cz+du$, $y=ax+cz+du$, $z=ax+by+du$ and $u=ax+by+cz$, show that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1$$

Putting P for $ax+by+cz+du$, we have

$$x+ax=(by+cz+du)+ax$$

$$=P\text{, or, } x(1+a)=P\,; \qquad \therefore\ \frac{1}{1+a}=\frac{x}{P}\,; \quad \cdots \ (1)$$

$$y+by=(ax+cz+du)+by$$

$$=P\text{, or, } y(1+b)=P\,; \qquad \therefore\ \frac{1}{1+b}=\frac{y}{P}\,; \quad \cdots \ (2)$$

$$z+cz=(ax+by+du)+cz$$

$$=P\text{, or, } z(1+c)=P\,; \qquad \therefore\ \frac{1}{1+c}=\frac{z}{P}\,; \quad \cdots \ (3)$$

$$u+du=(ax+by+cz)+du$$

$$=P\text{, or, } u(1+d)=P\,; \qquad \therefore\ \frac{1}{1+d}=\frac{u}{P}\,; \quad \cdots \ (4)$$

Hence, from (1), (2), (3) and (4), we have

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=\frac{ax}{P}+\frac{by}{P}+\frac{cz}{P}+\frac{du}{P}=\frac{ax+by+cz+du}{P}=1.$$

MISCELLANEOUS EXERCISES VI

I

1. Find the value of $\sqrt{(x^2+y^3+z)(x-y-3z)}+\sqrt[3]{xy^3z^2}$, when $x=-1$, $y=-3$, $z=1$.

2. Simplify $3a-2(b-c)-\{2(a-b)-3(c+a)\}-\{9c-4(c-a)\}$.

3. Resolve into factors $3(a+b)^2-2(a^2-b^2)-a(a+b)$.

4. Divide $2x^4-10x^3y+25x^2y^2-31xy^3+20y^4$ by $x^2-3xy+4y^2$.

5. Simplify $\dfrac{b}{a+b}-\dfrac{ab}{(a+b)^2}-\dfrac{ab^2}{(a+b)^3}$.

6. Solve the equation $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}$.

7. If $\left(x+\dfrac{1}{x}\right)^2=3$, prove that $x^3+\dfrac{1}{x^3}=0$.

8. Simplify $\dfrac{3\sqrt{3}-2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

II

1. Find the value of $(2a+b)(a-b)+(2b+c)(b-c)+(2c+a)(c-a)$, when $a=1$, $b=2$, $c=-3$.

2. Divide $1+3x-24x^2+8x^4$ by $2x^2+3x-1$.

3. If x^2+7x+c is exactly divisible by $x+4$, what is the value of c?

4. Simplify $\dfrac{1}{2\sqrt{7}-3\sqrt{2}}-\dfrac{1}{2\sqrt{7}+3\sqrt{2}}$.

5. Find the H.C.F. of
$$x^4-3x^3-2x^2+12x-8 \text{ and } x^3-7x+6.$$

6. Simplify $\left(1+\dfrac{35}{x-7}-\dfrac{15}{x-3}\right)\left(\dfrac{1}{5}-\dfrac{7}{x+7}+\dfrac{3}{x+3}\right)$.

7. Solve the equation $\dfrac{x+1}{6}+\dfrac{3x-1}{8}-\dfrac{5x-7}{12}+1=\dfrac{7x-5}{24}$.

8. If $x-\dfrac{1}{x}=1$, prove that $x^3-\dfrac{1}{x^3}=4$.

III

1. Find the value of $\{a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)\}$
$\div(bc+ca+ab)$, when $a=3$, $b=-2$, $c=4$.

2. Simplify $\dfrac{1+x}{1-x}+\dfrac{1-x}{1+x}-\dfrac{1+x^2}{1-x^2}-\dfrac{1-x^2}{1+x^2}$.

3. Resolve into factors $a^2-b^2+6bc-9c^2$.

4. Find the H.C.F. of
$x^3+5ax^2-5a^2x-a^3$ and $5x^3-3ax^2-5a^2x+3a^3$.

5. Find the L.C.M. of x^2-5x+6, x^2-4x+3 and x^2-3x+2.

6. Reduce to its lowest terms $\dfrac{x^5+5x^4+8x^3+4x^2}{x^5+x^4+8x^2+8x}$.

7. Solve $\dfrac{1}{x+3}+\dfrac{1}{x-2}=\dfrac{2}{x-7}$.

8. If $a : b :: x : y$, show that $ab : xy :: a^2+b^2 : x^2+y^2$.

IV

1. Simplify $\dfrac{\sqrt{x^2-y^2}+x}{\sqrt{x^2+y^2}+y}+\dfrac{\sqrt{x^2+y^2}-y}{x-\sqrt{x^2-y^2}}$.

2. If the product of two expressions be $x^8+x^4y^4+y^8$ and one of them be x^2-xy+y^2, find the other.

3. Resolve into factors :
(i) x^3+x^2-x-1 ; (ii) $a^2b^2-a^2-b^2+1$.

4. Show that $(ax+by)^2+(bx-ay)^2=(a^2+b^2)(x^2+y^2)$.

5. Find the L.C.M. of $8x^3+27$, $16x^4+36x^2+81$ and $6x^2-5x-6$.

6. Solve $\dfrac{3x-4}{x}+\dfrac{2}{4x+3}=3$.

7. Find x and y, if $\dfrac{bx+ay}{2ab}=\dfrac{by-ax}{b^2-a^2}=ab$.

8. If $\dfrac{x}{a+b-c}=\dfrac{y}{a-b+c}=\dfrac{z}{b+c-a}$, show that each of these fractions $=\dfrac{x+y+z}{a+b+c}$

V

1. Simplify $\dfrac{x^2-25y^2}{x^2+3xy-10y^2}\times\dfrac{x^2-4y^2}{x^2-3xy-10y^2}$.

2. Divide $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a+b+c$, and find the factors of the quotient.

3. Find the value of $\dfrac{x^3-y^3}{x^3+y^3}$, when $x=a+3$, $y=a-3$.

4. Find the square root of $24+\dfrac{x^2}{y}+8\left(\dfrac{2y}{x^2}-xy^{-\frac{1}{2}}\right)-\dfrac{32\sqrt{y}}{x}$.

5. Show that $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$
$$=(ay-bx)^2+(bz-cy)^2+(cx-az)^2.$$

6. Subtract $\dfrac{7-2\sqrt{5}}{4-\sqrt{5}}$ from $\dfrac{15+6\sqrt{5}}{2+\sqrt{5}}$.

7. Solve $\left.\begin{aligned}2^x\times4^y&=32\\3^x+9^y&=3\end{aligned}\right\}$.

8. If $a:b::c:d$, show that $(a^2+c^2)(b^2+d^2)=(ab+cd)^2$.

VI

1. Reduce to its simplest form the expression
$$\frac{2a(1-x^2)^2}{yz}+\frac{(1+x)^2(1-x)}{y^2}+\frac{2ay^2(1-x)}{z}.$$

2. Multiply $a+b+\dfrac{b^2}{a}+\dfrac{a^2}{b}$ by $a-b+\dfrac{b^2}{a}-\dfrac{a^2}{b}$.

3. Divide $x^4-2bx^3-(a^2-b^2)x^2+2a^2bx-a^2b^2$ by $x^2-(a+b)x+ab$.

4. If $a=y+z$, $b=z+x$, $c=x+y$, then
$$a^2+b^2+c^2-bc-ca-ab=x^2+y^2+z^2-yz-zx-xy.$$

5. Reduce $\dfrac{5x^3-14x^2+16}{3x^3-2x^2+16x-48}$ to its lowest terms.

6. Solve $\dfrac{2}{x}+\dfrac{7}{y}=29$, $\dfrac{5}{x}-\dfrac{6}{y}=2$.

7. Solve $2x+3y-8z+35=0$, $7x-4y+z-8=0$, $12x-5y-3z+10=0$.

8. If $a:b=c:d=e:f$, prove that
$$a:b::\sqrt{m^2a^2+n^2c^2-p^2e^2}:\sqrt{m^2b^2+n^2d^2-p^2f^2}.$$

VII

1. Divide $-2x^5y^{-8}+17x^6y^{-4}-5x^7-24x^8y^4$
by $-x^2y^{-5}+7x^3y^{-1}+8x^4y^3$.

2. Find the H.C.F. of
$$e^{2x}a^3+e^{2x}-a^3-1 \text{ and } e^{2x}a^2+2e^xa^2-e^{2x}-2e^x+a^2-1.$$

3. Show that $1-\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}=\dfrac{(a+c+d-b)(b+c+d-a)}{2(ab+cd)}$.

4. Simplify $\dfrac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$.

5. Solve $\dfrac{x-4}{x-5}-\dfrac{x-5}{x-6}=\dfrac{x-7}{x-8}-\dfrac{x-8}{x-9}$.

6. Show that if each of the expressions x^2+px+q and $x^2+p'x+q'$ be divisible by $x+m$, then $m=\dfrac{q-q'}{p-p'}$.

7. A bill of £100 was paid with guineas and half-crowns, and 18 more half-crowns than guineas were used; find how many of each were paid.

8. If $a : b :: c : d$, prove that $4a^6+5b^6 : 4c^6+5d^6 :: a^3b^3 : c^3d^3$.

VIII

1. Show that $(ax+by+cz)^3+(cx-by+az)^3$ is divisible by $(a+c)(x+z)$.

2. Resolve into factors:

(i) $(b+c)^2-6a(b+c)+5a^2$; (ii) $x^2+2xy-a^2-2ay$.

3. Simplify $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$.

4. If $a+b+c=0$, show that $a^2-bc=b^2-ca=c^2-ab$.

5. Solve $3(x+3)^2+5(x+5)^2=8(x+8)^2$.

6. Extract the square root of $25x^{-2}-12x+16x^{-8}+4x^4-24x^{-5}$.

7. Find the value of x, y, z, if $yz=4$, $zx=9$, $xy=25$.

8. If $a : b :: c : d$, show that $a(a+b+c+d)=(a+b)(a+c)$.

IX.

1. Find the value of $\{a^2-(b-c)^2\}-\{b^2-(c-a)^2\}-\{c^2-(a-b)^2\}$, when $a=1$, $b=2$ and $c=-3$.

2. Simplify $\dfrac{x}{(x-1)^2}-\dfrac{1}{(x+1)^2}-\dfrac{x(x^2+3)}{(x^2-1)^2}$.

3. Resolve into factors $a^3-b^3+3ab+1$.

4. Solve $\dfrac{4x+3}{9}+\dfrac{7x-29}{5x-12}=\dfrac{8x+19}{18}$.

5. Show that $\dfrac{1}{(y-z)^2}+\dfrac{1}{(z-x)^2}+\dfrac{1}{(x-y)^2}=\left(\dfrac{1}{y-z}+\dfrac{1}{z-x}+\dfrac{1}{x-y}\right)^2$.

6. Solve $x+y : x-y=5 : 3$, $x+5y=36$.

7. Find the time between 8 and 9 o'clock, when the hands of a clock are at right angles to each other.

8. If $a : b :: b : c$, show that $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

X

1. Divide $27a^3-8b^3-27c^3-54abc$ by $3a-2b-3c$.

2. Find the H.C.F. of x^5+11x^3-54 and $x^5+11x+12$.

3. Resolve into factors $(a^2-b^2)(x^2+y^2)+2(a^2+b^2)xy$.

4. Simplify $\dfrac{\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}-2}{\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}+2}+\dfrac{\dfrac{a}{b}\left(1-\dfrac{b^2}{a^2}\right)}{\dfrac{(a+b)^2}{ab}-2}$.

5. Show that $a^3(b+c)+b^3(c+a)+c^3(a+b)+abc(a+b+c)$
$$=(a^2+b^2+c^2)(bc+ca+ab).$$

6 Solve $\sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}$.

7. One man and two boys can do in 12 days a piece of work which would be done in 6 days by 3 men and 1 boy. How long would it take one man to do it ?

8. If $a:b::b:c$, prove that
$$a^4+a^2c^2+c^4=b^2\left(\frac{b^2}{c^2}-1+\frac{b^2}{a^2}\right)(a^2+b^2+c^2).$$

XI

1. Show that $(x^2+xy+y^2)^2-4xy(x^2+y^2)=(x^2-xy+y^2)^2$.

2. Resolve into factors :
   (i) $a^2-b^2-c^2+d^2-2(ad-bc)$;
   (ii) $x^2-y^2-z^2+2yz+x+y-z$.

3. Extract the square root of $\dfrac{9x^2}{a^2}+\dfrac{a^2}{9x^2}-\dfrac{6x}{a}-\dfrac{2a}{3x}+3$.

4. Solve $x+2y+3z=6,\ 2x+4y+z=7,\ 3x+2y+9z=14$.

5. Find the H.C.F. of $x^4y-x^3y^2-15x^2y^3+38xy^4-14y^5$ and $x^5-7x^4y+21x^3y^2-34x^2y^3+28xy^4$.

6. A man buys 570 oranges, some at 16 for a shilling and the rest at 18 for a shilling ; he sells them all at 15 for a shilling and gains three shillings ; how many of each sort does he buy ?

7. Simplify $\dfrac{1}{\left(1-\dfrac{c}{a}\right)\left(1-\dfrac{b}{a}\right)}+\dfrac{1}{\left(1-\dfrac{a}{b}\right)\left(1-\dfrac{c}{b}\right)}+\dfrac{1}{\left(1-\dfrac{b}{c}\right)\left(1-\dfrac{a}{c}\right)}$

8. If $a:b=c:d=e:f$, prove that
$$(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2$$

XII

1. If $x=a+d,\ y=b+d,\ z=c+d$, show that
$$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

2. Simplify $\dfrac{y+z}{(y^2-xz)(z^2-xy)}+\dfrac{z+x}{(z^2-xy)(x^2-yz)}+\dfrac{x+y}{(x^2-yz)(y^2-xz)}$

3. Resolve into factors :
   (i) $x^2-2ax-b^2+2ab$. (ii) $x^2+(a+b+c)x+ab+ac$.

4. Find the H.C.F. of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$.

5. Solve $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}-\dfrac{2x}{5}=0$.

6. A and B can together do a work in 12 days; A and C in 15 days; B and C in 20 days; find in how many days they will do the work, all working together.

7. Simplify $4\sqrt{147}-3\sqrt{75}-6\sqrt{\frac{1}{3}}+18\sqrt{\frac{1}{27}}$.

8. Show that, if $x : y :: a : b$, then will

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}=\frac{(x+y)^2+(a+b)^2}{x+y+a+b}.$$

XIII

1. If $2s=a+b+c$, show that

$$a(b-c)(s-a)^2+b(c-a)(s-b)^2+c(a-b)(s-c)^2=0.$$

2. Show that $x^6+x^3a^3+a^6$ is divisible by $x^{\frac{3}{4}}+x^{\frac{3}{8}}a^{\frac{3}{8}}+a^{\frac{3}{4}}$.

3. Simplify $\frac{x^2-yz}{(x+y)(x+z)}+\frac{y^2-zx}{(y+z)(y+x)}+\frac{z^2-xy}{(z+x)(z+y)}$.

4. Solve $\frac{x^2-a^2}{x-a}+\frac{x^2-b^2}{x-b}+\frac{x^2-c^2}{x-c}=a+b+c-3x$.

5. Find how many gallons of water must be mixed with 80 gallons of spirit which cost 15 shillings a gallon, so that by selling the mixture at 12 shillings a gallon there may be a gain of 10 per cent. on the outlay.

6. Simplify $\frac{(x^{a+b})^2.(x^{b+c})^2.(x^{c+a})^2}{(x^a.x^b.x^c)^4}$.

7. Simplify $3\sqrt[3]{128}-4\sqrt[3]{-686}+2\sqrt[3]{54}$.

8. If $a : b :: b : c$, prove that $a^2+ab+b^2 : b^2+bc+c^2 :: a : c$.

XIV

1. If $a+b+c=2s$, and $a^2+b^2+ab+s^2=2s(a+b)$, show that $(a-s)^2+(b-s)^2+(c-s)^2=s^2$.

2. If $x+a$ be a common factor of x^2+px+q and x^2+lx+m, show that $a=\frac{m-q}{l-p}$.

3. Simplify $\frac{7+3\sqrt{5}}{7-3\sqrt{5}}+\frac{7-3\sqrt{5}}{7+3\sqrt{5}}$.

4. Solve $\frac{a-b}{x-a}+\frac{a-b}{x-b}=\frac{a}{x-a}-\frac{b}{x-b}$.

5. Solve $\frac{y+z-x}{b+c}=\frac{z+x-y}{c+a}=\frac{x+y-z}{a+b}=1$.

6. A can do a piece of work in 20 days, which B can do in 12 days. A begins the work, but after a time B takes his place, and the whole work is finished in 14 days from the beginning. How long did A work?

7. Express $(x+a)(x+2a)(x+3a)(x+4a)$ as the difference of two squares.

8. Show that if $a(y+z)=b(z+x)=c(x+y)$, then

$$\frac{-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

XV

1. For what value of b will $x^4+2ax^3+(a^2+8)x^2+(4a+ab)x+4b$ be a perfect square?

2. Prove that $(b-c)(1+ab)(1+ac)+(c-a)(1+bc)(1+ba)$
$$+(a-b)(1+ca)(1+cb)=(b-c)(c-a)(a-b).$$

3. Simplify $\dfrac{2x^2+2}{x^4+x^2+1}+\dfrac{1}{x+\sqrt{x}+1}+\dfrac{1}{x-\sqrt{x}+1}-\dfrac{1}{x^2-x+1}$.

4. Find the H.C.F. of
$$2x^3+(2a-3b)x^2-(2b+3ab)x+3b^2 \text{ and } 2x^2-(3b-2c)x-3bc.$$

5. Find the value of $\dfrac{a^n}{2na^n-2nx}+\dfrac{b^n}{2nb^n-2nx}$, when $x=\dfrac{a^n+b^n}{2}$.

6. Solve $\dfrac{20x+36}{25}+\dfrac{5x+20}{9x-16}=\dfrac{4x}{5}+\dfrac{86}{25}$.

7. A vessel is filled with a mixture of spirit and water, 70 per cent. of which is spirit. After 9 gallons are taken out and the vessel is filled up with water, there remains 58⅓ per cent. of spirit; find the contents of the vessel.

8. If $x-z:y-z::x^2:y^2$, show that
$$x+z:y+z::\frac{x}{y}+2:\frac{y}{x}+2.$$

XVI

1. Find the H.C.F. of
$$x^5+2x^4-5x^2-7x+3 \text{ and } 3x^6-3x^4-18x^3+x^2+2x+3.$$

2. Solve $\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}$.

3. If $(a+b+c)x=(-a+b+c)y=(a-b+c)z=(a+b-c)w$, show that
$$\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{1}{x}.$$

4. Solve $\left.\begin{aligned} \frac{5\sqrt{x+y}}{x}+\frac{5\sqrt{x+y}}{y}&=10\tfrac{3}{8} \\ \frac{3\sqrt{x-y}}{y}-\frac{3\sqrt{x-y}}{x}&=\frac{4}{5} \end{aligned}\right\}$

5. Resolve into factors $ax(y^3+b^3)+by(bx^3+a^3y)$.

6. Find the continued product of $\sqrt{a}+\sqrt{b}+\sqrt{c}$, $\sqrt{a}+\sqrt{b}-\sqrt{a}$, $\sqrt{a}-\sqrt{b}+\sqrt{c}$, $\sqrt{b}+\sqrt{c}-\sqrt{a}$.

7. If $\frac{a+b}{a-b}=\frac{c}{d}$, show that $\frac{a^2+ab}{ab-b^2}=\frac{c^2+cd}{cd-d^2}$.

8. Each of two vessels contains a mixture of wine and water; a mixture consisting of equal measures from the two vessels contains as much wine as water, and another mixture consisting of four measures from the first vessel and one from the second is composed of wine and water in the ratio of 2 : 3. Find the proportion of wine and water in each of the vessels.

XVII

1. Find the H.C.F. of x^5+x^2+2x+2 and x^4+x^2+1.

2. Solve $\left.\begin{aligned} \sqrt{y}-\sqrt{y-x}&=\sqrt{20-x} \\ \sqrt{y-x}:\sqrt{20-x}&::3:2 \end{aligned}\right\}$.

3. Find the value of $\left(\frac{x}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2$, when $x=\sqrt{\frac{n-1}{n+1}}$.

4. Show that $\frac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}=ab+bc+ca$.

5. If $a+b+c=0$, show that $4(b^2c^2+c^2a^2+a^2b^2)=(a^2+b^2+c^2)^2$.
Hence, prove that $(y-z)^2(z-x)^2+(z-x)^2(x-y)^2+(x-y)^2(y-z)^2$
$=(x^2+y^2+z^2-yz-zx-xy)^2$.

6. One of the digits of a number is greater by 5 than the other. When the digits are inverted the number becomes $\frac{3}{8}$ of the original number. Find the number.

7. Simplify $\frac{3x^3+x^2-5x+21}{6x^3+29x^2+26x-21}$.

8. If $3(a^2+b^2+c^2)=(a+b+c)^2$, show that $a=b=c$.

XVIII

1. Show that $\{(x-y)^2+(y-z)^2+(z-x)^2\}^2$
$=2\{(x-y)^4+(y-z)^4+(z-x)^4\}$.

2. Solve $\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=4\sqrt{2}\left\{\frac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}\right\}^{\frac{1}{2}}$.

3. Resolve into factors :

(i) $14x^2-37x+5$. (ii) $(1+a)^2(1+c^2)-(1+c)^2(1+a^2)$.

(iii) $m^4-n^4+2n(m^3+n^3)-(m+n)^2(m-n)^2$.

4. A baker charges $9\frac{1}{2}d.$ for a loaf which he represents as weighing 4 lbs., but which really weighs 3 lbs. 12 oz. After he has sold a certain number of loaves, he is detected and fined £5, and thus loses 5 shillings more than he has cleared by selling short weight. How many loaves did he sell ?

5. Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$

6. If $\dfrac{a-b}{ay+bx}=\dfrac{b-c}{bz+cy}=\dfrac{c-a}{cx+az}=\dfrac{a+b+c}{ax+by+cz}$, then each of these atios $=\dfrac{1}{x+y+z}$, supposing $a+b+c$ not to be zero.

7. Solve $x(x+y+z)=24$, $y(x+y+z)=48$, $z(x+y+z)=72$.

8. Eliminate x from the equations $a+c=\dfrac{b}{x}-dx$, $a-c=\dfrac{d}{x}-bx$.

XIX

1. Solve $(x^2-2ax+3a^2)^{\frac{1}{2}}+(x^2-4ax+5a^2)^{\frac{1}{2}}$

$$=(x^2-5ax+7a^2)^{\frac{1}{2}}+(x^2-7ax+9a^2)^{\frac{1}{2}}.$$

2. Show that $\dfrac{a(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{b(b+a)(b+c)}{(b-a)(b-c)}+\dfrac{c(c+a)(c+b)}{(c-a)(c-b)}=a+b+c.$

3. Simplify $\dfrac{(b-c)a^3+(c-a)b^3+(a-b)c^3}{c^2-bc-ca+ab}$.

4. If m gold coins are equal in weight to n silver coins and p of e former equal in value to q of the latter, compare the values of equal eights of gold and silver.

5. If $x=b+c$, $y=c+a$, $z=a+b$, show that

$$x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc).$$

6. If $\dfrac{1}{b^3(a-c)}+\dfrac{1}{a^3(b-c)}=\dfrac{1}{ab(a-c)(b-c)}$, prove that

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}, \text{ or, } a^2+b^2=ab.$$

7. Simplify $\sqrt{\dfrac{(\sqrt{12}-\sqrt{8})(\sqrt{3}+\sqrt{2})}{5+\sqrt{24}}}$.

8. Eliminate x and y from the equations

$$(b+c)x+(c+a)y+(a+b)=0,\ (c+a)x+(a+b)y+(b+c)=0,$$
$$(a+b)x+(b+c)y+(c+a)=0.$$

XX

1. Show that $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$
$$=(b+c)(c+a)(a+b).$$

2. If $x+a$ be a factor of $a^2x^3-b^3x^2+ac^3x+3a^3bc$, and if a is not equal to zero, show that $a^3+b^3+c^3=3abc$.

3. Simplify $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+\dfrac{ca}{b(b^2-a^2)(b^2-c^2)}+\dfrac{ab}{c(c^2-b^2)(c^2-a^2)}$

4. Divide $a^4(b-c)+b^4(c-a)+c^4(a-b)$ by $(a-b)(b-c)(c-a)$.

5. If $a+b+c=0$, show that $a^5+b^5+c^5=5abc(c^2-ab)$.

6. Solve $\dfrac{a}{x+a-c}+\dfrac{b}{x+b-c}=2$.

7. Solve $ax+by+cz=a+b+c$, $\dfrac{ax}{b+c}+\dfrac{by}{a+c}=1$, $\dfrac{2x}{b+c}+\dfrac{2y}{a+c}=\dfrac{1}{a}+\dfrac{1}{b}$.

8. A person starts to walk at a uniform speed without stopping from Cuttack to Jobra and back; at the same time another starts to walk at a uniform speed without stopping from Jobra to Cuttack and back. They meet $2\frac{2}{5}$ kilometres from Jobra and again, an hour after, $1\frac{3}{5}$ kilometres from Cuttack. Find their rates of walking, and the distance between Cuttack and Jobra.

XXI

1. Show that
$$\{(b+c)^2+(c+a)^2+(a+b)^2\}\times\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$$
$$=2\{a^4(b-c)+b^4(c-a)+c^4(a-b)\}.$$

2. Show that $\dfrac{(b-c)^2}{(c-a)(a-b)}+\dfrac{(c-a)^2}{(a-b)(b-c)}+\dfrac{(a-b)^2}{(b-c)(c-a)}=3.$

3. If $a+b+c=0$, show that $a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2$.

4. If $s=a+b+c$, prove that
$$(s-3a)^2+(s-3b)^2+(s-3c)^2=3\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

5. Resolve into factors $a^2+2ab-2ac-3b^2+2bc$.

6. Find the H.C.F. of $x^4-2x^3+5x^2-4x+3$ and
$$2x^4-x^3+6x^2+2x+3.$$

7. Find the condition that ax^2+bx+c and $a'x^2+b'x+c'$ may have a common factor of the *form* $x+f$.

8. If $a:b=b:c=c:d$, prove that
$$a:d=\sqrt{a^5+b^2c^2+a^3c^2}:\sqrt{b^4c+d^4+b^2cd^2}.$$

XXII

1. Show that $a(b-c)(1+ab)(1+ac)+b(c-a)(1+bc)(1+ba)$
$+c(a-b)(1+ca)(1+cb)=abc(a-b)(a-c)(b-c)$.

2. If $a+b+c=0$, show that $a^7+b^7+c^7=7abc(c^2-ab)^2$.

3. Show that if ax^2+bx+c and $a'x^2+b'x+c'$ have a common factor of the *form* $x+f$, then will $(ac'-a'c)^2=(bc'-b'c)(ab'-a'b)$.

4. *A* and *B* run a race; *B* has 50 metres start, but *A* runs 20 metres while *B* runs 19. What must be the length of the course that *A* may come in a metre ahead of *B*?

5. Show that $\dfrac{p+q+r}{p-q}-\dfrac{q(4p+3r)-r(p+r)}{p^2-q^2}=\dfrac{(p-q+r)^2}{p^2-q^2}$.

6. Show that $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$.

7. Solve $x+y+z=2a+2b+2c$, $ax+by+cz=2bc+2ca+2ab$,
$(b-c)x+(c-a)y+(a-b)z=0$.

8. Eliminate x, y, z from the equations
$ax+cy+bz=0$, $cx+by+az=0$, $bx+ay+cz=0$.

XXIII

1. Show that $(b-c)(1+a^2b)(1+a^2c)$
$+(c-a)(1+b^2c)(1+b^2a)+(a-b)(1+c^2a)(1+c^2b)$
$=abc(a+b+c)(a-b)(a-c)(b-c)$.

2. Find the L.C.M. of
$21x^2-13x+2$, $28x^2-15x+2$ and $12x^2-7x+1$.

3. Show that $(x+y)^7-x^7-y^7$ is divisible by $(x^2+xy+y^2)^2$.

4. If $2s=a+b+c$ and $2t^2=a^2+b^2+c^2$, show that
$(t^2-a^2)(t^2-b^2)+(t^2-b^2)(t^2-c^2)+(t^2-c^2)(t^2-a^2)$
$=4s(s-a)(s-b)(s-c)$.

5. If $(1+xx'+yy')^2=(1+x^2+y^2)(1+x'^2+y'^2)$, show that
$x=x'$ and $y=y'$.

6. Simplify $\dfrac{ab(a-b)(a^2+b^2)+bc(b-c)(b^2+c^2)+ca(c-a)(c^2+a^2)}{a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)}$.

7. If $a+b+c=0$, prove that $\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^3+b^3+c^3}{3}\cdot\dfrac{a^2+b^2+c^2}{2}$.

8. Eliminate x and y from the equations

$$ax+by=\sqrt{a^2+b^2},\ \frac{x^2}{p^2}+\frac{y^2}{q^2}=\frac{1}{a^2+b^2},\ x^2+y^2=1.$$

XXIV

1. Solve $x+y+z=a+b+c,\ bx+cy+az=cx+ay+bz=ab+bc+ca.$

2. Divide 243 into three parts such that one-half of the first, one-third of the second and one-fourth of the third part shall all be equal to one another.

3. If $4(a^2+b^2+c^2+d^2)=(a+b+c+d)^2$, show that $a=b=c=d$

4. If $2s=a+b+c$, show that

$$a(b-c)(s-a)^2+b(c-a)(s-b)^2+c(a-b)(s-c)^2=0.$$

5. If $bz+cy=a,\ az+cx=b$ and $ay+bx=c$, prove that

$$\frac{a^2}{1-x^2}=\frac{b^2}{1-y^2}=\frac{c^2}{1-z^2}.$$

6. Eliminate x and y from the equations

$$ax+by=x+y+xy=x^2+y^2-1=0.$$

7. If ax^2-bx+c and dx^3-bx+c have a common factor, show that $a^3-abd+cd^2=0.$

8. If $a^3+b^3+c^3=(a+b+c)^3$, then will

$$a^{2n+1}+b^{2n+1}+c^{2n+1}=(a+b+c)^{2n+1},$$

where n is any positive integer.

XXV

1. If $x=a^2-bc,\ y=b^2-ca,\ z=c^2-ab$, prove that

$$\frac{x^2-yz}{a}=\frac{y^2-zx}{b}=\frac{z^2-xy}{c}=(a+b+c)(x+y+z).$$

2. If $2s=a+b+c+d$, show that

$$4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d)$$

3. Prove that $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3$

$$-3(b+c-a)(c+a-b)(a+b-c)=4(a^3+b^3+c^3-3abc).$$

4. Show that, if $a+b+c=0$, then

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9.$$

5. If $x:a=y:b=z:c$, prove that

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}$$

6. Prove that, if $ax+by+cz=0$ and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$,

$$ax^3+by^3+cz^3+(a+b+c)(y+z)(z+x)(x+y)=0.$$

7. Eliminate x, y, z from the equations :

$$\text{(i)}\ \left.\begin{aligned} ax+hy+gz&=0\\ hx+by+fz&=0\\ gx+fy+cz&=0 \end{aligned}\right\} \qquad \text{(ii)}\ \left.\begin{aligned} a(y+z)&=x\\ b(z+x)&=y\\ c(x+y)&=z \end{aligned}\right\}$$

8. Eliminate l, m, n from the equations

$$\left.\begin{aligned} al=bm&=cn\\ l^2+m^2+n^2&=1\\ a^2l^3+b^2m^3+c^2n^3&=a'^2l+b'^2m+c'^2n \end{aligned}\right\}$$

CHAPTER XXXIV

QUADRATIC EQUATIONS AND EXPRESSIONS

We have already explained in Chapter XX what quadratic equations are and how easy types of such equations can be solved. We shall in the present article consider some examples of a harder type.

I. Pure Quadratic Equations

233. Such equations may, after suitable reduction and transformation, be expressed in the standard form

$$ax^2=c.$$

∴ the required solutions are

$$x=\pm\sqrt{\frac{c}{a}}.$$

The following examples will serve as illustrations

Example 1. If $\frac{35-2x}{9}+\frac{5x^2+7}{5x^2-7}=\frac{17-\frac{2}{3}x}{3}$, find x.

By transposition, we have

$$\frac{5x^2+7}{5x^2-7}=\frac{51-2x}{9}-\frac{35-2x}{9}=\frac{16}{9};$$

$$\therefore\quad \frac{5x^2}{7}=\frac{16+9}{16-9}=\frac{25}{7};\qquad \text{[componendo and dividendo]}$$

$$\therefore\quad x^2=5;\qquad \therefore\quad x=\pm\sqrt{5}$$

Example 2. Solve $3\left(\frac{x^2-9}{x^2+3}\right)+4\left(\frac{22\frac{1}{2}+x^2}{x^2+9}\right)=7.$

By transposition, we have

$$4\left(\frac{22\frac{1}{2}+x^2}{x^2+9}\right)-4=3-3\left(\frac{x^2-9}{x^2+3}\right),$$

or, $$4\left\{\frac{22\frac{1}{2}+x^2}{x^2+9}-1\right\}=3\left\{1-\frac{x^2-9}{x^2+3}\right\},$$

or, $$4\times\frac{13\frac{1}{2}}{x^2+9}=3\times\frac{12}{x^2+3};$$

$$\therefore\ \frac{3}{x^2+9}=\frac{2}{x^2+3};$$ [removing the factor 18 from both sides]

$\therefore\ 3x^2+9=2x^2+18;\ \therefore\ x^2=9;\ \therefore\ x=\pm3.$ [arguing, as before]

Example 3. If $a+b=\frac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, find x.

We have $(a+b)(x+\sqrt{1+x^2})=2a\sqrt{1+x^2}$;

$\therefore\ (a+b)x=(a-b)\sqrt{1+x^2}$, or, $(a+b)^2x^2=(a-b)^2(1+x^2)$.

$\therefore\ x^2\{(a+b)^2-(a-b)^2\}=(a-b)^2$, or, $x^2.4ab=(a-b)^2$;

$$\therefore\ x^2=\frac{(a-b)^2}{4ab};\qquad \therefore\ x=\pm\frac{a-b}{2\sqrt{ab}}.$$

Example 4. If $\frac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}}=\frac{\sqrt{x^2-1}-1}{x^2-2}$, find x.

Put y for $\sqrt{x^2-1}$ and $\therefore\ y^2-1$ for x^2-2.

Thus, we have $\frac{1+y}{1+2ay}=\frac{y-1}{y^2-1}=\frac{1}{y+1}$.

Therefore, $(1+y)^2=1+2ay$, or, $1+2y+y^2=1+2ay$;

$\therefore\ y+2=2a$, or, $y=2(a-1)$;

i.e., $\sqrt{x^2-1}=2(a-1)$; $\therefore\ x^2-1=4(a-1)^2$;

$$\therefore\ x=\pm\sqrt{1+4(a-1)^2}.$$

Example 5. Solve $(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}=b.$

Since $\{(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}\}^3$

$$=(a+x)+(a-x)+3(a^2-x^2)^{\frac{1}{3}}\{(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}\}$$

$=2a+3(a^2-x^2)^{\frac{1}{3}}\times b;$ [because $(a+x)^{\frac{1}{3}}+(a-x)^{\frac{1}{3}}=b$]

therefore, cubing both sides of the equation, we get

$$2a + 3(a^2 - x^2)^{\frac{1}{3}} \times b = b^3, \qquad \text{or,} \quad 3b(a^2 - x^2)^{\frac{1}{3}} = b^3 - 2a;$$

$$\therefore \quad a^2 - x^2 = \left\{\frac{b^3 - 2a}{3b}\right\}^3;$$

$$\therefore \quad x^2 = a^2 - \left\{\frac{b^3 - 2a}{3b}\right\}^3;$$

$$\therefore \quad x = \pm\left\{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3\right\}^{\frac{1}{2}}.$$

Example 6. Solve $\dfrac{a+x}{a^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}} + \dfrac{a-x}{a^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}} = a^{\frac{1}{2}}.$

Since, $(a+x)\{a^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\} = a^{\frac{1}{2}}(a+x) + (a+x)^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}},$

and $(a-x)\{a^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\} = a^{\frac{1}{2}}(a-x) + (a-x)^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}};$

therefore, clearing the equation of fractions, we have

$$\begin{aligned}
&2a^{\frac{3}{2}} + (a^2 - x^2)^{\frac{1}{2}}\{(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\} \\
&\quad = a^{\frac{1}{2}}\{a^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}\}\{a^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\} \\
&\quad = a^{\frac{1}{2}}\left[a + a^{\frac{1}{2}}\{(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\} + (a^2 - x^2)^{\frac{1}{2}}\right] \\
&\quad = a^{\frac{3}{2}} + a\{(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\} + a^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}}.
\end{aligned}$$

Hence, removing $a^{\frac{3}{2}}$ from both sides and transposing, we get

$$a^{\frac{1}{2}}\{a - (a^2 - x^2)^{\frac{1}{2}}\} = \{(a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}}\} \times \{a - (a^2 - x^2)^{\frac{1}{2}}\} \quad \cdots \quad \text{(A)}$$

whence $a^{\frac{1}{2}} = (a+x)^{\frac{1}{2}} + (a-x)^{\frac{1}{2}};$

squaring both sides, $a = 2a + 2(a^2 - x^2)^{\frac{1}{2}},$

$$\text{or,} \quad -a = 2(a^2 - x^2)^{\frac{1}{2}};$$

$$\therefore \quad a^2 = 4(a^2 - x^2); \quad \therefore \quad 4x^2 = 3a^2; \quad \therefore \quad x = \pm\frac{a\sqrt{3}}{2}.$$

Note. *It must be observed that the above equation admits of another solution which has been overlooked; for* $a - (a^2 - x^2)^{\frac{1}{2}}$ *being a factor common to both sides of (A), if this be taken equal to zero, the given equation is evidently satisfied. Hence,* $(a^2 - x^2)^{\frac{1}{2}} = a$*, or,* $x = 0$ *is another solution. The same remark applies to example 4, which the student will very easily see for himself.*

EXERCISE 127

Find the value of x in each of the following equations :

1. $24x+\dfrac{7}{x}=\dfrac{169}{7}x.$ 2. $\dfrac{8x^2+10}{15}=7-\dfrac{50+4x^2}{25}.$

3. $\dfrac{14x^2+16}{21}-\dfrac{2x^2+8}{8x^2-11}=\dfrac{2x^2}{3}.$ 4. $\dfrac{x+7}{x(x-7)}-\dfrac{x-7}{x(x+7)}=\dfrac{7}{x^2-73}.$

5. $\dfrac{x^3-1}{(x-1)^2}-\dfrac{x^3+1}{(x+1)^2}=6.$ 6. $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1}=\dfrac{1}{x}.$

[Rationalise both the terms of the left-hand side and then proceed.]

7. $(1+x+x^2)^{\frac{1}{2}}=a-(1-x+x^2)^{\frac{1}{2}}.$

8. $\dfrac{(x-a)(x-b)}{(x-ma)(x-mb)}=\dfrac{(x+a)(x+b)}{(x+ma)(x+mb)}.$ 9. $\dfrac{ax+1+(a^2x^2-1)^{\frac{1}{2}}}{ax+1-(a^2x^2-1)^{\frac{1}{2}}}=\dfrac{b^2x}{2}.$

10. $(a+x)^{\frac{2}{3}}+(a-x)^{\frac{2}{3}}=3(a^2-x^2)^{\frac{1}{3}}.$

11. $\dfrac{5x^2+17}{x^2-11}+\dfrac{14x^2-117}{2x^2-9}=12.$ 12. $\dfrac{x^2-1}{x^2-4}-\dfrac{x^2-5}{x^2-8}=\dfrac{x^2-2}{x^2-5}-\dfrac{x^2-6}{x^2-9}.$

13. $\{a+(a^2-x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}+\{a-(a^2-x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$

$$=n\left\{\frac{a+x}{a+(a^2-x^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.$$

[Since, $a+(a^2-x^2)^{\frac{1}{2}}=\dfrac{(a+x)+(a-x)+2(a^2-x^2)^{\frac{1}{2}}}{2}=\dfrac{\{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}\}^2}{2}$;

and similarly, $a-(a^2-x^2)^{\frac{1}{2}}=\dfrac{\{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}\}^2}{2}$;

$\therefore$ the left-hand side $=\dfrac{2(a+x)^{\frac{1}{2}}}{\sqrt{2}}=\sqrt{2}(a+x)^{\frac{1}{2}}.$

Hence, squaring both sides, &c.]

14. $\dfrac{(1+2x)^{\frac{1}{2}}-1}{(1-2x)^{\frac{1}{2}}+1}+\dfrac{(1-2x)^{\frac{1}{2}}+1}{(1+2x)^{\frac{1}{2}}-1}=2\sqrt{2}.$

II. Solution of Affected Quadratic Equations by factorisation

234. Affected quadratic equations can, by suitable transformation and reduction, be expressed in the standard form

$$ax^2+bx+c=0.$$

If the left-hand side can be easily factorised, then by equating to zero either of these factors, we get a solution of the quadratic.

The following are the illustrative examples.

Example 1. Solve $10(2x+3)(x-3)+(7x+3)^2=20(x+3)(x-1)$.

We have $10(2x^2-3x-9)+(49x^2+42x+9)=20(x^2+2x-3)$;

$\therefore\ 49x^2-28x-21=0$; $\quad\therefore\ 7x^2-4x-3=0$,

or, $(7x^2-7x)+(3x-3)=0$, $\quad$ or, $(7x+3)(x-1)=0$.

$$\text{Hence, either } \left.\begin{array}{l} 7x+3=0 \\ \text{and } \therefore\ x=-\frac{3}{7} \end{array}\right\} \quad \text{or, } \left.\begin{array}{l} x-1=0 \\ \text{and } \therefore\ x=1 \end{array}\right\}$$

Thus, $-\frac{3}{7}$ and 1 are roots of the equation.

Example 2. Solve $(7-4\sqrt{3})x^2+(2-\sqrt{3})x=2$.

Since, $7-4\sqrt{3}=(2-\sqrt{3})^2$,

we have. $(2-\sqrt{3})^2x^2+(2-\sqrt{3})x=2$.

Hence, putting z for $(2-\sqrt{3})x$, we have

$$z^2+z-2=0, \quad \text{or, } (z+2)(z-1)=0.$$

$$\text{Hence, either, } \left.\begin{array}{l} z+2=\ 0 \\ \text{and } \therefore\ z=-2 \end{array}\right\} \quad \text{or, } \left.\begin{array}{l} z-1=0 \\ \text{and } \therefore\ z=1 \end{array}\right\}$$

$$\left.\begin{array}{ll} \text{Thus,} & x=\dfrac{-2}{2-\sqrt{3}}=-2(2+\sqrt{3}) \\ \text{or,} & \quad=\dfrac{1}{2-\sqrt{3}}=2+\sqrt{3}. \end{array}\right|$$

Example 3. Solve $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$. ... (1)

We have *identically*

$$(3x^2-7x-30)-(2x^2-7x-5)=x^2-25 \quad \cdots \quad \cdots (2)$$

i.e., this relation is true for every value of x, and hence it is also true for the particular value which x has in the proposed equation.

From (1) and (2), by division,

$$\frac{(3x^2-7x-30)-(2x^2-7x-5)}{\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}}=\frac{x^2-25}{x-5};$$

or, $\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5$. ... ... (3)

From (1) and (3), by addition, $2\sqrt{3x^2-7x-30}=2x$;

$\therefore\ 3x^2-7x-30=x^2$, $\quad$ or, $2x^2-7x-30=0$,

or, $(2x+5)(x-6)=0$; $\quad\therefore\ x=-\frac{5}{2}$, or, 6.

N. B. We might as well as have subtracted (1) from (3) and got the same result.

Example 4. Solve $\frac{1}{(x-b)(x-c)}+\frac{1}{(a+c)(a+b)}$

$$=\frac{1}{(a+c)(x-c)}+\frac{1}{(a+b)(x-b)}.$$

By transposition, $\frac{1}{x-c}\left\{\frac{1}{x-b}-\frac{1}{a+c}\right\}=\frac{1}{a+b}\left\{\frac{1}{x-b}-\frac{1}{a+c}\right\}.$

Therefore, either, $\frac{1}{x-b}-\frac{1}{a+c}=0,$ whence $x=a+b+$

or, $\frac{1}{x-c}=\frac{1}{a+b},$ whence also $x=a+b+$

Thus, the equation has got two *equal roots.*

Example 5. Solve $\frac{a+c(a+x)}{a+c(a-x)}+\frac{a+x}{x}=\frac{a}{a-2cx}.$

Since, $\frac{a+c(a+x)}{a+c(a-x)}=\frac{a}{a+c(a-x)}+\frac{c(a+x)}{a+c(a-x)},$

we have by transposition,

$$(a+x)\left\{\frac{c}{a+c(a-x)}+\frac{1}{x}\right\}=a\left\{\frac{1}{a-2cx}-\frac{1}{a+c(a-x)}\right\},$$

or, $$(a+x)\cdot\frac{a(1+c)}{x\{a+c(a-x)\}}=a\cdot\frac{c(a+x)}{(a-2cx)\{a+c(a-x)\}},$$

or, $$\frac{(a+x)(1+c)}{x}=\frac{c(a+x)}{a-2cx}.$$

Hence, either, $a+x=0$, and ∴ $x=-a$,

or, $\frac{1+c}{x}=\frac{c}{a-2cx},$ whence $x=\frac{a(1+c)}{c(3+2c)}.$

Thus, $-a$ and $\frac{a(1+c)}{c(3+2c)}$ are the roots of the equation.

EXERCISE 128

Solve the following equations :

1. $x^2+9x+18=6-4x.$ 2. $(x-2)(x+1)=208.$ 3. $x^2+3a^2=4ax.$

4. $\frac{x^2-b^2}{2}+ab=ax.$ 5. $abx^2-(a+b)cx+c^2=0.$

6. $12x^2+23ax-24a^2=0.$ 7. $10(x-a)^2-41(x-a)b+21b^2=0.$

8. $12(x-a)^2+28(x-a)(x-b)-5(x-b)^2=0.$

9. $20x^2+x(a+2b)=30(a+b)^2+bx.$

10. $\frac{3(10+x)}{95}-\frac{40}{3(10-x)}=\frac{x}{15}.$ 11. $(a-b)x^2-(a+b)x+2b=0.$

12. $\dfrac{x^2}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)x=\dfrac{1}{(ab^2)^{-\frac{1}{2}}+(a^2b)^{-\frac{1}{2}}}.$ 13. $\dfrac{2x(a-x)}{3a-2x}=\dfrac{a}{4}.$

14. $\dfrac{16}{x^{\frac{3}{2}}}+\dfrac{x^{\frac{1}{2}}}{2}=\dfrac{6}{x^{\frac{1}{2}}}.$ 15. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{2c}{x-c}.$

16. $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\dfrac{x}{a-x}.$ 17. $\sqrt{2x^2+5x-2}-\sqrt{2x^2+5x-9}=1.$

18. $\sqrt{3x^2+7x-1}+\sqrt{3x^2+7x-10}=9.$

19. $\sqrt{4x^2-7x+16}+\sqrt{4x^2-7x-1}=17.$

20. $\sqrt{5x^2-6x+8}-\sqrt{5x^2-6x-7}=1.$

235. If in the process of solving an adfected quadratic by factorisation, the factors are *not* easily obtained, any one of the following methods should be adopted

236. The ordinary method of solving an Adfected Quadratic. Bring the terms containing the unknown quantity to the left-hand side of the equation, and the known quantities to the right-hand side; if the coefficient of x^2 be negative, change the sign of every term of the equation and *then* divide every term by the coefficient of x^2; thus, the equation is reduced to the form $x^2+px=q$.

Now, add $\dfrac{p^2}{4}$ (*i.e.*, square of half the coefficient of x) to both sides, on which the left-hand side becomes a complete square and we get $\left(x+\dfrac{p}{2}\right)^2=q+\dfrac{p^2}{4}$, whence $x+\dfrac{p}{2}=\pm\sqrt{q+\dfrac{p^2}{4}}$,

and, therefore, $x=-\dfrac{p}{2}\pm\sqrt{q+\dfrac{p^2}{4}}.$

EXERCISE 129

Solve the following equations :

1. $70x-63=7x^2$.

[By transposition, we have $-7x^2+70x=63$.

Since, the coefficient of x^2 is negative, changing the sign of every term,

$$\text{we get } 7x^2-70x=-63.$$

Dividing both sides by 7, $x^2-10x=-9$.

Now, adding $(\frac{10}{2})^2$, or, 25 to both sides,

$$x^2-10x+25=25-9=16, \quad \text{or,} \quad (x-5)^2=16.$$

Hence, $x-5=\pm 4$, (because $x-5$ is a quantity of which the square is 16);

$\therefore$ $x=5+4$, or, $5-4$, *i.e.*, $x=9$, or, 1.]

2. $2x^2 - 11x + 5 = 0.$

[By transposition, $2x^2 - 11x = -5$;

dividing both sides by 2, $x^2 - \frac{11}{2}x = -\frac{5}{2}.$

Adding $(\frac{11}{4})^2$ to both sides,

$x^2 - \frac{11}{2}x + (\frac{11}{4})^2 = \frac{121}{16} - \frac{5}{2}$, i.e., $(x - \frac{11}{4})^2 = \frac{81}{16}$;

$\therefore\ x - \frac{11}{4} = \pm\frac{9}{4}$; $\therefore\ x = \frac{11}{4} \pm \frac{9}{4} = 5$, or, $\frac{1}{2}$.]

3. $87 - 98x - 30x - 16x^2.$

4. $17x^2 - 85x + 216 = 65x - 8x^2.$ 5. $\frac{x^2+8}{11} = 5x - x^2 - 5.$

6. $4(x^2 - 3\frac{3}{8}x) = 10(x^2 - 4\frac{2}{8}x - 6) + 3(\frac{x}{8} - \frac{6}{5}).$

7. $4(5x^2 - 3\frac{7}{9}x) = 5(x^2 - 7x + 12) + \frac{8(x-9)}{9}$

8. $2x + \cdot 02 = 2\cdot 45x - x^2.$ 9. $4(x^2 + 23x - 24) = 29x^2 - 8x + 1.$

10. $(3x-1)(x-4) + (x-2)(2x-3) = 4x(x-3) - 5.$

[The left-hand side $= (3x^2 - 13x + 4) + (2x^2 - 7x + 6) = 5x^2 - 20x + 10.$

Hence, we have $5x^2 - 20x + 10 = 4x^2 - 12x - 5$,

$\therefore\ x^2 - 8x = -15$; (by transposition)

$\therefore\ x^2 - 8x + (4)^2 = 16 - 15$; or, $(x-4)^2 = 1$;

$\therefore\ x - 4 = \pm 1$; $\therefore\ x = 4 \pm 1 = 5$, or, 3.]

11. $(2x-5)(3x-7) - (x-1)(4x-5) = x^2 - 3(x+14).$

12. $(3x-11)(x-2) + (2x-3)(x+4) + 13x = 10(2x-1)^2 + 19.$

13. $(x - \frac{1}{2})(x - \frac{1}{3}) + (x - \frac{1}{3})(x - \frac{1}{4}) = (x - \frac{1}{4})(x - \frac{1}{2}).$

14. $\frac{x}{15} + \frac{40}{3(10-x)} = \frac{3(10+x)}{95}.$

[By transposition, $\frac{40}{3(10-x)} = \frac{3(10+x)}{95} - \frac{x}{15} = $ &c.]

15. $\frac{2x}{15} + \frac{3x-50}{3(10+x)} = \frac{12x+70}{190}.$

16. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3}.$

[Subtracting 2 from both sides, we have

$\left(\frac{x+4}{x-4} - 1\right) + \left(\frac{x-4}{x+4} - 1\right) = \frac{4}{3}$, or, $\frac{8}{x-4} - \frac{8}{x+4} = \frac{4}{3}$,

or, $2\left(\frac{1}{x-4} - \frac{1}{x+4}\right) = \frac{1}{3}$, or, $\frac{2 \times 8}{x^2 - 16} = \frac{1}{3}$;

$\therefore\ x^2 - 16 = 48$; $\therefore\ x^2 = 64$; $\therefore\ x = \pm 8$.]

17 $\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6}.$ [Proceed as in the last example.]

18. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$. [Proceeding as in example 16, we get $x^2-4x=0$, whence $(x-2)^2=4$; $\therefore x=2\pm2=4$, or, 0.]

19. $\frac{x-2}{x+2}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$.

20. $\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{5}{6}$.

[We have $\left(\frac{x+2}{x-2}-1\right)-\left(\frac{x-2}{x+2}-1\right)=\frac{5}{6}$, or, &c. &c.]

21. $\frac{x-6}{x-12}-\frac{x-12}{x-6}=\frac{5}{6}$.

22. $\frac{2x-9}{2x-7}-\frac{2x-7}{2x-9}=\frac{7}{12}$.

23. $\frac{x+6}{x+7}-\frac{x+1}{x+2}=\frac{1}{3x+1}$. [C. U. 1878]

24. $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}$. [We have $\left(\frac{2x}{x-4}-2\right)+\left(\frac{2x-5}{x-3}-2\right)=4\frac{1}{3}$.]

25. $\frac{x}{x+5}-\frac{11x}{11x-8}+\frac{7}{6-4x}=0$.

26. $\frac{1}{x+a}+\frac{1}{x+2a}+\frac{1}{x+3a}=\frac{3}{x}$.

[We have $\left(\frac{1}{x+a}-\frac{1}{x}\right)+\left(\frac{1}{x+2a}-\frac{1}{x}\right)+\left(\frac{1}{x+3a}-\frac{1}{x}\right)=0$,

whence $\frac{1}{x+a}+\frac{2}{x+2a}+\frac{3}{x+3a}=0$, or, $\frac{1}{x+a}+\frac{1}{x+3a}=-2\left(\frac{1}{x+3a}+\frac{1}{x+2a}\right)$,

whence $\frac{x+2a}{x+a}=-\frac{2x+5a}{x+2a}$; $\therefore$ &c.]

237. General expression for the roots of a quadratic.

N. B. The roots of any equation are those values of the unknown quantity that satisfy the equation.

As every quadratic equation can be written in the form $ax^2+bx+c=0$ *(after suitable reduction, if necessary), we must regard this equation as the general type of all quadratics. Let us solve it.*

By transposition, $ax^2+bx=-c$.

Dividing both sides by a, $x^2+\frac{b}{a}x=-\frac{c}{a}$.

Adding $\left(\frac{b}{2a}\right)^2$ to both sides,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}, \quad \text{or,} \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

$$\therefore \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}; \qquad \therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Thus, the roots of the quadratic $ax^2+bx+c=0$, are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, and, therefore, we must regard the expression $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ as the *general expression*, for the roots sought.

By the application of this formula we can find out the roots of a quadratic equation without going through the process explained in Art. 236.

Example 1. Write down the roots of $2x^2-13x+15=0$.

Comparing this with the equation $ax^2+bx+c=0$, we have $a=2$, $b=-13$, $c=15$.

Hence, the roots of the given equation are

$$=\frac{-(-13)\pm\sqrt{(-13)^2-4\times2\times15}}{2\times2}$$

$$=\frac{13\pm\sqrt{169-120}}{4}=\frac{13\pm\sqrt{49}}{4}=\frac{13\pm7}{4}$$

That is, $x=5$, or, $\frac{3}{2}$.

Example 2. Write down the roots of $-3x^2=11x-4$.

Bringing all the terms to one side, we have $-3x^2-11x+4=0$.

Here $a=-3$, $b=-11$, $c=4$.

Hence, $x=\frac{-(-11)\pm\sqrt{(-11)^2-4\times(-3)\times4}}{2\times(-3)}$

$$=\frac{11\pm\sqrt{121+48}}{-6}=\frac{11\pm\sqrt{169}}{-6}$$

$$=\frac{11\pm13}{-6}=-4\text{, or, }\frac{1}{3}.$$

EXERCISE 130

Write down the roots of the following equations :

1. $3x^2-17x+24=0$.
2. $x^2+9x+20=0$.
3. $6x^2=20-7x$.
4. $-9x^2+25=6x-10$.
5. $8x^2=14x+15$.
6. $-3x^2+20x=25$
7. $5+x-4x^2=0$.

238. **Sreedharacharyya's (or Hindu) method of solving a quadratic.** Reduce the equation to the form $px^2+qx=r$; multiply both sides of this by $4p$ (*i.e.*, by four times the coefficient of x^2) and *then* add q^2 to both sides; we thus get $4p^2x^2+4pqx+q^2=4pr+q^2$, the left-hand side of which is evidently a complete square, being equal to $(2px+q)^2$.

Hence, we get $(2px+q)^2=4pr+q^2$,

or, $(2px+q)=\pm\sqrt{4pr+q^2}$,

or, $2px=-q\pm\sqrt{4pr+q^2}$;

$$\therefore \quad x=\frac{-q\pm\sqrt{4pr+q^2}}{2p}.$$

Example 1. Solve $5x^2-17x+6=0$.

By transposition, $5x^2-17x=-6$.

Multiplying both sides by 4×5,

$$4\times(5x)^2-4\times(5x)\times 17=-120.$$

Adding $(17)^2$ to both sides, we have

$$4\times(5x)^2-4\times(5x)\times 17+(17)^2=289-120$$

or, $(2\times 5x-17)^2=169$; $\therefore$ $10x-17=\pm 13$;

$$\therefore \quad x=\frac{17\pm 13}{10}=3, \text{ or, } \tfrac{2}{5}.$$

Example 2. Solve $-8x^2+10x=3$.

Multiplying both sides by $4\times(-8)$, $4\times 64x^2-4\times 8\times 10x=-96$.

Adding $(10)^2$ to both sides,

$$4\times 64x^2-4\times 8\times 10x+(10)^2=100-96,$$

or, $(2\times 8x-10)^2=4$; $\therefore$ $16x-10=\pm 2$;

$$\therefore \quad x=\frac{10\pm 2}{16}=\frac{3}{4}, \text{ or, } \frac{1}{2}.$$

Example 3. Solve $6x^2+23x=12x+10$.

By transposition, $6x^2+11x=10$.

Multiplying both sides by 4×6, $4\times(6x)^2+4\times(6x)\times 11=240$.

Adding $(11)^2$ to both sides,

$$4\times(6x)^2+4\times(6x)\times 11+(11)^2=121+240,$$

or, $(2\times 6x+11)^2=361$; $\therefore$ $12x+11=\pm 19$;

$$\therefore \quad x=\frac{-11\pm 19}{12}=\tfrac{2}{3}, \text{ or, } -\tfrac{5}{2}.$$

EXERCISE 131

Solve the following equations by Sreedharacharyya's method :

1. $2x^2+9x=18$.
2. $15x^2-28=x$.
3. $16x^2+100x=3x^2+x+40$.
4. $x^2+50x=102-15x-x^2$.
5. $17x^2+19x=1848$.
6. $2cx^2-acx=3(2x-a)$.
7. $x^2+ax=ab(3x+a)-2x^2$.

239. Equations solved like Quadratics. Some equations though not actually quadratic themselves, may by suitable substitutions, be expressed as quadratics, and thus solved.

Example 1. Solve $x^4-10x^2+9=0$.

Putting y for x^2, the equation is $y^2-10y+9=0$,

or, $(y-1)(y-9)=0$.

Hence, either, $y-1=0$, or, $y-9=0$,

i.e., $y=1$, or, 9,

i.e., $x^2=1$, or, 9,

i.e., $x=\pm1$, or, ±3.

Example 2. Solve $\dfrac{25a^4}{x^2}+x^2=26a^2$.

Multiplying both sides by x^2, $25a^4+x^4=26a^2x^2$,

or, $x^4-26a^2x^2+25a^4=0$.

Putting y for x^2, we have $y^2-26a^2y+25a^4=0$,

or, $(y-a^2)(y-25a^2)=0$.

Hence, either, $y-a^2=0$, or, $y-25a^2=0$,

i.e., $y=a^2$, or, $25a^2$,

i.e., $x^2=a^2$, or, $25a^2$,

i.e., $x=\pm a$, or, $\pm5a$.

Example 3. Solve $(x^2+3x)^2-(x^2+3x)-6=0$.

Putting y for x^2+3x, we have $y^2-y-6=0$,

or, $(y+2)(y-3)=0$;

$\therefore$ either, (i) $y+2=0$, or, (ii) $y-3=0$.

(i) If $y+2=0$, we have $x^2+3x+2=0$,

i.e., $(x+1)(x+2)=0$,

i.e., $x=-1$, or, -2.

(ii) If $y-3=0$, we have $x^2+3x-3=0$.

Solving the quadratic, $x=\dfrac{-3\pm\sqrt{21}}{2}$;

$\therefore$ $x=-1,\ -2$, or, $\dfrac{-3\pm\sqrt{21}}{2}$.

Example 4. Solve $(x+2)(x+3)(x+4)(x+5)=24(x^2+7x+7)$.

Re-arranging the factors on the left side, we have

$$\{(x+2)(x+5)\}\{(x+3)(x+4)\}=24(x^2+7x+7).$$

or, $(x^2+7x+10)(x^2+7x+12)=24(x^2+7x+7)$,

or, $(y+10)(y+12)=24(y+7)$, [putting y for x^2+7x]

or, $y^2+22y+120=24y+168$,

or, $y^2-2y-48=0$; $\therefore$ $(y-8)(y+6)=0$.

Hence, either, (i) $y-8=0$, or, (ii) $y+6=0$.

(i) If $y-8=0$, we have $x^2+7x-8=0$,

or, $(x+8)(x-1)=0$;

$\therefore$ $x+8=0$, or, $x-1=0$, *i.e.*, $x=-8$, or, 1.

(ii) If $y+6=0$, we have $x^2+7x+6=0$,

or, $(x+1)(x+6)=0$;

$\therefore$ $x+1=0$, or, $x+6=0$, *i.e.*, $x=-1$, or, -6.

$\therefore$ $x=-8, 1, -1$, or, -6.

Example 5. Solve $3x^2-4x+\sqrt{3x^2-4x-6}=18$.

Adding -6 to both sides, $3x^2-4x-6+\sqrt{3x^2-4x-6}=12$.

Putting z for $\sqrt{3x^2-4x-6}$,

the given equation reduces to $z^2+z=12$,

i.e., $z^2+z-12=0$,

or, $(z-3)(z+4)=0$;

$\therefore$ either, (i) $z=3$, or, (ii) $z=-4$.

(i) If $z=3$, $\sqrt{3x^2-4x-6}=3$,

or, $3x^2-4x-6=9$, or, $3x^2-4x-15=0$,

or, $(x-3)(3x+5)=0$; $\therefore$ $x=3$, or, $-\frac{5}{3}$.

(ii) If $z=-4$, $\sqrt{3x^2-4x-6}=-4$,

or, $3x^2-4x-6=16$, or, $3x^2-4x-22=0$.

Solving the quadratic, $x=\dfrac{4\pm\sqrt{16+4.66}}{6}=\dfrac{2\pm\sqrt{70}}{3}$;

$\therefore$ $x=3$, $-\frac{5}{3}$, or, $\dfrac{2\pm\sqrt{70}}{3}$.

240. Equations of higher degrees solved by factorisation.

Example 1. Solve $x^3-7x+6=0$.

By inspection, $x-1$ is a factor of the left side.

Hence, factorising the left side, the equation may be written as

$(x-1)(x^2+x-6)=0$,

or, $(x-1)(x-2)(x+3)=0$; [factorising the quadratic factor

$\therefore$ either, $x-1=0$, or, $x-2=0$, or, $x+3=0$,
i.e., $x=1, 2$, or, -3.

Example 2. Solve $x^3+1=0$.

Here, we have $(x+1)(x^2-x+1)=0$;

$\therefore$ either, (i) $x+1=0$, or, (ii) $x^2-x+1=0$.

(i) If $x+1=0$, $x=-1$.

(ii) If $x^2-x+1=0$, solving the quadratic, we have

$$x=\frac{1\pm\sqrt{-3}}{2}; \quad \therefore \quad x=-1, \text{ or, } \frac{1\pm\sqrt{-3}}{2}.$$

Note. *The square root of -3 is an impossible operation. Such square roots are, however, frequently used in Algebra and are called* ***imaginary quantities.***

Example 3. Solve $x^4+7x^3+8x^2+7x+1=0$.

The left side of this equation is a *reciprocal expression* and may be put into factors, as in Art. 143.

Here, re-arranging the terms of the left side, we have

$$(x^4+1)+7(x^3+x)+8x^2=0,$$
$$\text{or,} \quad (x^2+1)^2+7x(x^2+1)+6x^2=0,$$
$$\text{or,} \quad \{(x^2+1)+x\}\{(x^2+1)+6x\}=0,$$
$$\text{or,} \quad (x^2+x+1)(x^2+6x+1)=0.$$

$\therefore$ either, (i) $x^2+x+1=0$, or, (ii) $x^2+6x+1=0$.

(i) If $x^2+x+1=0$, solving we have

$$x=\frac{-1\pm\sqrt{-3}}{2}.$$

(ii) If $x^2+6x+1=0$, solving we have

$$x=\frac{-6\pm\sqrt{6^2-4}}{2}=-3\pm\sqrt{8}.$$

$$\therefore \quad x=\frac{-1\pm\sqrt{-3}}{2}, \text{ or, } -3\pm\sqrt{8}.$$

241. Exponential equations solved as a quadratic.

Example 1. Solve $5^{x-1}+5^{-x}=1\frac{1}{5}$.

Here, we have $\frac{5^x}{5}+\frac{1}{5^x}=\frac{6}{5}$, or, $\frac{y}{5}+\frac{1}{y}=\frac{6}{5}$, [putting y for 5^x]

or, $y^2-6y+5=0$,

or, $(y-1)(y-5)=0$, whence $y=1$, or, 5,

i.e., $5^x=1$, or, 5, i.e., $5^x=5^0$, or, 5^1;

$\therefore$ $x=0$, or, 1.

Example 2. Solve $2^{x-2}+2^{3-x}=3$.

Here, we have $\frac{2^x}{2^2}+\frac{2^3}{2^x}=3$, or, $\frac{y}{4}+\frac{8}{y}=3$, [putting y for 2^x]

or, $y^2-12y+32=0$,

or, $(y-4)(y-8)=0$, ∴ $y=4$, or, 8,

i.e., $2^x=4$, or, 8, *i.e.*, $2^x=2^2$, or 2^3;

∴ $x=2$, or, 3.

EXERCISE 132

Solve the following equations :

1. $x^3-6x^2+11x-6=0$.
2. $x^3-4x^2+x+2=0$.
3. $2x^3+5x^2-4x-3=0$.
4. $x^3+5x^2-2x-6=0$.
5. $x^4-5x^3+6x^2-5x+1=0$.
6. $x^4-5x^3+14x^2-20x+16=0$.
7. $x^4+8x^3+24x^2+32x-20=0$.
8. $(x+2)(x+3)(x+4)(x+5)=360$.
9. $(x-1)(x-2)(x+3)(x+4)+4=0$.
10. $x^4-4x^3-x^2+10x+4=0$.
11. $x^4-6x^3+15x^2-18x+5=0$.
12. $2x^5-5x^4-3x^3+9x^2-x-2=0$.
13. $x^4-1=0$.
14. $x^4-37x^2+36=0$.
15. $3^{x-2}+3^{3-x}=4$.
16. $7^{x-3}+7^{2-x}=1\frac{1}{7}$.
17. $2^x-2^{3-2x}=7(1-2^{1-x})$.
18. $x^6-1=0$.
19. $11^x+11^{-x}=121\frac{1}{121}$.
20. $2x^2-5x-6\sqrt{2x^2-5x+3}=-8$.
21. $9x-4x^2+\sqrt{4x^2-9x+11}=5$.
22. $2(x^2-3x+1)^2+5(x^2-3x+1)+3=0$.
23. $(x+4)(x+1)+\sqrt{(x+5)(x-3)}=3x+31$.
24. $10x^4-63x^3+52x^2+63x+10=0$.

242. The Nature of Roots of a Quadratic. If α, β denote the roots of the quadratic equation $ax^2+bx+c=0$, we have by Art. 237,

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Three distinct cases do, therefore, arise according as the expression under the radical (b^2-4ac) is (1) zero, (2) positive and (3) negative.

Case I. Equal Roots. If $b^2-4ac=0$, $\sqrt{b^2-4ac}=0$;

$$\therefore \alpha=\frac{-b+0}{2a}=-\frac{b}{2a} \text{ and } \beta=\frac{-b-0}{2a}=-\frac{b}{2a}.$$

Hence, *the roots of* $ax^2+bx+c=0$ *are real and equal if* $b^2-4ac=0$.

Example. Examine the roots of $4x^2-12x+9=0$.

Here, $a=4$, $b=-12$ and $c=9$;

$\therefore\ b^2-4ac=(-12)^2-4.4.9=144-144=0$.

Hence, the roots of $4x^2-12x+9=0$ are real and equal and are found to be $\frac{3}{2}, \frac{3}{2}$.

Case II. Real and Unequal Roots. If b^2-4ac is a *positive* quantity, $\sqrt{b^2-4ac}$ is *real*.

$\therefore$ α and β are *real* but *unequal*.

Hence, *the roots of $ax^2+bx+c=0$ are real and unequal if b^2-4ac is positive.*

(i) If b^2-4ac is a *perfect square*, $\sqrt{b^2-4ac}$ is *rational* and *real*.

In this case, the roots are also *rational*, *real* and *unequal*.

(ii) If b^2-4ac is *positive* but *not a perfect square*, $\sqrt{b^2-4ac}$ is *real* but *irrational*.

Hence, the roots are also *real*, *irrational* and *unequal*.

Example 1. The roots of $2x^2+7x-4=0$ are real and unequal as well as rational, since $7^2-4.2.(-4)=49+32=81$ is positive and a perfect square. The roots are found to be $\frac{1}{2}$ and -4.

Example 2. The roots of $2x^2-9x+8=0$ are real, unequal but irrational, since, $(-9)^2-4.2.8=81-64=17$ is positive but not a perfect square.

Thus, the roots are $\frac{9\pm\sqrt{17}}{4}$.

Case III. Imaginary roots. If b^2-4ac is negative, $\sqrt{b^2-4ac}$ = the square root of a negative quantity, which is an impossible operation. Such square roots are however, frequently used in Algebra and are called **imaginary quantities**.

Hence, *if b^2-4ac is negative, the roots of $ax^2+bx+c=0$ are imaginary quantities.*

Thus, the roots of $x^2-x+1=0$ are imaginary, since $(-1)^2-4.1.1=-3$ and is, therefore, a negative quantity.

The roots are $\frac{1\pm\sqrt{(-1)^2-4.1.1}}{2}$, *i.e.*, $\frac{1\pm\sqrt{-3}}{2}$.

EXERCISE 133

Examine the roots of the following equations :

1. $3x^2+20x-19=0$. **2.** $3x^2-18x+9=0$. **3.** $x^2+5x+4=0$.

4. $4x^2-12x+9=0$. **5.** $-3x^2-2x+6=0$. **6.** $-4x^2+5x-8=0$.

7. $3x^2+7x+8=0$. 8. $4x^2-8x+(4-a^2-b^2)=0$.

9. $(a-b)x^2+2(a+b)x-(a-b)=0$.

10. For what value of m will the equation $2x^2+8x+m=0$ have equal roots ?

11. If $4x^2-px+9=0$ has equal roots, find p.

12. For what value of m will the equation $x^2-2(5+2m)x+3(7+10m)=0$ have equal roots ?

[By the condition of the problem,

$$\{-2(5+2m)\}^2-4.1.3(7+10m)=0, \quad i.e., \quad 4(5+2m)^2-4.3(7+10m)=0,$$
$$\text{or,} \quad (25+20m+4m^2)-3(7+10m)=0, \quad \text{or,} \quad 2m^2-5m+2=0,$$
$$\text{or,} \quad (2m-1)(m-2)=0\ ; \quad \therefore \quad m=\tfrac{1}{2}, \text{ or, } 2. \,]$$

13. Find the greatest and least values of $\dfrac{x^2+14x+9}{x^2+2x+3}$ for real values of x.

[Let $\dfrac{x^2+14x+9}{x^2+2x+3}=m$. Then $x^2+14x+9=m(x^2+2x+3)$,

$$\text{or,} \quad (1-m)x^2+2(7-m)x+3(3-m)=0\ ;$$
$$\therefore \quad x=\frac{-2(7-m)\pm\sqrt{4.(7-m)^2-4(1-m).3(3-m)}}{2(1-m)}$$

The expression under the radical sign

$$=4(49-14m+m^2)-12(3-4m+m^2)$$
$$=-8(m^2+m-20)=-8(m-4)(m+5).$$

Since x is real, the expression must be positive or zero,

i.e., $-8(m-4)(m+5)$ must be positive or zero.

∴ m cannot be *greater than* 4, but *may be equal to* 4 (since for *any* value of m greater than 4, say 5, the expression is negative).

Hence, the greatest value of the expression $=4$.

Similarly, m *cannot be* less than -5, but *may be equal to* (-5) (since for *any* value of m less than -5, say -6, the expression is negative).

Hence, the least value required $=-5$.]

14. Prove that $\dfrac{x}{x^2-5x+9}$ must lie between 1 and $-\frac{1}{11}$ for all real values of x.

15. Prove that the value of $\dfrac{x^2+8x+80}{2x+8}$ must not lie between -8 and 8, if x be real.

[Let $\dfrac{x^2+8x+80}{2x+8}=m$ and proceed as in Ex. 13.]

16. If x be real, prove that $\dfrac{11x^2+12x+6}{x^2+4x+2}$ cannot lie between -5 and 3.

17. If x be real, prove that $\dfrac{x^2-2x+21}{6x-14}$ cannot lie between 2 and $-\frac{10}{9}$.

18. If x be real, the value of $\dfrac{(x-1)(x+3)}{(x-2)(x+4)}$ does not lie between $\frac{4}{9}$ and 1.

***243. A quadratic equation cannot have more than two roots.**

Let $ax^2+bx+c=0$ be any quadratic equation. To prove that it cannot have more than two roots.

Proof. Since ax^2+bx+c

$$=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{(b^2-4ac)}{4a^2}\right\}$$

$$=a\left\{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right\}\left\{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right\} \quad [\text{ factorising }]$$

$$=a(x-\alpha)(x-\beta),$$

$$\left[\text{putting } \alpha \text{ for } \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta \text{ for } \frac{-b-\sqrt{b^2-4ac}}{2a}\right]$$

and since a is not zero, we have $ax^2+bx+c=0$, when and only when any one of the two factors $x-\alpha$, $x-\beta$ is zero,

i.e., when and only when $x=\alpha$, or, β.

Thus, the quadratic equation $ax^2+bx+c=0$ has got the two roots α and β and no more.

244. *If a quadratic equation in x is satisfied by three different values of x, the equation will be satisfied by every value of x.*

Let the quadratic equation $ax^2+bx+c=0$ be satisfied by three different values α, β, γ of x.

$$\therefore \quad a\alpha^2+b\alpha+c=0, \quad \cdots \quad (1)$$

$$a\beta^2+b\beta+c=0, \quad \cdots \quad (2)$$

$$\text{and} \quad a\gamma^2+b\gamma+c=0. \quad \cdots \quad (3)$$

Subtracting (2) from (1), we have $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$,

or, $(\alpha-\beta)\{a(\alpha+\beta)+b\}=0$.

Now, $\because$ $\alpha-\beta$ is *not* zero (α and β being different),

$$\therefore \quad a(\alpha+\beta)+b=0 \quad \cdots \quad (4)$$

Similarly, from (1) and (3),

$$a(\alpha+\gamma)+b=0. \quad \cdots \quad (5)$$

Hence, subtracting (5) from (4),

$$a(\beta-\gamma)=0.$$

But, $\beta-\gamma$ is *not* zero (since β and γ are different).

$$\therefore \quad a=0.$$

Hence, from (4), $\quad 0.(\alpha+\beta)+b=0, \quad$ *i.e.*, $\quad b=0.$

Since, $a=0$, $b=0$, we have from (1), $c=0$.

$$\therefore \quad ax^2+bx+c=0.x^2+0.x+0=0 \text{ for every value of } x.$$

245. Relations between roots and coefficients of a quadratic.

If α and β be the roots of the quadratic $ax^2+bx+c=0$, to prove that

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

Solving the equation as in Art. 237, we have

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

and $$\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Hence, by addition, $\alpha+\beta=\dfrac{-2b}{2a}=-\dfrac{b}{a}$;

and by multiplication, $\alpha\beta=\dfrac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$

$$=\frac{(-b)^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Since, the equation $ax^2+bx+c=0$ can also be written as $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, we may express the results as follows :

In a quadratic equation of the form $x^2+px+q=0$ (*i.e.*, *where the coefficient of $x^2=1$ and the terms are all on one side*),

(i) the sum of the roots $=-$the coefficient of x ;

(ii) the product of the roots $=$ the constant term,

i.e., the term independent of x.

Example 1. If α, β denote the roots of the quadratic $x^2+6x+9=0$ prove that $\alpha+\beta=-6$ and $\alpha\beta=9$.

Here, the coefficient of $x^2=1$ and the terms are all on one side.

Hence, we have $\alpha+\beta=-$the coefficient of $x=-6$ and $\alpha\beta=$the constant term$=9$.

Example 2. If α, β be the roots of $3x^2-17x+19=0$, prove that $\alpha+\beta=\frac{17}{3}$ and $\alpha\beta=\frac{19}{3}$.

Re-writing the equation in the form $x^2+(-\frac{17}{3})x+\frac{19}{3}=0$, so that the coefficient of $x^2=1$, and the terms are all on one side, we have

$$\alpha+\beta=-\text{the coefficient of } x=-(-\tfrac{17}{3})=\tfrac{17}{3},$$

and $\alpha\beta=$ the constant term $=\frac{19}{3}$.

Example 3. If α, β are the roots of $x^2+px+q=0$, find

(i) $\alpha-\beta$; (ii) $\alpha^3+\beta^3$; (iii) $\alpha^{-1}+\beta^{-1}$.

We have $\alpha+\beta=-$ the coefficient of x in $x^2+px+q=-p$,

and $\alpha\beta=$ the constant term $=q$.

(i) Since $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(-p)^2-4q=p^2-4q$.

$$\therefore\ \alpha-\beta=\pm\sqrt{p^2-4q}.$$

(ii) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(-p)^3-3q(-p)=-p^3+3pq$.

(iii) $\alpha^{-1}+\beta^{-1}=\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\alpha+\beta}{\alpha\beta}=\dfrac{-p}{q}$.

246. Formation of equations with given roots.

Let α, β be the given roots and let $x^2-px+q=0$ be the equation sought.

$\therefore\ \alpha+\beta=-$(the coefficient of x in x^2-px+q) $=-(-p)=p$,

and $\alpha\beta=$ the constant term $=q$.

Substituting for p and q in $x^2-px+q=0$,

the required equation is $x^2-(\alpha+\beta)x+\alpha\beta=0$, ... (A)

or, $(x-\alpha)(x-\beta)=0$. ... (B)

Otherwise: The expression $(x-\alpha)(x-\beta)$ is zero, if any one of its factors $x-\alpha$, $x-\beta$ is zero,

i.e., if x has any one of the values α and β.

Hence, *the equation whose roots are* α, β is $(x-\alpha)(x-\beta)=0$.

[Evidently the equation has no other roots; for, if the left-hand side is zero, one of its factors must be zero, so that x must have one of the values α or β.]

Note. *Similarly, the equation whose roots are* α, β, γ is $(x-\alpha)(x-\beta)(x-\gamma)=0$, *and so on.*

Example 1. Form the quadratic whose roots are 4 and -5.

By (B), the equation is $(x-4)\{x-(-5)\}=0$,

i.e., $(x-4)(x+5)=0$,

or, $x^2+x-20=0$.

Example 2. Form the quadratic whose roots are $3+\sqrt{5}$ and $3-\sqrt{5}$.

Since, $(3+\sqrt{5})+(3-\sqrt{5})=6$,

and $(3+\sqrt{5})(3-\sqrt{5})=3^2-5=4$;

$\therefore$ by (A), the equation sought is $x^2-6x+4=0$.

Example 3. If α, β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

By (A), the required equation is

$$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=0, \quad \text{or,} \quad x^2-\frac{\alpha^2+\beta^2}{\alpha\beta}x+1=0.$$

Since, $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$, we have

$$\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{\left(-\frac{b}{a}\right)^2-2\cdot\frac{c}{a}}{\frac{c}{a}}=\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c}{a}}=\frac{b^2-2ac}{ac}.$$

Hence, the required equation is $x^2-\frac{b^2-2ac}{ac}x+1=0$,

or, $acx^2-(b^2-2ac)x+ac=0$.

Example 4. Form the quadratic whose roots are the reciprocals of the roots of the equation $x^2+3x+4=0$.

Let α, β be the roots of $x^2+3x+4=0$.

Find the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

By (A), the equation required is

$$x^2-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x+\frac{1}{\alpha}\cdot\frac{1}{\beta}=0,$$

$$\text{or,} \quad x^2-\frac{\alpha+\beta}{\alpha\beta}x+\frac{1}{\alpha\beta}=0. \qquad \cdots \quad (1)$$

But since α, β are the roots of $x^2+3x+4=0$, we have $\alpha+\beta=-3$ and $\alpha\beta=4$.

$\therefore$ $\frac{\alpha+\beta}{\alpha\beta}=\frac{-3}{4}$, and $\frac{1}{\alpha\beta}=\frac{1}{4}$.

Hence, from (1), the required equation is $x^2-(-\frac{3}{4})x+\frac{1}{4}=0$,

or, $x^2+\frac{3}{4}x+\frac{1}{4}=0$, or, $4x^2+3x+1=0$.

247. Common Root of two equations.

Let α = the common root of the equations $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$.

We have $a\alpha^2+b\alpha+c=0$, and $a'\alpha^2+b'\alpha+c'=0$.

By cross-multiplication,

$$\frac{\alpha^2}{bc'-b'c}=\frac{\alpha}{ca'-c'a}=\frac{1}{ab'-a'b};$$

$$\therefore \quad \alpha^2=\frac{bc'-b'c}{ab'-a'b} \text{ and } \alpha=\frac{ca'-c'a}{ab'-a'b}; \qquad \cdots \quad (1)$$

$$\therefore \quad \frac{bc'-b'c}{ab'-a'b}=\left(\frac{ca'-c'a}{ab'-a'b}\right)^2,$$

$$\text{or,} \quad (ca'-c'a)^2=(bc'-b'c)(ab'-a'b) \qquad \cdots \quad (2)$$

which is the condition that the equations shall have a common root.

From (1), the common root

$$=\alpha=\frac{\alpha^2}{\alpha}=\frac{\dfrac{bc'-b'c}{ab'-a'b}}{\dfrac{ca'-c'a}{ab'-a'b}}=\frac{bc'-b'c}{ca'-c'a}, \text{ or, } \frac{ca'-c'a}{ab'-a'b}.$$

EXERCISE 134

Form the equations whose roots are :

1. 3 and 1. 2. 5 and -7. 3. 3 and $\frac{1}{3}$.

4. (i) $4+\sqrt{5}$ and $4-\sqrt{5}$; (ii) $2a+\sqrt{b}$ and $2a-\sqrt{b}$.

5. Find the sum and the product of the roots of :

(i) $x^2-5x+6=0$; (ii) $x^2+9x-13=0$;

(iii) $-3x^2+20x+15=0$; (iv) $5x^2=7x+3$;

(v) $3x+1=-15x^2$.

6. If α and β are the roots of the equation $x^2+px+q=0$, form the equation whose roots are :

(i) $\alpha^2+\alpha\beta$ and $\beta^2+\alpha\beta$;

[$(\alpha^2+\alpha\beta)+(\beta^2+\alpha\beta)=\alpha^2+2\alpha\beta+\beta^2=(\alpha+\beta)^2=(-p)^2=p^2$,

and $(\alpha^2+\alpha\beta)(\beta^2+\alpha\beta)=\alpha(\alpha+\beta)\beta(\alpha+\beta)=\alpha\beta(\alpha+\beta)^2=p^2q$;

since, $\alpha+\beta=-p$ and $\alpha\beta=q$.

Hence, the required equation is $x^2-p^2x+p^2q=0$.]

(ii) $\alpha^2+\beta^2$ and $2\alpha\beta$; (iii) $\alpha^{-2}+\beta^{-2}$ and $\frac{2}{\alpha\beta}$; (iv) $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$.

7. If the equations $x^2+bx+ca=0$ and $x^2+cx+ab=0$ have a common root, their other roots will satisfy the equation $x^2+ax+bc=0$.
[C. U. F. A. 1879]

[Let α = the common root of the two equations.

Then, $\alpha^2+b\alpha+ca=0$, ... (1)

and $\alpha^2+c\alpha+ab=0$.

Subtracting, $(b-c)\alpha+a(c-b)=0$

Dividing by $(b-c)$, $\alpha-a=0$, *i.e.*, $\alpha=a$.

Since, the product of the roots of the first equation $=ca$, and one of these roots $=a$.

$\therefore$ the other root of the 1st equation $=\dfrac{ca}{a}=c$.

Similarly, the remaining root of the 2nd equation $=\dfrac{ab}{a}=b$.

Hence, the required equation has the roots b and c, and is, therefore,

$$x^2-(b+c)x+bc=0. \qquad \cdots \qquad \cdots \quad (2)$$

Since, $\alpha=a$, we have from (1).

$$a^2+ba+ca=0, \quad i.e., \quad a(a+b+c)=0, \quad \text{or,} \quad a+b+c=0.$$

$\therefore$ from (2), we have $x^2+ax+bc=0$. ($\because$ $b+c=-a$.)]

8. If x be real, show that $\dfrac{m^2}{1+x}-\dfrac{n^2}{1-x}$ can have any real value.
[M. U. 1883]

[Let the given expression $=y$.

$$\therefore \frac{m^2(1-x)-n^2(1+x)}{1-x^2}=y,$$

or, $yx^2-(m^2+n^2)x-(y-m^2+n^2)=0$.

Solving, $$x=\frac{m^2+n^2\pm\sqrt{(m^2+n^2)^2+4y(y-m^2+n^2)}}{2y}$$

Since, x is real, the expression under the radical sign must be positive,

or, $(m^2+n^2)^2-4y(m^2-n^2)+4y^2$ is positive,

or, $(m^2-n^2)^2-4y(m^2-n^2)+4y^2+4m^2n^2$ is positive,

or, $(m^2-n^2-2y)^2+4m^2n^2$ must be positive.

This condition can evidently be satisfied by giving any real value to y, *i.e.*, to the expression.]

9. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be

$$\text{either,} \ \frac{pq'-p'q}{q-q'}, \quad \text{or,} \ \frac{q-q'}{p'-p}.$$

10. Form the equations whose roots are the reciprocals of the roots of (i) $3x^2+8x+91=0$; (ii) $ax^2+bx+c=0$.

11. If one root of the equation $ax^2+bx+c=0$, be the square of the other, prove that $b^3+a^2c+ac^2=3abc$.

12. If $ax^2+bx+c=a'x^2+b'x+c'$, when $x=183$, 281 and 397 respectively, prove that $a=a'$, $b=b'$ and $c=c'$.

[$\because$ $(ax^2+bx+c)-(a'x^2+b'x+c')=0$,

i.e., $(a-a')x^2+(b-b')x+(c-c')=0$ for three *distinct* values of x.

$\therefore$ by Art. 244, $a-a'=0$, $b-b'=0$ and $c-c'=0$.]

13. Find a, b, c, if $(a-12)x^2+(b-31)x=181-c$ for any value of x.

14. Find k, if the roots of $5x^2+7kx+3=0$ be the reciprocals of the roots of $3x^2+(8-k)x+5=0$.

15. Find a and k, if the roots of $3x^2+2kx+k+2=0$ be the reciprocals of the roots of $2ax^2+(k+a)x+3=0$.

CHAPTER XXXV

EQUATIONAL PROBLEMS

248. What are eggs a dozen when two more in a shilling's worth lowers the price one penny per dozen ?

Let $x=$ the number of eggs we get for a shilling.

Then the price of each egg $=\frac{12}{x}$ pence,

and $\therefore$ the price of a dozen $=\frac{144}{x}$ pence. ... (1)

If two more were obtained for a shilling, *i.e.*, if $(x+2)$ eggs were worth a shilling, the price of a dozen would, for a similar reason, be $\frac{144}{x+2}$ pence.

But by the condition of the problem, the latter price is one penny less than the former price, hence,

$$\frac{144}{x+2}=\frac{144}{x}-1;$$

$\therefore$ $x^2+2x=288$,

$\therefore$ $x^2+2x+1=289$;

$\therefore$ $x+1=17$.

$\therefore$ $x=16$.

Hence, from (1), the price per dozen $=9d$.

11. If one root of the equation $ax^2+bx+c=0$, be the square of the other, prove that $b^3+a^2c+ac^2=3abc$.

12. If $ax^2+bx+c=a'x^2+b'x+c'$, when $x=183$, 281 and 397 respectively, prove that $a=a'$, $b=b'$ and $c=c'$.

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$\therefore$ $x^2+2x+1=289$;

$\therefore$ $x+1=17$.

$\therefore$ $x=16$.

Hence, from (1), the price per dozen $=9d$.

249. Find two numbers, whose difference multiplied by the difference of their squares $=160$; and whose sum, multiplied by the sum of their squares gives the number 580.

Let $x+y$ and $x-y$ be the numbers.

Then, by the 1st condition of the problem,

$$2y.(4xy)=160,$$

$$\text{or,} \qquad xy^2=20. \qquad \cdots \qquad \cdots \quad (1)$$

By the 2nd condition of the problem,

$$2x\{2(x^2+y^2)\}=580,$$

$$\text{or,} \qquad x(x^2+y^2)=145. \qquad \cdots \qquad \cdots \quad (2)$$

From (1) and (2), by subtraction,

$$x^3=125=5^3;$$

$$\therefore \qquad x=5.$$

$$\text{Hence, from (1),} \qquad xy^2=5.y^2=20,$$

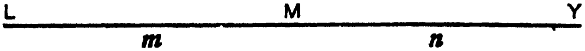
$$\textit{i.e.,} \qquad y^2=4;$$

$$\therefore \qquad y=2.$$

$$\therefore \qquad x=5, \text{ and } y=2.$$

Hence, the required numbers are 7 and 3.

250. *A* sets off from London to York and *B* at the same time from York to London, and they travel uniformly; *A* reaches York 16 hours and *B* reaches London 36 hours, after they have met on the road. Find in what time each has performed the journey.



Let *L* and *Y* represent London and York respectively, and *M* the place where the travellers meet. Let m, n be the measures of LM, MY respectively in kilometres.

Now, since *A* travels n kilometres (*i.e.*, from M to Y) in 16 hours he travels 1 kilometre in $\frac{16}{n}$ hours and $\therefore$ m kilometres in $\frac{16}{n} \cdot m$ hours;

hence, the time in which *A* travelled from L to $M=\frac{16}{n} \cdot m$ hours.

Similarly, the time in which *B* travelled from Y to $M=\frac{36}{n} \cdot m$ hours.

Now, since they started at the same instant, the time in which A travelled from L to M is evidently equal to the time in which B travelled from Y to M.

$$\therefore \quad \frac{16}{n}\cdot m = \frac{36}{m}\cdot n, \text{ whence } \frac{m}{n} = \frac{3}{2}.$$

Hence, the time in which A performed the journey

$$= \left(\frac{16}{n}\cdot m + 16\right) \text{ hours} = 40 \text{ hours};$$

and the time in which B performed the journey

$$= \left(\frac{36}{m}\cdot n + 36\right) \text{ hours} = 60 \text{ hours}.$$

251. A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article, thereby gaining 11 *per cent.* more on his outlay than he would gain, were the balance *true*. If, however, the scale-pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the transaction. Determine the legitimate gain per cent. on the article.

[In a *false* balance if any weight be placed on one of the scale-pans, the weight to be put on the other pan in order to make the beam horizontal will be *different*. For instance, if in buying rice a five-kilogram counterpoise be put on the pan, the quantity of rice put on the other will be either more or less than 5 kilograms. Suppose when the five-kilogram counterpoise is put on the scale-pan A, we are required to put on the pan B, a quantity of rice whose real weight is greater than 5 kilograms; but whatever may be its real weight, as its weight now is supposed to be equal to the weight of the counterpoise, we take it to be 5 kilograms. Thus, we take for 5 kilograms what is really more than 5 kilograms. Hence, if the merchant contrives to put the counterpoise on A and the article bought on B, he will evidently take away more of the article than he is supposed to do; let the supposed weight of the article, so bought, be w kilograms; if then W kilograms be the *real* weight of the article, w is less than W. Again, in selling the article if he puts the counterpoise on B and the article on A and if W' be the weight of the counterpoise, then W' is greater than W. By this contrivance then the merchant buys W kgs. of the article at the price of w kgs. and sells away these W kgs. again at the price of W' kgs. Hence, in such a transaction the merchant's gain is two-fold, he buys more of the article than he pays for and the whole quantity thus bought he sells away at the price of a still greater quantity.]

Let w and W' be the *apparent* weights of the article when bought and sold respectively.

Then, evidently w is less, and W' greater, than the true weight.

Let p = prime cost of unit of weight,
x = the legitimate gain per cent.

Then, the selling price of a unit of weight

$$=p+x \textit{ hundredths of } p=p\left(1+\frac{x}{100}\right).$$

Hence, the price paid by the merchant in buying the article, *i.e.* his outlay $=w.p$, and the price realised by selling it $=W.p.\left(1+\frac{x}{100}\right)$;

∴ by the condition of the problem,

$$W.p.\left(1+\frac{x}{100}\right)=w.p+(x+11) \textit{ hundredths of } w.p.$$

$$=w.p.\left(1+\frac{x+11}{100}\right). \quad \cdots \quad (1)$$

If the scale-pans were interchanged, the cost of buying the article would be $W'.p$ and the price realised by sale, $w.p.\left(1+\frac{x}{100}\right)$; hence by the 2nd condition of the problem,

$$w.p.\left(1+\frac{x}{100}\right)=W'.p. \quad \cdots \quad \cdots \quad (2)$$

From (1) and (2),

$$\frac{1+\frac{x+11}{100}}{1+\frac{x}{100}}=1+\frac{x}{100},$$

or, $$\Big(\text{[illegible]}\,\frac{x}{100}\Big)^2\ \text{[illegible]}+\frac{x+11}{100},$$

or, $$\left(\frac{x}{100}\right)^2+\frac{x}{100}+\frac{1}{4}=\frac{11}{100}+\frac{1}{4}=\frac{36}{100},$$

or, $$\left(\frac{x}{100}+\frac{1}{2}\right)^2=\left(\frac{6}{10}\right)^2;$$

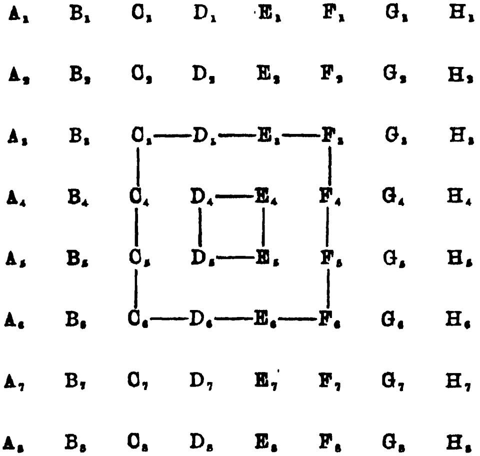
$$\therefore \quad \frac{x}{100}=\frac{6}{10}-\frac{1}{2}=\frac{1}{10};$$

$$\therefore \quad x=10,$$

i.e., the legitimate gain is 10 *per cent.*

252. A body of men were formed into a hollow square, three deep, when it was observed that with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.

[A number of men are said to be arranged in a *solid* square when they are arranged in parallel rows and the number of rows is equal to the number of men in each row. The following diagram, in which A_1, B_1, C_1, &c. represent men, will clearly illustrate the matter.



A_1	B_1	C_1	D_1	E_1	F_1	G_1	H_1
A_2	B_2	C_2	D_2	E_2	F_2	G_2	H_2
A_3	B_3	C_3	D_3	E_3	F_3	G_3	H_3
A_4	B_4	C_4	D_4	E_4	F_4	G_4	H_4
A_5	B_5	C_5	D_5	E_5	F_5	G_5	H_5
A_6	B_6	C_6	D_6	E_6	F_6	G_6	H_6
A_7	B_7	C_7	D_7	E_7	F_7	G_7	H_7
A_8	B_8	C_8	D_8	E_8	F_8	G_8	H_8

The above diagram represents an arrangement in which there are 8 rows, each containing 8 men. This is a *solid* square. If the square $C_3F_3F_6C_6$ be removed from inside, the remainder will be a *hollow* square *two deep* having 8 men in each side ; if, however, the square $D_4E_4E_5D_5$ be removed, the remainder will be a *hollow* square *three deep*.

Hence, the number of men in a *hollow* square *two deep* having x men in each side $= x^2-(x-4)^2$; in one *three deep* $= x^2-(x-6)^2$; and so on ; thus, the number of men in a hollow square n deep having x men in each side $= x^2-(x-2n)^2$.]

Let $x =$ the number of men in a side of the hollow square ; then the whole number of men $= x^2-(x-6)^2$ ... ... (1)

Hence, by the 2nd condition of the problem,

$$x^2-(x-6)^2+25=(x^{\frac{1}{2}}+22)^2,$$

or, $$12x-11=x+44x^{\frac{1}{2}}+484.$$

$\therefore$ $$11x-44x^{\frac{1}{2}}=495,$$

or, $$x-4x^{\frac{1}{2}}=45.$$

$\therefore$ $$x-4x^{\frac{1}{2}}+4=49.$$

$\therefore \quad x^{\frac{1}{2}}-2=7$; whence $x=81$.

Hence, from (1), the whole number of men

$$=81^2-75^2=156\times 6=936.$$

253. K engages to play a game of chess with B on the following condition that B should name a certain number and put into K's possession twenty-four rupees together with as many rupees as equal to the square of this number and that at the conclusion of the game K should return to B only a number of rupees equal to eight times the number named. What number could B name with the greatest advantage possible to himself ?

Let $x=$the number which B should name ; then he has to deposit with K, $(24+x^2)$ rupees and get back at the end of the game only $8x$ rupees ;

hence, B has altogether to lose $(x^2+24-8x)$ rupees,

$\therefore$ x must be such that this loss may be as small as possible.

Now, since $x^2-8x+24=(x-4)^2+8$, which is always greater than 8 except when $x=4$, the loss will for all values of x be greater than Rs. 8 except when x has this value.

Hence, in order that the loss may be a minimum B should name the number 4.

254. With the object of examining a student of the 1st year as regards his progress in Algebra, I undertake to engage in a certain contract with him, which is as follows : he is to give me a certain number of books, each worth as many rupees as the number of books, and to get from me in return six times as many rupees as any of those books is worth and also 21 rupees more. How many books should he bring me, with the greatest possible advantage to himself ?

Let $x=$the number of books that the student brings me ; then, since the price of each book is x rupees, evidently I get x^2 rupees from him ; and in return I give him $(6x+21)$ rupees.

Hence, his gain (or loss as the case may be)$=(21+6x-x^2)$ rupees.

Now, $21+6x-x^2=21-(x^2-6x)=30-(x^2-6x+9)=30-(x-3)^2$.

Evidently, therefore, the student is a loser if $x-3$ be greater than 5, *i.e.*, if x be greater than 8 ; and he is a gainer if x be 8 or less than 8.

But not only should the student be a gainer but his gain must be the greatest possible, which evidently is the case when $(x-3)^2$ is the least possible, *i.e.*, when $x=3$.

Hence, the student should bring me only three books.

255. Rama, Lakshmana and Bharata went to visit a Rishi and brought their wives with them. The Rishi knew the wives' names to be Urmila, Mandavi and Sita, but forgot which was the wife of each hero. They told the Rishi that they had given presents to Pandits, and that each of the six had rewarded as many Pandits, as he or she had given gold mudras to each Pandit. Rama had rewarded 23 more Pandits than Urmila, and Lakshmana had rewarded 11 Pandits more than Mandavi, likewise each hero had given away 63 gold mudras more than his wife. The Rishi having thought on what they said, dismissed them with his blessing, naming correctly the wife of each hero. From the conditions given, do you also find out the names of the wives ?

Let x = the number of Pandits rewarded by any hero,

and y = the number of Pandits rewarded by his wife ;

then the number of gold mudras given away by the hero $=x^2$;

and the number of gold mudras given away by his wife $=y^2$.

Hence, by the last condition of the problem, we have

$$x^2-y^2=63, \quad \text{or,} \quad (x+y)(x-y)=63.$$

But $\quad 63=63\times 1$, or, 21×3, or, 9×7 ;

hence, since $x+y$ and $x-y$ are positive integers, and $x+y$ is necessarily greater than $x-y$, we get the following three pairs of values for $x+y$, and $x-y$ and *no other*.

$$(1)\ \left.\begin{aligned}x+y&=63\\x-y&=1\end{aligned}\right\},\qquad (2)\ \left.\begin{aligned}x+y&=21\\x-y&=3\end{aligned}\right\},\qquad (3)\ \left.\begin{aligned}x+y&=9\\x-y&=7\end{aligned}\right\}.$$

Hence, we have the following three pairs of values for x and y :

$$(1)\ \left.\begin{aligned}x&=32\\y&=31\end{aligned}\right\},\qquad (2)\ \left.\begin{aligned}x&=12\\y&=9\end{aligned}\right\},\qquad (3)\ \left.\begin{aligned}x&=8\\y&=1\end{aligned}\right\}\ \cdots\quad (A)$$

i.e., the wife of the hero who rewarded 32 Pandits, rewarded 31 Pandits ;

the wife of the hero who rewarded 12 Pandits, rewarded 9 Pandits ; ... ... (*a*)

and the wife of the hero who rewarded 8 Pandits, rewarded only one Pandit. ... ... (β)

Now, let us find out the names of the wives from the other conditions of the problem.

The number of Pandits rewarded by Rama may be 32, 12 or 8 ; but since he is known to have rewarded 23 *more* Pandits than somebody else, the number of Pandits rewarded by him *must be* 32.

The number of Pandits rewarded by Lakshmana may then be either 12 or 8, but as he is known to have rewarded 11 *more* Pandits than somebody else, the number of Pandits rewarded by him *must be* 12. ... (*a*)

Hence, the number of Pandits rewarded by Bharata *must be* 8. ... (*b*

Again, since the number of Pandits rewarded by Urmila is 23 less than the number rewarded by Rama, it *must be* 9 ; hence, by (*a*) and (*a*), Urmila is the wife of Lakshmana ;

also, since the number of Pandits rewarded by Mandavi is 11 less than the number rewarded by Lakshmana, it *must be* 1 ; and, therefore, by (β) and (*b*), Mandavi is the wife of Bharata ; evidently therefore, Sita is the wife of Rama.

Thus, we have

$$\left.\begin{matrix}\text{Rama}\\ \text{Sita}\end{matrix}\right\}, \qquad \left.\begin{matrix}\text{Lakshmana}\\ \text{Urmila}\end{matrix}\right\}, \qquad \left.\begin{matrix}\text{Bharata}\\ \text{Mandavi}\end{matrix}\right\}.$$

EXERCISE 135

1. A person bought a certain number of oxen for £80 ; if he had bought 4 more for the same sum, each ox would have cost £1 less ; find the number of oxen and the price of each.

2. A gentleman sends a lad into the market to buy a shilling's worth of oranges. The lad having eaten a couple, the gentleman pays at the rate of a penny for fifteen more than the market price. How many did the gentleman get for his shilling ?

3. The plate of a looking glass is 18 centimetres by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame.

4. *A* and *B* lay out some money on speculation. *A* disposes of his bargain for £11, and gains as much *per cent.* as *B* lays out ; *B*'s gain is £36, and it appears that *A* gains four times as much *per cent.* as *B*. Required the capital of each.

5. A boat's crew row $3\frac{1}{2}$ kilometres down a river and back again in 1 hour and 40 minutes. Supposing the river to have a current of 2 kilometres per hour, find the rate at which the crew would row in still water.

6. What two numbers are those whose sum multiplied by the greater is 204 ; and whose difference multiplied by the less is 35 ?

7. What two numbers are those whose sum added to the sum of their squares is 42 and whose product is 15 ?

8. *A* and *B* distribute £60 each among a certain number of persons. A relieves 40 persons more than *B* does, and *B* gives to each 5*s.* more than *A*. How many persons did *A* and *B* respectively relieve ?

9. The product of two numbers added to their sum is 23 ; and five times their sum taken from the sum of their squares leaves 8 ; required the numbers.

10. A horse dealer buys a horse, and pays a certain sum for it ; he afterwards sells it again for Rs. 171, and gains exactly as much *per cent.* as the horse had cost him. How much did he pay for the horse ?

11. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 260 metres ; if the circumference of each

were one-third metre more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 metres. Find the circumference of each wheel.

12. By lowering the price of apples and selling them one penny a dozen cheaper, an apple-woman finds that she can sell 60 more than she used to do for 5s. At what price per dozen did she sell them at first ?

13. There is a number between 10 and 100 ; when multiplied by the digit on the left the product is 280, if the sum of the digits be multiplied by the same digit the product is 55 ; required the number.

14. A and B are two stations 300 kilometres apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B nine hours, the train from B reaches A four hours, after they meet. Find the rate at which each train travels.

15. By selling a horse for £24, I lose as much *per cent.* as it costs me. What was the prime cost of it ?

16. Find three numbers, such that if the first be multiplied by the sum of the second and the third, the second by the sum of the first and the third and the third by the sum of the first and the second, the products shall be 408, 480 and 504 respectively.

17. There are two square buildings that are paved with stones, a metre square each. The side of one building exceeds that of the other by 12 metres, and both their pavements taken together contain 2120 stones. What are the lengths of them separately ?

18. There are three numbers, the difference of whose differences is 5 ; their sum is 44, and continued product 1950 ; find the numbers.

19. A train A starts to go from P to Q, two stations 240 kilometres apart, and travels uniformly, An hour later, another train B starts from P, and after travelling for 2 hours, comes to a point that A had passed 45 minutes previously. The pace of B is now increased by 5 kilometres an hour, and it overtakes A just on entering Q. Find the rates at which they started.

20. A square court-yard has a rectangular gravel walk round it inside. The side of the court wants 2 metres of being 6 times the breadth of the gravel walk ; and the number of square metres in the walk exceeds the number of metres in the periphery of the court by 92. Required the area of the court.

21. Divide the number 26 into three such parts that their squares may have equal differences, and that the sum of those squares may be 300.

22. The number of soldiers present at a review is such that they could all be formed into a solid square and also could be formed into four hollow squares each 4 deep and each containing 24 more men in the front rank than when formed into a solid square ; find the whole number.

23. A and B run a race round a two-kilometre course. In the first hit B reaches the winning post 2 minutes before A. In the second hit A increases his speed 2 kilometres an hour, and B diminishes his by the same quantity; and A then reaches the winning post 2 minutes before B. Find at what rate each ran in the first hit.

24. From a vessel of wine containing a gallons, b gallons are drawn off and the vessel is filled up with water. Find the quantity of wine remaining in the vessel when this has been repeated 4 times.

25. A wall was built round a rectangular court to a certain height. Now the length of one side of the court was two metres less, whilst three times the length of the other was 25 metres greater than 8 times the height of the wall; and the number of square metres in the court was greater than the number in the wall by 178. Required the dimensions of the court, and the height of the wall.

26. A person bought a number of £20 railway shares when they were at a certain rate *per cent.* discount for £1,500; and afterwards when they were at the same rate *per cent.* premium sold them all but 60 for £1,000. How many did he buy and what did he give for each of them?

27. The sum of 4 numbers is 44; the sum of the product of the 1st and 2nd, and 3rd and 4th is 250; of the 1st and 3rd, and 2nd and 4th is 234; and of the 1st and 4th, and 2nd and 3rd is 225. Find them.

28 To complete a certain work A requires m times as long a time as B and C together; B requires n times as long as A and C together, and C requires p times as long as A and B together. Compare the times in which each would do it and prove that,

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

29. In a certain village there lived in the year 1872 a number of families each consisting of as many members as there were families. Ten years afterwards it was found that during this interval there were 670 births in the village and that on the average 50 lives were lost per family. Prove that the number of persons, living in the village at the time of this calculation, could not be less than 45, and if this number be actually 45, find out the number of souls that lived in the village in the year 1872.

30. Suppose you agree to give me out of your landed property a square plot of ground and receive in exchange a circular plot of land whose area is 76 square metres and also a rectangular plot, one of whose sides is 36 metres and the other is equal to a side of the piece of land you give me. What must be the area of the plot you give me, so that you can profit most by the exchange.

CHAPTER XXXVI

GRAPHS OF QUADRATIC EQUATIONS AND EXPRESSIONS AND THEIR APPLICATIONS

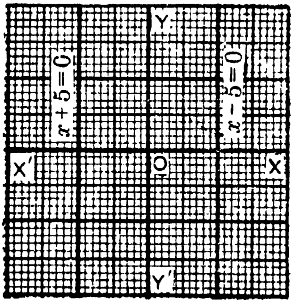
256. The graphs of $XY=0$, X and Y being expressions of the first degree in x and y.

Example 1. Draw the graph of the equation $x^2=25$.

The equation $x^2=25$ may be written as

$$\left.\begin{aligned} x^2-25&=0 \\ \text{or,}\quad (x-5)(x+5)&=0 \end{aligned}\right\}$$

Evidently, the given equation is satisfied (i) by all those points which satisfy the equation $x-5=0$; (ii) by all those points which satisfy the equation $x+5=0$.



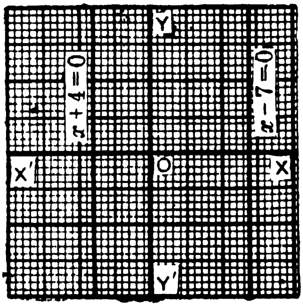
Hence, **the required graph consists of two straight lines,** one being the graph of the equation $x-5=0$ and the other being the graph of the equation $x+5=0$, as shown in the diagram with twice the length of a side of a small square as unit of length.

Example 2. Draw the graph of the equation $x^2-3x-28=0$.

Factorising the left-hand side of the equation, we have

$$(x-7)(x+4)=0.$$

Hence, proceeding as in example 1, we notice that **the required graph consists of two straight lines,** one being the graph of the equation $x-7=0$ and the other being the graph of $x+4=0$, as shown in the diagram with twice the length of a side of small a square as unit of length.



Example 3. Draw the graph of the equation $y^2=4x^2$.

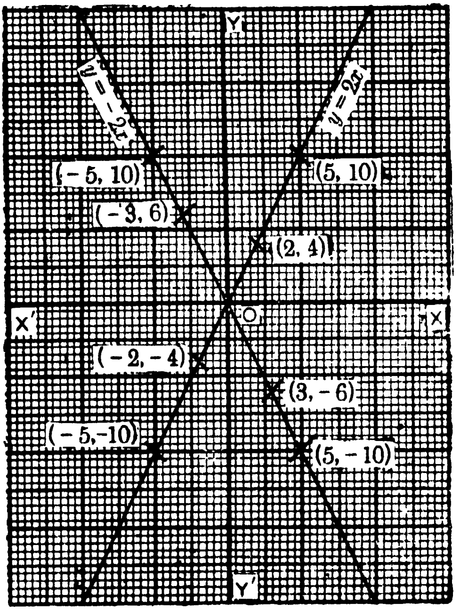
From the given equation, we have

$$\left.\begin{aligned} y^2-4x^2&=0\\ \text{or,}\quad (y+2x)(y-2x)&=0 \end{aligned}\right\}$$

Clearly, the given equation is satisfied by (i) all those points which satisfy the equation $y+2x=0$, and also (ii) by all those points which satisfy the equation $y-2x=0$.

Hence, **the required graph consists of two straight lines,** one being the graph of the equation $y+2x=0$, and the other being the graph of the equation $y-2x=0$.

Hence, the required graph is as shown below with twice the length of a side of a small square as unit of length.



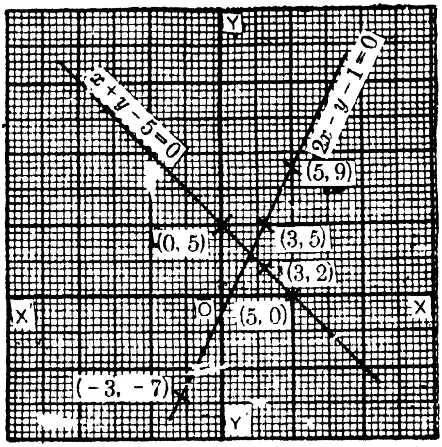
Example 4. Draw the graph of the equation $2x^2+xy-y^2-11x+4y+5=0$.

Factorising the left-hand side of the given equation, we have

$$(x+y-5)(2x-y-1)=0.$$

Obviously, the given equation is satisfied (i) by all **those points which** satisfy the equation $x+y-5=0$ as well as (ii) by all **those points which** satisfy the equation $2x-y-1=0$.

Hence, the **required graph consists of two straight lines, one** being the graph of the equation $x+y-5=0$ and the other **being the** graph of the equation $2x-y-1=0$, as shown in the diagram **with twice** the length of a side of a small square as unit of length.



257. Thus, it is clear from the above examples **that whenever a quadratic** equation can be expressed in the form $XY=0$, **where X and Y are expressions** of the **first** degree in x and y, **the graph consists of a pair** of straight lines, which are respectively the **graphs of the equations** $X=0$ and $Y=0$.

When, however, a quadratic equation cannot be **expressed in the** form $XY=0$, its graph is a **curve**. We shall now **proceed to consider** a few graphs of this nature.

258. The graph of a quadratic equation in which the **coefficients of x^2 and y^2** are equal and positive and there is no term **involving the product of x** and y, is a **Circle**. The equation of this type of **graph is** generally of the following forms :

(i) $x^2+y^2=a^2$, (ii) $(x-h)^2+(y-k)^2=a^2$,

(iii) $x^2+y^2+ax+by+c=0$.

Draw the graph of the equation $x^2+y^2=a^2$.

The process of drawing the graph is being explained, with 6 for the value of a.

$$x^2+y^2=36, \quad \text{or,} \quad y^2=36-x^2.$$

It is evident that for every value of x, there will be two equal and opposite values of y.

(i) If $x=0$, $y^2=36$,

$\therefore$ $y=\pm\sqrt{36}=\pm 6$. So the points (0, 6), (0, −6) will be on the required graph.

(ii) If $x=\pm 6$, $y=0$,

$\therefore$ the points (6, 0) and (−6, 0) will be on the required graph.

(iii) If $x=\pm 2$, $y^2=32$,

$\therefore$ $y=\pm 4\sqrt{2}=\pm 4\times 1·414\cdots=\pm 5·656\cdots=\pm 5·7$.

Hence the points (2, 5·7), (2, −5·7), (−2, 5·7), (−2, −5·7) are on the required graph.

(iv) If $x=\pm 3$, $y^2=27$,

$\therefore$ $y=\pm 3\sqrt{3}=\pm 3\times 1·732\cdots=\pm 5·196\cdots=\pm 5·2$.

Hence the points (3, 5·2), (3, −5·2), (−3, 5·2) and (−3, −5·2) are on the required graph.

(v) If $x=\pm 4$, $y^2=20$,

$\therefore$ $y=\pm 2\sqrt{5}=\pm 2\times 2·236\cdots=\pm 4·472\cdots=\pm 4·5$.

Hence the points (4, 4·5), (4, −4·5), (−4, 4·5) and (−4, −4·5) are on the required graph.

The corresponding values of x and y may be tabulated as follows :

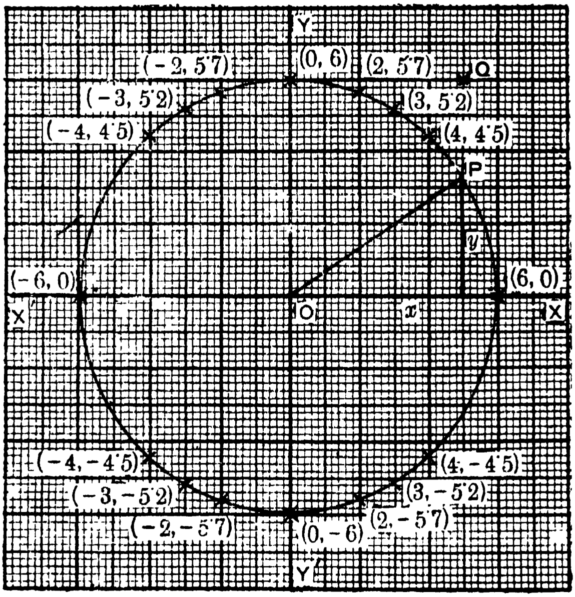
x	0	0	6	−6	2	2	−2	−2	3	3	−3	−3	4	4	−4	−4
y	6	−6	0	0	5·7	−5·7	5·7	−5·7	5·2	−5·2	5·2	−5·2	4·5	−4·5	4·5	−4·5

Let five times the length of a side of a small square represent the unit of length.

Plotting the points tabulated above and drawing a free-hand and continuous curve, we obtain the required graph as shown in the diagram at page 510.

Note. *In the diagram on page 510, if a circle is drawn with the origin O as centre and radius equal to 6 units of length with the help of a pencil compass, it will be found that the circle almost coincides with the free-hand curve. Thus the accuracy of the free-hand drawing can be verified by drawing a circle.*

Take *any* point P on the circle, and let its co-ordinates be denoted by x and y; evidently then $x^2+y^2=OP^2=36$. But if a point, such as Q, be taken anywhere *not on the circle*, it is easy to see that its co-ordinates will *not* satisfy the given equation.



Thus, it is shown that the co-ordinates of every point on the circle, and of no other point satisfy the given equation. Hence, the circle drawn is the required graph.

259. Draw the graph of the equation $(x-h)^2+(y-k)^2=a^2$.

With different values of h, k and a the process of drawing the graph of the equation $(x-h)^2+(y-k)^2=a^2$ is being explained. When $h=3$, $k=2$ and $a=5$, the equation becomes $(x-3)^2+(y-2)^2=25$.

$$(x-3)^2+(y-2)^2=25,$$

or, $$(y-2)^2=25-(x-3)^2\,;$$

$\therefore$ $$y-2=\pm\sqrt{25-(x-3)^2}\,;$$

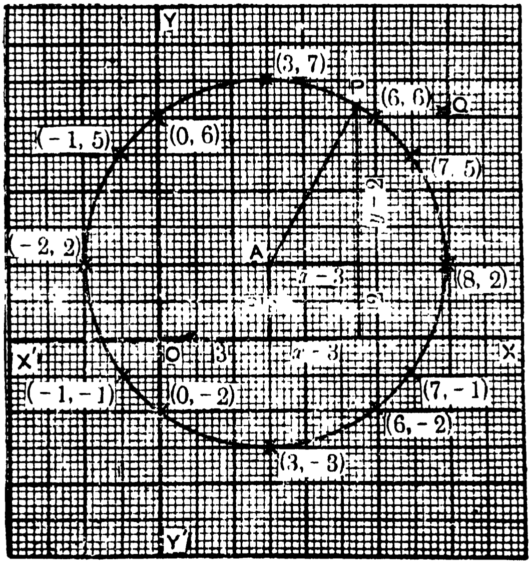
$\therefore$ $$y=2\pm\sqrt{25-(x-3)^2}.$$

The corresponding values of x and y in the equation $y=2\pm\sqrt{25-(x-3)^2}$ may be tabulated as follows :

x	0	0	3	3	−1	1	−2	6	6	7	7	8
y	6	−2	7	−3	5	−1	2	6	−2	5	−1	2

Taking five times the length of a side of a small square as the unit of length, let us plot the points tabulated above on squared paper. The continuous curve line joining the points is the required graph.

Let A be the point (3, 2). With centre A and a radius equal to 5 units of length describe a circle as in the diagram below. Then this circle and the curve passing through the points will be the same. Thus the accuracy of the free-hand drawing can be verified.



Take *any* point P on the circle, and let its co-ordinates be denoted by x and y. Now from the diagram, it is clear that AP is the hypotenuse of a right-angled triangle of which the sides are $(x-3)$ and $(y-2)$ units of length respectively.

Hence, $(x-3)^2+(y-2)^2=AP^2=25$, which shows that the co-ordinates of P satisfy the given equation. But if a point, such as Q, be taken anywhere *not on the circle*, it is easy to see that its co-ordinates will *not* satisfy the given equation.

Thus, it is clear that the co-ordinates of every point on the circle and of no other point, satisfy the given equation. Hence, the circle described is the required graph.

Note 1. ***The graph of*** $(x+2)^2+(y+5)^2=49$. *Draw the graph after plotting the points as shown at page 511 and verify its accuracy by drawing a circle of which the centre is the point* $(-2, -5)$, *and the radius is equal to 7 units of length.*

Note 2. ***The graph of*** $x^2+y^2-8x+10y+25=0$. *The equation* $x^2+y^2-8x+10y+25=0$ *can be easily reduced to the form* $(x-4)^2+(y+5)^2=16$. *Hence, its graph is a circle of which the centre is the point* $(4, -5)$ *and the radius is equal to 4 units of length.*

Example 1. Solve graphically $x^2-6x-12=0$.

The equation may be written in the form

$$(x^2-6x+9)+4=25, \text{ i.e., } (x-3)^2+2^2=25.$$

$\therefore$ the roots of the given equation are the abscissæ of the points where the line $y=0$ (*i.e.*, the x-axis) cuts the graph of the equation $(x-3)^2+(y-2)^2=25$ [for, putting $y=0$ in the equation of the circle, we have $(x-3)^2+(y-2)^2=25$, *i.e.*, $(x-3)^2+4=25$].

Hence, drawing the graph of the equation $(x-3)^2+(y-2)^2=25$ as in Art. 259, we notice from the diagram that these abscissæ are 7·6 and −1·6 approximately.

$\therefore$ the required roots are 7·6 and −1·6 approximately.

Example 2. Trace the graphs of (i) $x^2+y^2=169$ and (ii) $x+y=17$. Find the co-ordinates of their points of intersection.

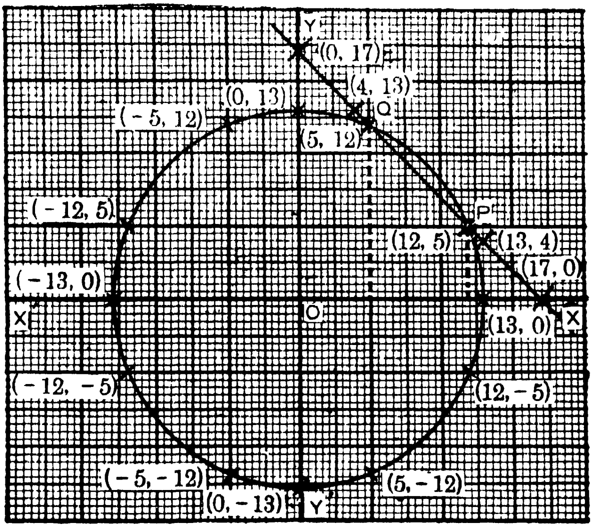
$$x^2+y^2=169, \text{ or, } y^2=169-x^2\ ;\ \therefore\ y=\pm\sqrt{169-x^2}.$$

The corresponding values of x and y in the equation $y=\pm\sqrt{169-x^2}$ may be tabulated as follows :

x	0	0	−13	13	5	5	−5	−5	12	12	−12	−12
y	13	−13	0	0	12	−12	12	−12	−5	5	5	−5

Similarly the corresponding values of x and y in the equation $x+y=17$ may be tabulated as follows :

x	17	0	13	4
y	0	17	4	13



Taking twice the length of a side of a small square as the unit of length and drawing the graphs of the two equations we shall find that they will intersect at $P(12, 5)$ and $Q(5, 12)$ as in the above diagram.

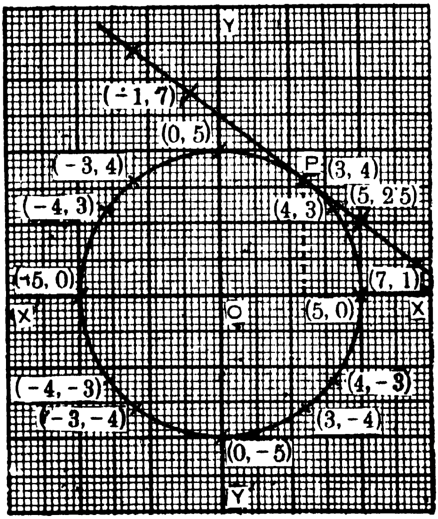
Note. ***To solve graphically the equations***

$$\left.\begin{aligned} x^2+y^2&=169 \\ x+y&=17 \end{aligned}\right\}$$

we notice that the co-ordinates of each of the above points P and Q satisfy both the equations and are, therefore, the required solutions.

Thus, *the roots are* $\left.\begin{aligned} x&=12 \\ y&=5 \end{aligned}\right\}$ and $\left.\begin{aligned} x&=5 \\ y&=12 \end{aligned}\right\}$

Example 3. Show that the graph of $3x+4y=25$ touches that of $x^2+y^2=25$, and find the co-ordinates of the point of contact. [C. U. 1911]



$$x^2+y^2=25,\ \text{or},\ y^2=25-x^2\ ;\ \therefore\ y=\pm\sqrt{25-x^2}.$$

The corresponding values of x and y in the equation $y=\pm\sqrt{25-x^2}$ may be tabulated as follows :

x	0	0	5	−5	3	3	−3	−3	4	4	−4	−4
y	5	−5	0	0	4	−4	4	−4	3	−3	3	−3

$3x+4y=25$,

or, $y=\dfrac{25-3x}{4}$; the corresponding values of x and y of the equation may be tabulated as follows :

x	7	5	−1	−3
y	1	2·5	7	8·5

Taking four times the side of a small square as the unit of length and drawing the graphs, we find that they touch at $P(3, 4)$ as in the diagram at page 514.

EXERCISE 136

Draw the graphs of the following equations :

1. $x^2+y^2=81.$
2. $(x-5)^2+(y-6)^2=49.$
3. $(x+6)^2+(y-7)^2=100.$
4. $x^2+y^2-8x-14y+1=0.$
5. $x^2+y^2+14x-16y+32=0.$
6. $x^2+y^2+12x+18y+92=0.$
7. $x^2+y^2-10x+16y-55=0.$

Solve graphically :

8. $\left.\begin{aligned} x^2+y^2 &= 100 \\ x+y &= 14 \end{aligned}\right\}.$

9. $\left.\begin{aligned} x^2+y^2 &= 25 \\ x-y &= 1 \end{aligned}\right\}.$

10. $\left.\begin{aligned} x^2+y^2-4x-6y-12 &= 0 \\ x+y &= 12 \end{aligned}\right\}.$

11. $x^2-4x-12=0.$

[The roots are the abscissæ of the points where the x-axis cuts $x^2+y^2-4x-6y-12=0$, etc.]

12. $x^2-6x-16=0.$

13. Draw the graphs of $x^2+y^2=36$ and $3x-4y=30$. Show that they touch at $(3{\cdot}6, -4{\cdot}8)$.

14. Draw the graph of $x^2+y^2-4x-6y-23=0$ and find its tangents parallel to the co-ordinate axes.

15. Draw the graph of $x^2+y^2-10x-10y+25=0$ and show that it touches the co-ordinate axes. Find the co-ordinates of the points of contact.

16. Draw the graphs of the following equations :

(1) $x^2=16$;
(2) $x^2-5x+6=0$;
(3) $5x^2-3x-2=0$;
(4) $y^2-3y=0$;
(5) $xy=0$;
(6) $x^2-3xy+2y^2=0$;
(7) $x^2-y^2+4y-4=0$;
(8) $(x+3)^2=4(y-5)^2$.

17. Draw the graph of $5x^2-24xy-5y^2=0$ and show that they are two perpendicular straight lines.

18. Find the angle between the straight lines which represent the graphs of :

(i) $xy=0$;
(ii) $(x-3)(y-2)=0$;
(iii) $(3x-2y+5)(2x+3y+2)=0$;
(iv) $(7x-6y+3)(6x+7y+8)=0$.

260. The graph of a quadratic equation in which the coefficients of x^2 and y^2 are positive and unequal is a curve called an Ellipse. The equation is generally of the form $a^2x^2+b^2y^2=c^2$.

Draw the graph of the equation $4x^2+9y^2=36$.

(1) When $x=0$, we have $y^2=4$, and, therefore, $y=\pm 2$. Hence, the points (0, 2) and (0, −2) are on the required graph.

(2) When $y=0$, we have $x^2=9$, and, therefore, $x=\pm 3$. Hence, the points (3, 0) and (−3, 0) are on the required graph.

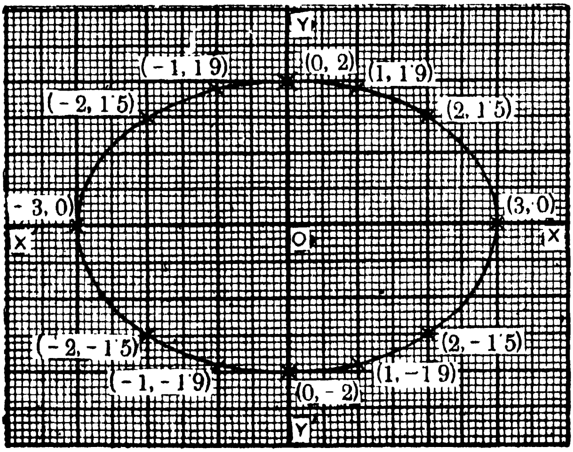
(3) When $x=\pm 1$, we have $9y^2=32$, and, therefore, $y=\pm \frac{4}{3}\sqrt{2}$ $=\pm\frac{4\times 1{\cdot}414\ldots}{3}=\pm\frac{5{\cdot}656\ldots}{3}=\pm 1{\cdot}885\cdots=\pm 1{\cdot}9$ approximately. Hence, the four points (1, 1·9), (1, −1·9), (−1, 1·9) and (−1, −1·9) are on the required graph.

(4) When $x=\pm 2$, we have $9y^2=20$, and, therefore, $y=\pm\frac{2}{3}\sqrt{5}$ $=\pm\frac{2\times 2{\cdot}236\ldots}{3}=\pm\frac{4{\cdot}472\ldots}{3}=\pm 1{\cdot}490\cdots\cdots=\pm 1{\cdot}5$ nearly.

Hence, the four points (2, 1·5), (2, −1·5), (−2, 1·5), and (−2, −1·5) are on the required graph.

Corresponding values of x and y may be tabulated as follows :

x	0	0	3	−3	1	1	−1	−1	2	2	−2	−2
y	2	−2	0	0	1·9	−1·9	1·9	−1·9	1·5	−1·5	1·5	−1·5



Let us now plot the twelve points as found above (taking 10 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram at page 516.

The curve so drawn is the required graph.

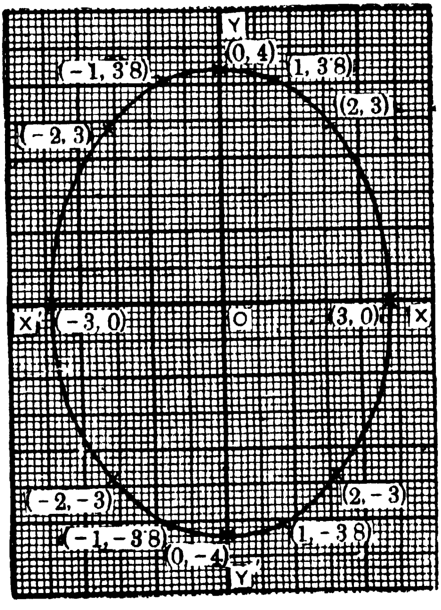
Note 1. *Evidently the curve is symmetrical about the axis of x, i.e., every chord at right angles to the axis of x is bisected by it. Similarly, the curve is also symmetrical about the axis of y.*

Note 2. *The curve lies entirely within the space enclosed by the four straight lines $x=3$, $x=-3$, $y=2$, $y=-2$, since from the given equation it is obvious that x is imaginary when, $y>2$ and <-2 and y is imaginary when, $x>3$ and <-3.*

Example 1. Draw the graph of the expression $\frac{4}{3}\sqrt{9-x^2}$.

Let $y=\frac{4}{3}\sqrt{9-x^2}$.

For each value of x, there will be two equal and opposite values of y. Thus, (1) when $x=0$, $y=\pm 4$; (2) when $x=\pm 3$, $y=0$; (3) when $x=\pm 1$, $y=\pm\frac{4}{3}\sqrt{8}=\pm 3{\cdot}8$ approximately; (4) when $x=\pm 2$, $y=\pm\frac{4}{3}\sqrt{5}=\pm 3{\cdot}0$ approximately.



The corresponding values of x and y may be arranged in a tabular form as follows :

x	0	0	3	−3	1	1	−1	−1	2	2	−2	−2
y	4	−4	0	0	3·8	−3·8	3·8	−3·8	3	−3	3	−3

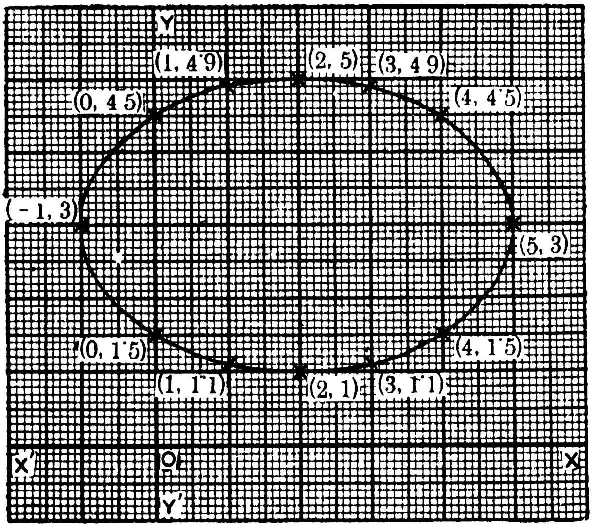
Plotting these twelve points (taking 8 times the side of a small square as the unit of length) and drawing a free-hand curve through them as shown in the diagram on the last page, we obtain the required graph.

Example 2. Draw the graph of $4(x-2)^2+9(y-3)^2=36$.

Re-writing the equation, we have

$$9(y-3)^2=36-4(x-2)^2,$$

or, $$y-3=\pm\tfrac{2}{3}\sqrt{9-(x-2)^2}.$$



Hence, for each value of $x-2$, we get two values of $y-3$ from which the corresponding values of x and y may be tabulated as follows :

x	2	2	5	−1	1	1	3	3	4	4	0	0
y	5	1	3	3	4·9	1·1	4·9	1·1	4·5	1·5	4·5	1·5

Plotting these twelve points (taking 10 times the side of a small square as the unit of length) and drawing a free-hand curve through them as shown in the diagram at page 518, we get the required graph.

Example 3. Draw the graph of $4x^2+9y^2-16x-54y+61=0$.

The left-hand side of the given equation

$$=4(x^2-4x)+9(y^2-6y)+61$$
$$=4\{(x-2)^2-4\}+9\{(y-3)^2-9\}+61$$
$$=4(x-2)^2+9(y-3)^2-36.$$

$\therefore$ the equation is $4(x-2)^2+9(y-3)^2-36=0$,

or, $4(x-2)^2+9(y-3)^2=36$.

To draw its graph see example 2 on page 518.

261. Draw the graph of the equation $x^2-y^2=1$.

(1) When $x=0$, we have $y^2=-1$, and, therefore, y is *imaginary*. This shows that the graph does not cut the axis of y.

(2) When $y=0$, we have $x^2=1$, and, therefore, $x=\pm 1$. Hence, the points (1, 0) and (−1, 0) are on the required graph.

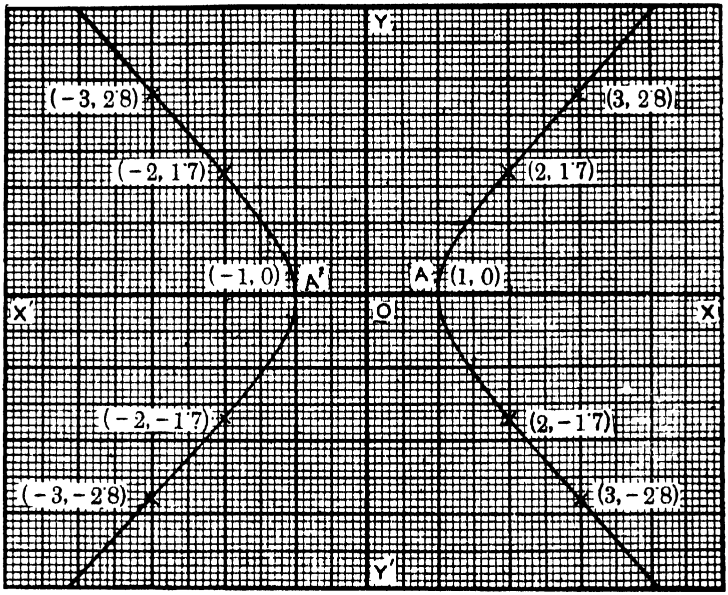
(3) When $x=\pm 2$, we have $y^2=3$, and, therefore, $y=\pm\sqrt{3}$ $=\pm 1·732\cdots\cdots=\pm 1·7$ approximately. Hence, the four points (2, 1·7), (2, −1·7), (−2, 1·7) and (−2, −1·7) are on the required graph.

(4) When $x=\pm 3$, we have $y^2=8$, and, therefore, $y=\pm 2\sqrt{2}$ $=\pm 2\times 1·414\cdots\cdots=\pm 2·828\cdots\cdots=\pm 2·8$ approximately. Hence, the four points (3, 2·8), (3, −2·8), (−3, 2·8) and (−3, −2·8) are on the required graph.

The corresponding values of x and y may be tabulated as follows :

x	1	−1	2	2	−2	−2	3	3	−3	−3
y	0	0	1·7	−1·7	1·7	−1·7	2·8	−2·8	2·8	−2·8

Let us now plot the ten points as found above (taking 10 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram below. A and A' are the points of intersection of the right and left branches of the hyperbola respectively, with the x-axis.



The curve so drawn is the required graph.

Note 1. *The curve so drawn is evidently symmetrical about the axis of x and also about the axis of y.*

Note 2. *The curve consists of two branches, one lying entirely on the right of the line $x=1$ and the other lying entirely on the left of the line $x=-1$.*

A curve of this class is called a ***Hyperbola.***

Example 1. Trace the graph of (i) $x^2-y^2=1$, and (ii) $x^2+y^2=1$. Show that they touch each other.

Draw the graph of $x^2-y^2=1$ as above and the graph of the circle $x^2+y^2=1$ on the *same scale.* It will be found that they touch each other at the points $(1, 0)$ and $(-1, 0)$.

Example 2. Trace the graph of (i) $x^2-y^2=1$ and (ii) $x=2y$. Find the co-ordinates of their points of intersection.

Draw the Hyperbola $x^2 - y^2 = 1$ and the straight line $x = 2y$ on the *same scale*. Produce the straight line, if necessary, to meet the Hyperbola. They will be found to intersect at *two* points whose co-ordinates are (1·2, ·6) and (−1·2, −·6) approximately.

262. Draw the graph of the equation $y = x^2$.

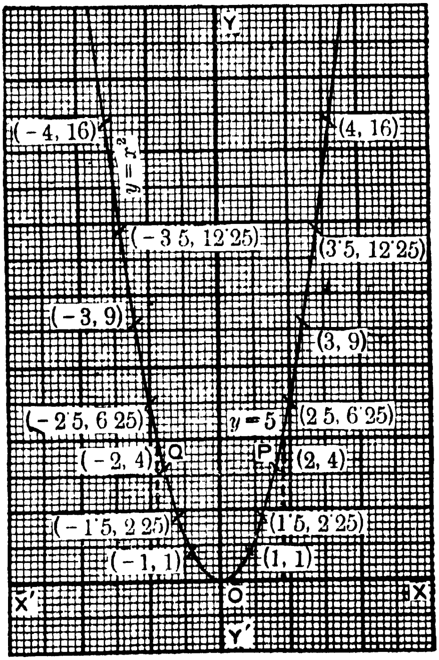
Evidently the following points are on the required graph and their co-ordinates may be tabulated as follows :

x	0	1	−1	1·5	−1·5	2	−2	2·5	−2·5	3	−3	3·5	−3·5	4	−4
y	0	1	1	2·25	2·25	4	4	6·25	6·25	9	9	12·25	12·25	16	16

Let 4 times the side of a small square be the unit of length.

Let us now plot the points found above and draw a curve through them free-hand, as in the following diagram.

The curve so drawn is the required graph.



Note 1. *Since, $y=x^2$, we have $x=\pm\sqrt{y}$; $\therefore$ x is imaginary when y is negative. Hence, no point of the curve can have a negative ordinate and, therefore, no part of the curve can lie below the x-axis. The curve passes through the origin, lies entirely above the x-axis and extends upwards to infinity.*

Note 2. *Every chord drawn perpendicular to OY is bisected by it as can be easily verified. Hence, the curve drawn above is symmetrical about the axis of y. This is also evident from the fact that if the paper be folded about OY, the left-hand portion of the curve entirely coincides with the right-hand portion.*

A curve of this class is called a ***Parabola.***

The general equation of a parabola is $y=ax^2+bx+c$.

In the equation of a parabola either of x and y will be of the first degree and there will be no term involving the product of x and y (i.e., xy).

In the above example, $a=1$, $b=0$ *and* $c=0$.

Note 3. ***The graph of $y=-x^2$.*** *The curve $y=x^2$ lies entirely above the axis of x, and extends upwards to infinity. It is easy to see that the graph of the equation $y=-x^2$ would be an equal curve being entirely* ***below*** *the axis of x and extending downwards to infinity.*

Note 4. ***To determine the square root of a number from the graph of $y=x^2$.*** *The abscissa of any point on the curve is evidently the square root of the ordinate. Hence, when the graph of the equation $y=x^2$ is drawn by measuring the abscissa of any point on the graph we can determine the square root of the number which represents the ordinate. Thus, in the diagram, the ordinates of P or Q represent 5. $\therefore$ the square root of $5=$ the abscissa of P or $Q=2{\cdot}25$, or, $-2{\cdot}25$ approximately.* [*4 sides of a small square $=1$ unit.*]

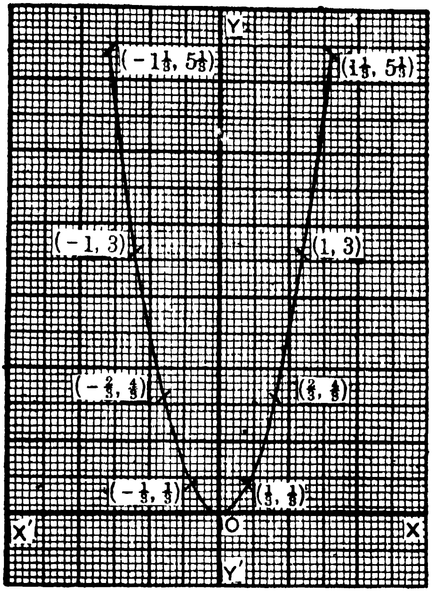
263. Draw the graph of the equation $y=3x^2$.

Evidently the following points are on the required graph and their co-ordinates may be tabulated as follows :

x	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	1	-1	$1\frac{1}{3}$	$-1\frac{1}{3}$
y	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	3	3	$5\frac{1}{3}$	$5\frac{1}{3}$

Taking 12 times the side of a small square as the unit of length, let us plot the points found above and draw a curve through them free-hand, as in the diagram on page 523.

The curve so drawn is the required graph.



Note 1. *Since* $y=3x^2$, *we have* $x^2=\frac{1}{3}y$. ∴ *x is imaginary for every negative value of y. Hence, as in the graph of Art.* 262, *the curve passes through the origin, lies entirely above the x-axis and extends upwards to infinity.*

Again, it may be easily verified that every chord drawn perpendicular to OY is bisected by it. Hence, the curve is symmetrical about the axis of Y.

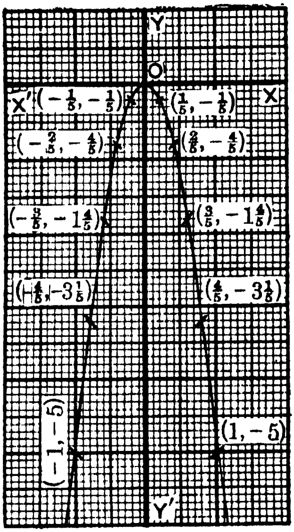
Note 2. *The graph of* $y=-3x^2$ *can be easily seen to be an equal curve passing through the origin, lying entirely below the x-axis and extending downwards to infinity*

264. Draw the graph of the equation $y=-5x^2$.

Evidently, the following points are on the required graph and their co-ordinates may be tabulated as follows :

x	0	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{2}{5}$	$\frac{3}{5}$	$-\frac{3}{5}$	$\frac{4}{5}$	$-\frac{4}{5}$	1	-1
y	0	$-\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{4}{5}$	$-\frac{4}{5}$	$-1\frac{4}{5}$	$-1\frac{4}{5}$	$-3\frac{1}{5}$	$-3\frac{1}{5}$	-5	-5

Taking 10 times the side of a small square as the unit of length, let us plot the points found above, and draw a curve through them free-hand, as in the diagram.



The curve so drawn is the required graph.

Note 1. *Since* $y=-5x^2$, *we have* $x^2=-\frac{1}{5}y$. ∴ *x is imaginary for every positive value of y. Hence, no point on the curve can have a positive ordinate and, therefore, no part of the curve can lie above the x-axis. The curve passes through the origin, lies entirely below the x-axis and extends downwards to infinity.*

Note 2. *It may be easily seen that every chord drawn perpendicular to OY' is bisected by it. Hence, the curve is symmetrical about the axis of y.*

Note 3. *The graph of the equation $y=5x^2$ can be easily seen to be an equal curve passing through the origin, lying entirely above the x-axis and extending upwards to infinity.*

265. It is clear, from Arts. 262, 263 and 264, that the graph of any equation of the form $y=ax^2$, where a is any numerical constant, positive or negative, is a curve which (i) is *symmetrical about the axis of y*, (ii) *lies entirely on one side of the axis of x*, and (iii) *extends up to infinity on that side.* A curve of this class is called a **Parabola.**

If a be a positive integer, the curve will be as in the figure of Art. 262 but will rise more steeply in the direction of OY. [See the fig. of Art. 263.] If a be a positive fraction, we shall have a flatter curve, extending more rapidly to the right and left of OY. If a be negative, as in Art. 264, the curve will lie below the x-axis and will be steeper or flatter than the graph of $y=x^2$, according as a is greater or less than unity. [See the fig. of Art. 264.]

In every case, the axis of x is a tangent to the curve at the origin.

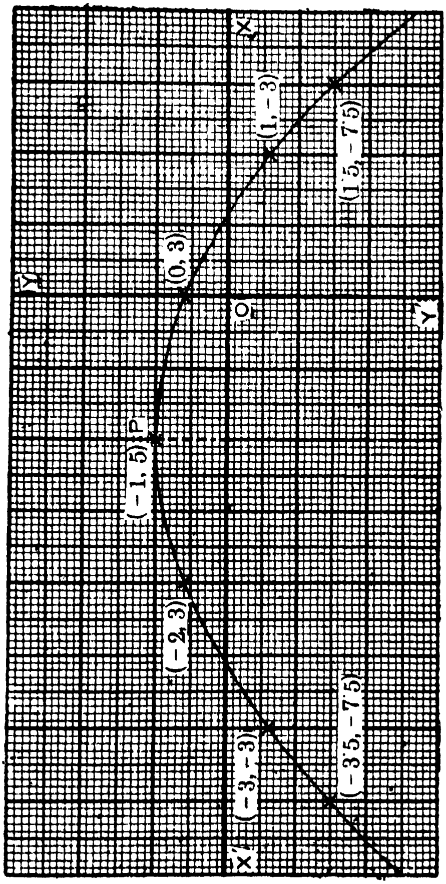
266. We shall now discuss the graphs of some quadratic functions of the form ax^2+bx+c. It will be seen, as in the next article, that the curve is always a parabola, differing in shape and position according to values of a, b, c.

267. Draw the graph of the expression $3-4x-2x^2$.

The required graph is the same as that of the equation

$$y=3-4x-2x^2.$$

It is easy to see that the following points are on the required graph:



$\left.\begin{matrix}x=0\\y=3\end{matrix}\right\}$, $\left.\begin{matrix}x=1\\y=-3\end{matrix}\right\}$, $\left.\begin{matrix}x=1·5\\y=-7·5\end{matrix}\right\}$, $\left.\begin{matrix}x=-1\\5=\end{matrix}\right\}$,

$\left.\begin{matrix}x=-2\\y=3\end{matrix}\right\}$, $\left.\begin{matrix}x=-3\\y=-3\end{matrix}\right\}$, $\left.\begin{matrix}x=-3·5\\y=-7·5\end{matrix}\right\}$, *Take twenty sides of a small square as the unit for measuring x, and two sides of a small square as the unit for measuring y.*

Let us now plot the above points and draw a curve through them free-hand, as in the diagram on the last page.

The curve so drawn is the required graph.

Note. *The graph of any expression of the form ax^2+bx+c is a parabola, provided the numerical value of a is not zero.*

268. Graphical solution of Quadratic Equations.

Example 1. To solve graphically the equation $3-4x-2x^2=0$.

Draw the graph of $y=3-4x-2x^2$ as in the last article.

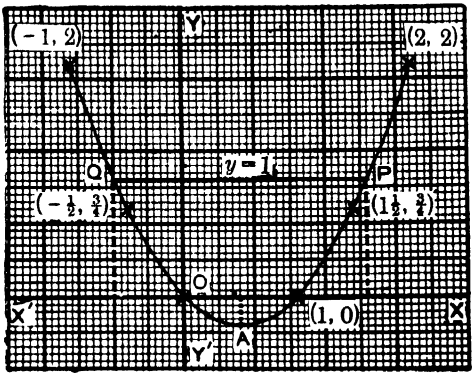
From the figure it is evident that $y=0$, when x is approximately equal to ·6 or $-2·6$. Hence, $3-4x-2x^2=0$, when $x=·6$ or $-2·6$ approximately, in other words, the roots of the equation $3-4x-2x^2=0$ are ·6 and $-2·6$ approximately. From this it is clear *that the roots of the equation $3-4x-2x^2=0$ are the abscissæ of the points where the graph of the expression $3-4x-2x^2$ cuts the axis of x.*

Example 2. Trace the graph of $y=x^2-x$ from $x=-1$ to $x=2$ and therefrom obtain an approximate solution of the equation

$$1=x^2-x. \qquad \text{[C. U. 1917]}$$

The following points evidently lie on the graph :

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
y	2	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	$\frac{3}{4}$	2



Taking 16 sides of a small square as the unit of length, the graph will be as shown in the diagram on page 526.

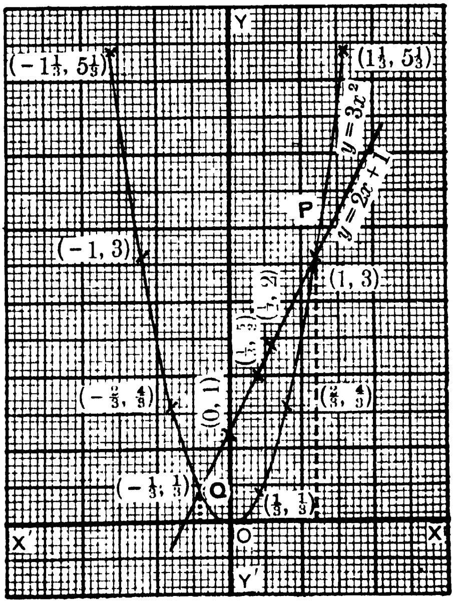
If we now put $y=1$, the equation $y=x^2-x$ becomes $1=x^2-x$. Hence, the roots of the equation $1=x^2-x$ are the abscissæ of the points P and Q of the graph of $y=x^2-x$, at which the ordinate is 1. P and Q are evidently the points where the line $y=1$ meets the graph. From the figure, we find that the abscissæ of P and Q are 1·6 and −·6 respectively, which are, therefore, the required solutions.

Example 3. Trace the graphs of (i) $y=3x^2$ and (ii) $y=2x+1$, and determine the points where they meet. [C. U. 1915]

Deduce the roots of the equation $3x^2=2x+1$.

Evidently the corresponding values of x and y on $y=3x^2$ may be tabulated as follows :

x	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	1	−1	$1\frac{1}{3}$	$-1\frac{1}{3}$
y	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	3	3	$5\frac{1}{3}$	$5\frac{1}{3}$



Also, the points $\left.\begin{matrix}x=0\\y=1\end{matrix}\right\}$, $\left.\begin{matrix}x=\frac{1}{3}\\y=\frac{5}{3}\end{matrix}\right\}$ and $\left.\begin{matrix}x=\frac{1}{2}\\y=2\end{matrix}\right\}$ lie on the straight line $y=2x+1$.

Taking twelve times the side of a small square as the unit of length the graphs will be as shown in the diagram on the last page.

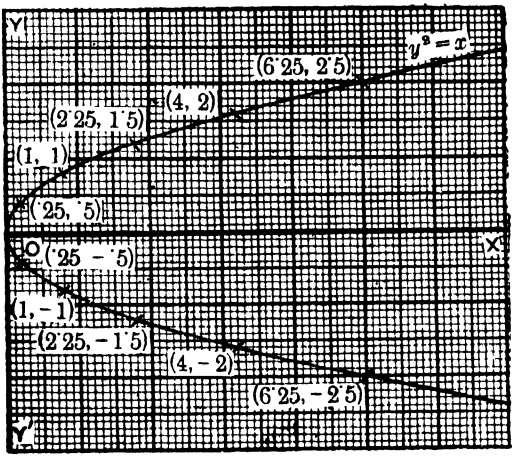
Let the straight line meet the parabola at P and Q whose co-ordinates are found from the diagram to be (1, 3) and $(-\frac{1}{3}, \frac{1}{3})$ respectively.

The abscissæ of the points common to the graphs of $y=3x^2$ and $y=2x+1$ are evidently the roots of $3x^2=2x+1$. But, from the figure, these abscissæ are 1 and $-\frac{1}{3}$, which are, therefore, the required roots of $3x^2=2x+1$.

269. Draw the graph of $y^2=x$.

We have $y=\pm\sqrt{x}$. The corresponding values of x and y may be tabulated as follows :

x	0	·25	·25	1	1	2·25	2·25	4	4	6·25	6·25
y	0	·5	−·5	1	−1	1·5	−1·5	2	−2	2·5	−2·5



Let eight sides of a small square be the unit of length. Now plotting the points found above and drawing a curve through them free-hand, the graph will be as in the diagram.

Note 1. *Since for every point of the graph,* $y = \pm\sqrt{x}$ *and is, therefore, imaginary when* x *is negative, it follows that no point of the graph can have a negative abscissa, i.e., no part of the graph lies on the negative side of the* x*-axis. This graph, therefore, lies on the positive side of the* x*-axis and extends to infinity on that side. It is easy to see that the curve is symmetrical about the* x*-axis.*

Note 2. *The graph of* $y^2 = -x$ *is evidently an equal curve turned in the opposite direction on the negative side of the* x*-axis.*

270. Maximum and minimum values of quadratic expressions.

Example 1. Show graphically that the expression $3-4x-2x^2$ is positive for all values of x between −2·6 and ·6 and find its maximum value.

Let $y=3-4x-2x^2$.

Drawing the graph of $y=3-4x-2x^2$ as in Art. 267, we find that for all values of x between −2·6 and ·6 the curve lies above the x-axis and ∴ the ordinates are positive, and for values of x greater than ·6 and less than −2·6, the curve is below the axis of x and ∴ the ordinates are negative. But the ordinate $(y)=3-4x-2x^2$.

Hence, $3-4x-2x^2$ is positive for all values of x between −2·6 and ·6.

Also, we notice from the figure that the ordinate is greatest at the point $P\,(-1, 5)$, its greatest value being 5.

∴ the maximum value required $=5$.

Example 2. Show graphically that the expression x^2-x is negative for all values of x between $x=0$ and $x=1$. Find its minimum value.

Let $y=x^2-x$.

Drawing the graph of $y=x^2-x$ as in Art. 268, Example 2 (see the diagram on page 526), we find that for all values of x between $x=0$ and $x=1$ the curve is below the x-axis and ∴ the ordinates are negative.

But the ordinate $(y)=x^2-x$.

Hence, x^2-x is negative for all values of x between $x=0$ and $x=1$.

Also, it is evident from the figure that y (*i.e.*, x^2-x) has the minimum value $-\frac{1}{4}$ at the point A.

271. Draw the graph of the equation $xy=1$.

It is easy to see that the following points are on the required graph :

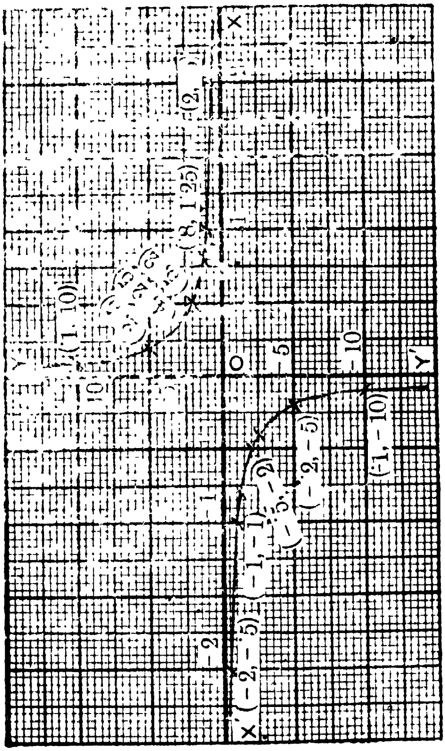
$$\left.\begin{matrix}x=\cdot1\\ y=10\end{matrix}\right\},\quad \left.\begin{matrix}x=\cdot2\\ y=5\end{matrix}\right\},\quad \left.\begin{matrix}x=\cdot4\\ y=2\cdot5\end{matrix}\right\},\quad \left.\begin{matrix}x=\cdot5\\ y=2\end{matrix}\right\},$$

$$\left.\begin{matrix}x=\cdot8\\ y=1\cdot25\end{matrix}\right\},\quad \left.\begin{matrix}x=1\\ y=1\end{matrix}\right\},\quad \left.\begin{matrix}x=2\\ y=\cdot5\end{matrix}\right\}.$$

Evidently also the following points are on the required graph :

$\left.\begin{array}{l} x=-\cdot1 \\ y=-10 \end{array}\right\}$, $\left.\begin{array}{l} x=-\cdot2 \\ y=-5 \end{array}\right\}$, $\left.\begin{array}{l} x=-\cdot4 \\ y=-2\cdot5 \end{array}\right\}$, $\left.\begin{array}{l} x=-\cdot5 \\ y=-2 \end{array}\right\}$,

$\left.\begin{array}{l} x=-\cdot8 \\ y=-1\cdot25 \end{array}\right\}$, $\left.\begin{array}{l} x=-1 \\ y=-1 \end{array}\right\}$, $\left.\begin{array}{l} x=-2 \\ y=-\cdot5 \end{array}\right\}$.



Let two centimetres be the unit for measuring x and 2 millimetres the unit for measuring y.

Let us now plot the points and draw a curve through them free-hand, as in the above diagram.

The curve so drawn is the required graph.

Note 1. *As x diminishes from 1 to zero, y increases from 1 to infinity ; and as x diminishes from zero to -1, y increases from negative infinity to -1.*

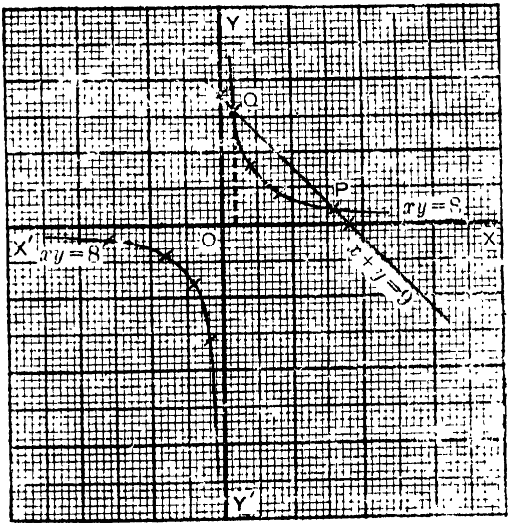
Note 2. *As x increases from 1 to infinity, y diminishes from 1 to zero ; and as x diminishes from -1 to negative infinity, y increases from -1 to zero.*

Note 3. *The graph consists of two branches, one lying between OX and OY and the other between OX' and OY'.*

Note 4. *The more we move towards the right or left of O, the nearer does the curve approach the axis of x ; whilst the more we move upwards and downwards from O, the nearer does the curve approach the axis of y. But in no case does the curve meet the axis except at an infinite distance from O. Hence, each of the axes is said to be an **Asymptote** to the curve.*

Note 5. *A curve of this kind is called a **Rectangular Hyperbola**.*

Example. Draw the graphs of (i) $xy=8$ and (ii) $x+y=9$. Find the co-ordinates of their points of intersection.



Drawing the graph of $xy=8$ by the above method and the graph of the straight line $x+y=9$ in the same figure on the same scale, as in the above diagram it will be found that they intersect at two points P and Q whose co-ordinates are

$$\left.\begin{array}{l}x=8\\y=1\end{array}\right\} \text{ and } \left.\begin{array}{l}x=1\\y=8\end{array}\right\} \text{ respectively.}$$

EXERCISE 137

Draw the graphs of the following equations :

1. $x^2+4y^2=4$. 2. $4x^2+9y^2=1$. 3. $25x^2+y^2=25$.

4. $16x^2+9y^2=1$. 5. $x^2-4y^2=4$. 6. $y^2-x^2=1$.

7. $4x^2-y^2=16$. 8. $y^2-9x^2=9$.

9. In one and the same diagram draw the graphs of $4x^2-9y^2=0$ and $4x^2-9y^2=36$.

10. In one and the same diagram draw the graphs of $9y^2-4x^2=0$ and $9y^2-4x^2=36$.

11. Draw the graph of the equation $5y^2=x^2-10$, taking the unit for measuring y five times as large as that for measuring x.

12. Draw the graph of the equation $x^2-4x+2y=0$, taking the unit for measuring y twice as large as that for measuring x.

13. Draw the graph of the equation $y^2+x=0$, taking the unit for measuring x equal to half that for measuring y.

14. Draw the graph of the equation $3y=x^2$, taking the same unit for measuring both x and y.

15. Find graphically, correct to the first figure after the decimal point, the square roots of :

(i) 3 ; (ii) 5 ; (iii) 7.

16. Find graphically, the minimum values of the expressions :

(i) $x^2+6x+10$; (ii) $4x^2+4x+5$; (iii) $\frac{1}{2}x^2+4x+1$;
(iv) $2x^2-6x+7$.

17. Find graphically, the maximum values of the expressions :

(i) $4x-x^2$; (ii) $3+6x-9x^2$; (iii) $12-3x-\dfrac{x^2}{4}$;
(iv) $1+2x-2x^2$.

18. Draw the graphs of the equations (i) $xy=4$ and (ii) $x+y=5$, and find where they intersect.

19. Show graphically that (i) the expression $4x-x^2$ is positive for all values of x between 0 and 4 ; (ii) the expression $x^2+6x+12$ is positive for all values of x and (iii) x^2-4x-5 is negative for all values of x between -1 and 5.

20. Draw the graphs of (i) $xy=-8$, and (ii) $x+y=2$ and find where they intersect.

Solve graphically :

21. $x^2=4x-3$. 22. $3x^2=x+2$.

23. $2x^2-7x+5=0$. 24. $7x^2-2x=5$.

25. (i) $\left.\begin{aligned} x^2-y^2&=1 \\ x&=2y \end{aligned}\right\}$; (ii) $\left.\begin{aligned} xy&=5 \\ x+y&=-6 \end{aligned}\right\}$;

(iii) $\left.\begin{aligned} y^2&=4x \\ y&=2x \end{aligned}\right\}$; (iv) $\left.\begin{aligned} x^2&=y \\ x&=-2y \end{aligned}\right\}$.

CHAPTER XXXVII

ARITHMETICAL PROGRESSION

272. Definition. Quantities are said to be in Arithmetical Progression when they increase continually by a common quantity (called the common difference).

Thus, each of the following series of quantities is in Arithmetical Progression :

$2, \quad 5, \quad 8, \quad 11, \quad 14,$ &c.

$9, \quad 5, \quad 1, \quad -3, \quad -7,$ &c.

$a, \quad a+b, \quad a+2b, \quad a+3b,$ &c.

$a, \quad a-b, \quad a-2b, \quad a-3b,$ &c.

In the first of the above examples the quantities increase by 3, whereas in the second the quantities decrease by 4 ; so the common differences of these two cases are said to be 3 and -4 respectively. Similarly, in the third example the common difference is b and in the fourth it is $-b$.

N. B. Arithmetical Progression is briefly written as A.P.

273. The **common difference** of the terms of an A. P. is found by subtracting any term of the series from the term following it.

Thus, in the series $a, a+b, a+2b, a+3b, \ldots,$ the common difference $=(a+b)-a=(a+2b)-(a+b)=(a+3b)-(a+2b)=\cdots\cdots=b.$

274. To find the nth term of an A. P.

If a be the first term and b, the common difference of a series of numbers in Arithmetical Progression, we have the 2nd term $=a+b$, the 3rd term $=a+2b$, the 4th term $=a+3b$, …… the 10th term $=a+9b$, …… the 21st term $=a+20b$; and so on. Hence, the nth term $=a+(n-1)b$.

Example 1. Find the 19th term of the series 10, 8, 6, 4, &c.

The first term $=10$, and the common difference $=-2$.

Hence, the 19th term $=10+18(-2)=10-36=-26$.

Example 2. What term of the series 5, 7, 9, 11, &c. is 25 ?

Let the rth term of the given series be the required term ; then, we must have

$$\begin{aligned} 25 &= 5+(r-1)2 \\ &= 3+2r, \text{ whence } r=11. \end{aligned}$$

Thus, the 11th term of the given series $=25$.

275. Given any two terms of an A. P., to find it completely.

Example 1. The 7th and 13th terms of an A. P. are 34 and 64 respectively. Find the series.

Let $a=$ the first term,

and $b=$ the common difference of the A. P.

$\therefore$ the 7th term $=a+(7-1)b=a+6b=34$, $\quad\cdots$ (1)

and the 13th term $=a+(13-1)b=a+12b=64$. $\quad\cdots$ (2)

From (1) and (2), by subtraction,

$$6b=30, \text{ i.e., } b=5.$$

Now from (1), $a+6\times5=34$, or, $a=34-30=4$.

Hence, the first term and the common difference of the required series are 4 and 5 respectively.

$\therefore$ the series is 4, 9, 14, 19, 24, ...

Example 2. The pth and qth terms of an A. P. are c and d respectively. Find the series completely.

Let $a=$ the first term,

and $b=$ the common difference of the A. P.

$\therefore$ the pth term $=a+(p-1)b=c$, $\quad\cdots\quad\cdots$ (1)

and the qth term $=a+(q-1)b=d$. $\quad\cdots\quad\cdots$ (2)

Solving equations (1) and (2), a and b can be obtained. Thus, by subtracting (2) from (1) we have,

$$(p-q)b=c-d, \quad\therefore\quad b=\frac{c-d}{p-q}.$$

Also, from (1), $a+(p-1)b=a+(p-1)\cdot\frac{c-d}{p-q}=c$.

$$\therefore\quad a=c-\frac{(p-1)(c-d)}{p-q}=\frac{d(p-1)-c(q-1)}{p-q}$$

Hence, a and b being known, the whole series may be written down.

EXERCISE 138

1. Find the 8th, 20th and $(n-3)$th terms of the series :
(i) 2, 4, 6, 8, &c. (ii) 1, 3, 5, 7, &c. (iii) $\frac{13}{8}$, $\frac{7}{8}$, $\frac{1}{8}$, $-\frac{5}{8}$, &c.
(iv) $\frac{9}{7}$, $\frac{5}{7}$, $\frac{1}{7}$, &c. (v) 5, 11, 17, ...

2. What terms of the series 9, 11, 13, 15, &c. are 65, 99 and $6n-13$?

3. The first term of a given series is 3 and the 7th term 39, find the common difference.

4. If there be 60 terms in A. P. of which the first term is 8 and the last term 185 ; find the 31st term.

5. The 3rd and 13th terms of a series in A. P. are -40 and 0 Find the series and determine its 20th term.

6. The 5th and 31st terms of an A. P. are 1 and -77. Obtain its 1st and 18th terms.

7. Find the 1st term and the common difference of a series whose 8th and 102th terms are 23 and 305 respectively.

8. The pth term of an A. P. is c and its qth term is d. Find the rth term.

9. *If every term of an A. P. be increased or diminished by the same quantity, the resulting terms will also be in A. P.*

10. Prove that *if each term of an A. P. be multiplied or divided by the same quantity, the resulting series will also be in A. P.*

11. If a be the first term and l the last term of a series of numbers in A. P., show that the 5th term from the beginning + the 5th term from the end $= a + l$.

12. In the preceding example, show that the rth term from the beginning + the rth term from the end $= a + l$.

13. Is 302 a term of the series 3, 8, 13, 18, &c. ?

[Here, the common difference $= 5$. If possible, let 302 = the rth term of the series, r being evidently an integer.

$$\therefore\ 302 = 3 + (r-1)5, \qquad \text{or,}\ \ r - 1 = \frac{302-3}{5}, \qquad \text{or,}\ \ r = \frac{304}{5}.$$

The value of r being fractional is inadmissible.

$\therefore$ 302 is not a term of the series.]

14. The pth term of an A. P. is q and the qth term is p. Show that the mth term is $p + q - m$.

276. To find the sum of n terms of an Arithmetic series of which the first term is a and the common difference, b.

Let S denote the required sum, and l, the last term (*i.e.*, the nth term).

Then, $S = a + (a+b) + (a+2b) + (a+3b) + \text{\&c.} + \{a + (n-1)b\}$.

And, by writing the series in the reverse order, we have also

$$S = l + (l-b) + (l-2b) + (l-3b) + \text{\&c.} + \{l - (n-1)b\}.$$

Therefore, by addition,

$$2S=(a+l)+(a+l)+(a+l)+\&c.\ldots \text{ to } n \text{ terms}=n(a+l).$$

$$\therefore \quad S=\frac{n}{2}(a+l) \cdots \qquad \cdots \qquad \cdots \qquad \cdots \quad (1)$$

Thus, the sum of n terms in A. P. is n times the semi-sum of the first and last terms, or, in other words, n times the average of the first and last terms.

Also, since $l=a+(n-1)b$,

$$\therefore \quad S=\frac{n}{2}\left[a+\left\{a+(n-1)b\right\}\right]=\frac{n}{2}\left[2a+(n-1)b\right] \quad \cdots \quad (2)$$

N. B. The formulæ (1) and (2) should be carefully remembered so that they might readily be applied in any suitable case.

Example 1. Find the sum of 20 terms of the series 5, $4\frac{1}{3}$, $3\frac{2}{3}$, &c.

The first term $=5$, and the common diff. $=\frac{13}{3}-5=-\frac{2}{3}$.

Hence, the required sum $=\frac{20}{2}\{2\times5+(20-1)\times(-\frac{2}{3})\}$

$$=10\,(10-\tfrac{19\times2}{3})=10(-\tfrac{8}{3})=-26\tfrac{2}{3}.$$

Example 2. Find the value of $1+2+3+4+$ &c. to 100 terms.

The last term of the series evidently $=100$.

Hence, the required sum $=\frac{100}{2}(1+100)=50\times101=5050$.

Example 3. Find, without assuming any formula, the sum of $1+4+7+10+\cdots\cdots+37$. [C. U. 1919]

Evidently, the common difference $=3$, and the number of terms in the series $=13$.

Let S denote the required sum.

$$\therefore \quad S=1+4+7+\cdots\cdots+31+34+37.$$

Also, re-writing the series in the reverse order,

$$S=37+34+31+\cdots\cdots+7+4+1.$$

Adding together the two series,

$$2S=38+38+38+\cdots\cdots \text{ to } 13 \text{ terms}=38\times13.$$

$$\therefore \quad S=\frac{38\times13}{2}=19\times13=247.$$

Example 4. Find, without assuming any formula, the sum of the series $1+3+5+7+\cdots\cdots$ to n terms.

Evidently, the common difference $=2$,

and the nth term $=1+(n-1)\times2=2n-1$.

Let $\quad S=$ the sum required.

$$\therefore \quad S=1+3+5+\cdots+(2n-5)+(2n-3)+(2n-1).$$

Re-writing the series in the reverse order,

$$S=(2n-1)+(2n-3)+(2n-5)+\cdots+5+3+1.$$

Adding the two series,

$$2S=2n+2n+2n+\cdots\cdots \text{ to } n \text{ terms}=2n.n.$$

$$\therefore \quad S=n^2.$$

EXERCISE 139

Find the sum of the following series :

1. $1+2+3+4+\cdots\cdots$&c. to 25 terms.
2. $1+3+5+7+\cdots\cdots$&c. to 30 terms.
3. $-3, 3, 9, 15, \ldots\ldots\ldots$ to 14 terms.
4. $\frac{3}{8}+\frac{5}{8}+\frac{7}{8}+\cdots\cdots\cdots\cdots$ to 20 terms.
5. $\frac{7}{11}+\frac{13}{22}+\frac{6}{11}+\cdots\cdots\cdots$ to 30 terms.
6. $1\frac{1}{7}+1+\frac{6}{7}+\frac{5}{7}+\cdots\cdots$ to 16 terms.
7. $3+4+8+9+13+14+18+19+\cdots$ to 20 terms. [C. U. F. A. 1881]

   [The given series $=(3+4)+(8+9)+(13+14)+(18+19)+\cdots$
   $=7+17+27+37+\cdots\cdots$ to 10 terms
   $=\frac{\{14+(10-1)\times 10\}}{2}\times 10=520.$]

8. $5+4\frac{3}{4}+4\frac{1}{2}+\cdots\cdots$ &c. to 21 terms.
9. $13+12\frac{1}{3}+11\frac{2}{3}+\cdots\cdots$ &c. to 40 terms.
10. $2+7+12+\cdots\cdots$ &c. to 101 terms.
11. $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\cdots\cdots$ &c. to n terms.
12. $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\frac{5a-3b}{a+b}+\cdots\cdots$ &c. to n terms.
13. $1+5+3+9+5+13+7+17+\cdots\cdots$ to 30 terms.
14. $\left(2-\frac{1}{n}\right)+\left(2-\frac{3}{n}\right)+\left(2-\frac{5}{n}\right)+\cdots\cdots$ to n terms.
15. $(a+b)^2+(a^2+b^2)+(a-b)^2+\cdots\cdots$ to n terms.

Find the sum of the following series without applying any formula :

16. $3+5+7+\cdots\cdots$ to 29 terms.
17. $-10-6-2+2+\cdots$ to 22 terms.
18. $(x-y)+(2x-3y)+(3x-5y)+\cdots\cdots$ to n terms.
19. $5+8+11+\cdots\cdots+155.$ 20. $8+3-2-7-12-\cdots\cdots$ to n terms.

277. Applications of the formulæ (1) and (2) of the preceding article. The following examples illustrate some important applications of those formulæ.

Example 1. The first term of a series in A. P. is 17, the last term $-12\frac{3}{8}$ and the sum $25\frac{7}{16}$; find the common difference.

Let n = the number of terms; then, we must have

$$25\frac{7}{16}=\frac{n}{2}\left\{17+\left(-12\frac{3}{8}\right)\right\}=\frac{n}{2}\left(17-12\frac{3}{8}\right)=\frac{n}{2}\times4\frac{5}{8},$$

$$\text{or,}\quad \frac{407}{16}=\frac{37n}{16}.\quad\therefore\quad n=\frac{407}{37}=11.$$

If, then, b be the required common difference, we must have

$$-12\tfrac{3}{8}(=\text{the 11th term})=17+10b.$$

$$\therefore\quad 10b=-12\tfrac{3}{8}-17=-29\tfrac{3}{8}=-\tfrac{235}{8}.$$

$$\therefore\quad b=-\frac{235}{8\times10}=-\frac{5\times47}{5\times2\times8}=-\frac{47}{16}.$$

Example 2. The sum of a series in A. P. is 72, the first term 17, and the common difference -2; find the number of terms, and explain the double answer.

Let n = the number of terms.

Then, we must have

$$72=\frac{n}{2}\{2\times17+(n-1)\times(-2)\}$$

$$=\frac{n}{2}\{34-2(n-1)\}=\frac{n}{2}(36-2n)=18n-n^2.$$

$$\therefore\quad n^2-18n+72=0,\qquad\text{or,}\quad(n-6)(n-12)=0.$$

$$\therefore\quad n=6,\ \text{or},\ 12.$$

The double answer shows that there are two sets of numbers, satisfying the conditions of the problem, and this can be easily verified. For the series to 6 terms is 17, 15, 13, 11, 9, 7; and to 12 terms it is 17, 15, 13, 11, 9, 7, 5, 3, 1, -1, -3, -5; now since the sum of the last 6 terms of the latter set of numbers = 0; evidently, therefore, the sum of 6 terms of the series, is exactly the same as that of 12 terms.

Example 3. How many terms of the series -8, -6, -4, &c. amount to 52 ?

Let $n=$ the required number.

Then, we must have

$$52=\frac{n}{2}\{2\times(-8)+(n-1)\times 2\}$$

$$=\frac{n}{2}(2n-18)=n^2-9n.$$

$$\therefore \quad n^2-9n-52=0;$$

$$\text{or,} \quad (n-13)(n+4)=0; \quad \therefore \quad n=13, \text{ or, } -4.$$

Hence, since the number of terms can only be a positive integer, we must reject the negative value and take 13 to be the answer to the question.

Example 4. The sum of p terms of an A. P. is q and the sum of q terms is p; find the sum of $p+q$ terms.

Let a be the first term, and b the common difference; then, since the sum of p terms $=q$, we must have

$$q=\frac{p}{2}\{2a+(p-1)b\},$$

$$\left.\begin{array}{ll} \text{or,} \quad 2q=p.2a+p(p-1)b. & \cdots \quad \cdots \quad (1) \\ \text{Similarly,} \quad 2p=q.2a+q(q-1)b. & \cdots \quad \cdots \quad (2) \end{array}\right\}$$

Subtracting (2) from (1), we have

$$2(q-p)=(p-q).2a+\{(p^2-q^2)-(p-q)\}b$$

$$=(p-q).2a+(p-q)(p+q-1)b.$$

$$\therefore \quad -2=2a+(p+q-1)b.$$

Hence, the sum of $(p+q)$ terms

$$=\frac{p+q}{2}\{2a+(p+q-1)b\}$$

$$=\frac{p+q}{2}\times(-2)=-(p+q).$$

EXERCISE 140

1. The first term of an A. P. is 5, the number of terms 30, and their sum 1455 ; find the common difference.

2. The first term of a series being 2, and the 5th term being 7, find how many terms must be taken so that the sum may be 63.

3. What is the common difference when the first term is 1, the last 50, and the sum 204 ?

4. How many terms of the series 19, 17, 15, &c., amount to 91 ?

5. The sum of a certain number of terms of the series 21, 19, 17, &c. is 120. Find the last term and the number of terms.

6. How many terms of the series 54, 51, 48, &c., must be taken to make 513 ? Explain the double answer.

7. If the sum of 8 terms of an A. P. is 64, and the sum of 19 terms is 361, find the sum of n terms.

8. Find the series of which the nth term is $\frac{3+n}{4}$; and also find the sum of the series to 105 terms.

9. Find the series whose rth term is $2r-1$; find the sum of the series to n terms.

10. The sum of n terms of an A. P. is $3n^2-n$, and the common difference 6 ; find the first term.

11. The sum of n terms of an A. P. is 40, the common difference 2, and the last term 13 ; find n.

12. Prove that the sum of the latter half of $2n$ terms of any arithmetical series $=\frac{1}{3}$ of the sum of $3n$ terms of the same series.

13. If $2n+1$ terms of the series 1, 3, 5, 7, 9, &c., be taken, then the sum of the alternate terms 1, 5, 9, &c., will be to the sum of the remaining terms 3, 7, 11, &c., as $n+1$ is to n.

14. Prove that (i) $b=\frac{l^2-a^2}{2s-(l+a)}$,

and (ii) $s=\frac{l+a}{2b}(l-a+b)$.

278. Arithmetic means.

Definitions : (1) When three quantities are in Arithmetical Progression, the middle one is said to be the **Arithmetic mean** between the other two.

Thus, 5 is the Arithmetic mean between 3 and 7.

(2) If A and B be any two quantities and x_1, x_2, x_3, x_4, &c., x_{n-1}, x_n, a number of others such that A, x_1, x_2, x_3, &c., x_{n-1}, x_n, B are in Arithmetical Progression, then x_1, x_2, x_3, &c. are called the **Arithmetic means** between A and B.

Thus, 3, 4, 5, 6, 7 are Arithmetic means between 2 and 8, and so are the numbers $3\frac{1}{2}$, 5 and $6\frac{1}{2}$; for both the series 2, 3, 4, 5, 6, 7, 8 and 2, $3\frac{1}{2}$, 5, $6\frac{1}{2}$, 8 are in A. P.

Note. *It is evident from the above example that between any two quantities the number of different sets of Arithmetic means is unlimited.*

279. To insert a given number of Arithmetic means between two given quantities.

Let a and c be the two given quantities, and n the number of Arithmetic means to be inserted.

Then, we have to find out n quantities x_1, x_2, x_3, &c., x_{n-2}, x_{n-1}, x_n such that a, x_1, x_2, x_3, &c., x_{n-1}, x_n, c may be in A. P. Evidently the series $[a, x_1, x_2, x_3, \text{\&c.}, x_{n-1}, x_n, c]$ consists of $n+2$ terms of which a is the first term and c the last.

Hence, if b be the common difference, we must have

$$c=a+(n+1)b,$$

whence $b=\dfrac{c-a}{n+1}$.

Hence,
$$x_1=a+b=a+\frac{c-a}{n+1},$$
$$x_2=a+2b=a+\frac{2(c-a)}{n+1},$$
&c. &c. &c.
$$x_n=a+nb=a+\frac{n(c-a)}{n+1}$$

Example 1. Find the Arithmetic mean between any two quantities a and b.

Let x = the quantity sought

Then, a, x, b are in A. P.; and $\therefore$ we must have $x-a=b-x$,

whence $x=\dfrac{a+b}{2}$.

Example 2. Insert 4 Arithmetic means between 3 and 18.

Let x_1, x_2, x_3, x_4 be the required means.

Then, 3, x_1, x_2, x_3, x_4, 18 are in A. P.

Hence, if b = the common difference,

we must have $18=3+5b$. $\therefore$ $b=3$.

Hence,
$$\left.\begin{aligned} x_1&=3+\ b=\ 6\\ x_2&=3+2b=\ 9\\ x_3&=3+3b=12\\ x_4&=3+4b=15\end{aligned}\right\}$$

Thus, the required means are 6, 9, 12 and 15.

EXERCISE 141

1. Find the Arithmetic means between (i) 5 and 8; (ii) -5 and 21; (iii) $m-n$ and $m+n$; (iv) $(a+x)^2$ and $(a-x)^2$.

2. Insert 2 Arithmetic means between (i) 8 and 12; (ii) -6 and 14.

3. Insert 3 Arithmetic means between 117 and 477.

4. Insert 4 Arithmetic means between 2 and -18

5. Insert 17 Arithmetic means between $3\frac{1}{2}$ and $-41\frac{1}{2}$.

6. There are n Arithmetic means between 1 and 31, such that the 7th mean : $(n-1)$th mean $=5:9$; required n.

280. The natural numbers. The numbers 1, 2, 3, &c. are called the natural numbers.

(i) To find the sum of the first n natural numbers.

Let S denote the sum ; then

$$S=1+2+3+\cdots\cdots\cdots\cdots+n$$

$$=\frac{n}{2}(1+n)=\frac{n(n+1)}{2}. \qquad \cdots \quad \text{(A)}$$

(ii) To find the sum of the first n odd natural numbers.

Let S denote the sum ; then

$$S=1+3+5+7+\cdots\cdots \text{ to } n \text{ terms}$$

$$=\frac{n}{2}\{2+(n-1)\times 2\}$$

$$=\frac{n}{2}\times 2n=n^2. \quad \cdots \qquad \cdots \quad \text{(B)}$$

(iii) To find the sum of the squares of the first n natural uumbers.

Let S denote the sum ; then

$$S=1^2+2^2+3^2+4^2+\cdots\cdots+n^2.$$

We have, $n^3-(n-1)^3=3n^2-3n+1$.

Hence, putting 1, 2, 3, &c., for n, we have

$$1^3-0^3=3.1^2-3.1+1,$$
$$2^3-1^3=3.2^2-3.2+1,$$
$$3^3-2^3=3.3^2-3.3+1,$$
$$4^3-3^3=3.4^2-3.4+1,$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$(n-1)^3-(n-2)^3=3.(n-1)^2-3.(n-1)+1,$$
$$n^3-(n-1)^3=3.n^2-3.n+1.$$

Hence, by addition,

$$n^3=3(1^2+2^2+3^2+\cdots+n^2)-3(1+2+3+\cdots+n)+n$$

$$=3S-3.\frac{n(n+1)}{2}+n\ ;$$

$$\therefore \quad 3S=n^3-n+\frac{3n(n+1)}{2}=n(n+1)\{(n-1)+\tfrac{3}{2}\}\ ;$$

$$\therefore \quad S=\frac{n(n+1)(2n+1)}{6} \qquad \cdots \quad \text{(C)}$$

(iv) To find the sum of the cubes of the first n natural numbers.

Let S denote the sum ; then

$$S = 1^3 + 2^3 + 3^3 + \cdots\cdots\cdots + n^3.$$

We have, $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1.$

Hence, putting 1, 2, 3, &c., for n, we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1,$$
$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1,$$
$$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1,$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$(n-1)^4 - (n-2)^4 = 4.(n-1)^3 - 6.(n-1)^2 + 4.(n-1) - 1,$$
$$n^4 - (n-1)^4 = 4.n^3 - 6.n^2 + 4.n - 1.$$

Hence, by addition,

$$n^4 = 4(1^3 + 2^3 + 3^3 + \text{\&c.} + n^3) - 6(1^2 + 2^2 + 3^2 + \text{\&c.} + n^2)$$
$$+ 4(1 + 2 + 3 + \text{\&c.} + n) - n$$
$$= 4S - 6.\frac{n(n+1)(2n+1)}{6} = 4.\frac{n(n+1)}{2} - n ;$$

$$\therefore \quad 4S = n^4 + n + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)\{(n^2 - n + 1) + (2n+1) - 2\} = n(n+1)(n^2 + n) ;$$

$$\therefore \quad S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2. \quad \cdots \quad \text{(D)}$$

Thus, *the sum of the cubes of the first n natural numbers is equal to the square of the sum of these numbers.*

Example 1. Sum the series $1.2 + 2.3 + 3.4 +$ &c. to n terms.

The nth term of the series evidently $= n(n+1) = n^2 + n$.

Hence, putting $n = 1$, the 1st term $= 1^2 + 1$,

„ „ $n = 2$, „ 2nd term $= 2^2 + 2$,

„ „ $n = 3$, „ 3rd term $= 3^2 + 3$,

... ...

and so on.

Hence, if S denote the sum of the given series, we have

$$S = (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + \text{\&c. to } n \text{ terms}$$
$$= (1^2 + 2^2 + 3^2 + \text{\&c.} + n^2) + (1 + 2 + 3 + \text{\&c.} + n)$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left\{\frac{2n+1}{3} + 1\right\} = \frac{n(n+1)(n+2)}{3}.$$

Example 2. Sum the series $1^2+3^2+5^2+7^2+$ &c. to n terms.

Since evidently each term of the given series is equal to the square of the *corresponding* term of the series 1, 3, 5, 7, &c., $\therefore$ the nth term of the given series = the square of the nth term of the series 1, 3, 5, 7, &c.; and $\therefore$ the nth term $=\{1+(n-1)\times 2\}^2=(2n-1)^2=4n^2-4n+1$.

Hence, putting $n=1, 2, 3,$ &c., we have

$$\begin{aligned} \text{the 1st term} &= 4.1^2-4.1+1,\\ \text{,, 2nd ,,} &= 4.2^2-4.2+1,\\ \text{,, 3rd ,,} &= 4.3^2-4.3+1,\\ \ldots \quad \ldots &\quad \ldots \quad \ldots \end{aligned}$$

and so on.

Hence, if S denote the sum of the given series, we must have

$$\begin{aligned} S &= 4(1^2+2^2+3^2+\text{\&c.}+n^2)-4(1+2+3+\text{\&c.}+n)+n\\ &= 4.\frac{n(n+1)(2n+1)}{6}-4.\frac{n(n+1)}{2}+n\\ &= 2n(n+1)\left\{\frac{(2n+1)}{3}-1\right\}+n=\frac{2n(n+1)\times 2(n-1)}{3}+n\\ &= \frac{n}{3}\{4(n^2-1)+3\}=\frac{n}{3}(4n^2-1) \end{aligned}$$

Example 3. Sum the series:

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+\text{\&c. to } n \text{ terms}$$

The nth term of the given series

$$\begin{aligned} &= 1^2+2^2+3^2+\text{\&c.}+n^2\\ &= \frac{n(n+1)(2n+1)}{6}=\frac{n(2n^2+3n+1)}{6}=\tfrac{1}{3}n^3+\tfrac{1}{2}n^2+\tfrac{1}{6}n. \end{aligned}$$

$$\begin{aligned} \text{Hence, the 1st term} &= \tfrac{1}{3}.1^3+\tfrac{1}{2}.1^2+\tfrac{1}{6}.1,\\ \text{2nd ,,} &= \tfrac{1}{3}.2^3+\tfrac{1}{2}.2^2+\tfrac{1}{6}.2,\\ \text{3rd ,,} &= \tfrac{1}{3}.3^3+\tfrac{1}{2}.3^2+\tfrac{1}{6}.3,\\ \ldots \quad &\ldots \quad \ldots \end{aligned}$$

and so on.

Hence, if S denote the required sum, we must have

$$\begin{aligned} S &= \tfrac{1}{3}(1^3+2^3+3^3+\text{\&c.}+n^3)\\ &\qquad +\tfrac{1}{2}(1^2+2^2+3^2+\text{\&c.}+n^2)+\tfrac{1}{6}(1+2+3+\text{\&c.}+n)\\ &= \frac{1}{3}.\frac{n^2(n+1)^2}{4}+\frac{1}{2}.\frac{n(n+1)(2n+1)}{6}+\frac{1}{6}.\frac{n(n+1)}{2}\\ &= \frac{n(n+1)}{12}\{n(n+1)+(2n+1)+1\}\\ &= \frac{n(n+1)}{12}(n^2+3n+2)=\frac{n(n+1)^2(n+2)}{12}. \end{aligned}$$

Example 4. Sum the series

$$3.7+5.10+7.13+9.16+\cdots n \text{ terms.} \qquad [\text{ D. B. } 1936]$$

The nth term of the series evidently $=(2n+1)(3n+4)$

$$=6n^2+11n+4.$$

Hence, putting $n=1$, the 1st term $=6.1^2+11.1+4$,

„ „ $n=2$, „ 2nd term $=6.2^2+11.2+4$,

„ „ $n=3$, „ 3rd term $=6.3^2+11.3+4$,

and so on.

Hence, if S denote the sum of the given series, we have

$$S=6(1^2+2^2+3^2+\cdots n \text{ terms})+11(1+2+3+4+\cdots n \text{ terms})+4n$$

$$=\frac{6.n(n+1)(2n+1)}{6}+\frac{11.n(n+1)}{2}+4n$$

$$=n(n+1)(2n+1)+\frac{11}{2}n(n+1)+4n$$

$$=n\left\{2n^2+3n+1+\frac{11}{2}n+\frac{11}{2}+4\right\}$$

$$=n\left(2n^2+\frac{17}{2}n+\frac{21}{2}\right)=\frac{n}{2}(4n^2+17n+21).$$

EXERCISE 142

Sum the series :

1. $2^2+5^2+8^2+$ &c. to n terms.
2. $1.2^2+2.3^2+3.4^2+$ &c. to n terms.
3. $1.3+3.5+5.7+7.9+$ &c. to n terms.
4. $2.3+3.4+4.5+$ &c. to n terms. [E. B. S. B. 1949]
5. $3\times 8+6\times 11+9\times 14+$ &c. to nth term. [W. B. S. F. 1954 (Suppl.)]
6. $2.3.4+3.4.5+4.5.6+5.6.7+\cdots n$ terms.
7. $1\times 3^2+2\times 4^2+3\times 5^2+\cdots\cdots$ 100th term. [W. B. S. F. 1954]
8. $1^3+3^3+5^3+$ &c. to n terms.
9. $1+(1+2)+(1+2+3)+$ &c. to n terms.
10. $(1)+(1+3)+(1+3+5)+$ &c. to n terms.
11. $1.2.3+2.3.4+3.4.5+$ &c. to n terms.
12. $2.3.1+3.4.4+4.5.7+$ &c. to n terms.
13. $1-2+3-4+5-6+$ &c. to n terms.
14. $1^2-2^2+3^2-4^2+5^2-6^2+$ &c. to n terms.

281. Miscellaneous Examples and Problems.

Example 1. Prove that if the number of terms of an A. P. be *odd*, twice the middle term is equal to the sum of the first and last terms.

Since the number of terms is odd, let it be denoted by $2n+1$.

Evidently, the middle term is one which has n terms on either side of it; hence, it is the $(n+1)$th term from the beginning and also the $(n+1)$th term from the end.

Hence, putting M for the middle term, we must have

$$M=a+(\overline{n+1}-1)b=a+nb \qquad \cdots \qquad \cdots \quad (1)$$

and also $$M=l-(\overline{n+1}-1)b=l-nb. \qquad \cdots \qquad \cdots \quad (2)$$

Hence, by addition, $2M=a+l$.

Example 2. Prove that the sum of an odd number of terms in A. P. is equal to the middle term multiplied by the number of terms.

Let $2n+1=$ the number of terms.

Then, the sum of the terms

$$=\frac{2n+1}{2}(a+l)=\frac{2n+1}{2}\times 2M \quad [\text{ last example }]$$

$$=(2n+1)\times M.$$

Example 3. Find the first five terms of the series of which the sum to n terms $=5n^2+3n$.

Let t_1, t_2, t_3, &c., t_n denote respectively the 1st, 2nd, 3rd, &c., nth terms of the series;

and let s_1, s_2, s_3, &c., s_n denote respectively the sums of 1, 2, 3, &c., n terms of the series.

Evidently then $s_1=t_1$; $s_2=t_1+t_2$; $s_3=t_1+t_2+t_3$; and so on.

Now, by the question, we have $s_n=5n^2+3n$, (*i.e.*, the sum of *any number* of terms $=5$ times the square of *that number* $+3$ times *that number*).

Hence, putting $n=1$, we have $s_1=5+3=8$,
" $n=2$, " $s_2=20+6=26$,
" $n=3$, " $s_3=45+9=54$,
" $n=4$, " $s_4=80+12=92$,
" $n=5$, " $s_5=125+15=140$, and so on.

Hence, $t_1=s_1=8$,
$t_2=s_2-s_1=26-8=18$,
$t_3=s_3-s_2=54-26=28$,
$t_4=s_4-s_3=92-54=38$,
$t_5=s_5-s_4=140-92=48$, and so on.

Thus, the first five terms of the series are 8, 18, 28, 38 and 48.

Example 4. Sum the series : $1+5+12+22+35+$ &c. to n terms.

[The peculiarity of the series is that the successive differences of the terms are in A. P.]

Let S denote the required sum and let t_n denote the nth term of the series. Then, we have

$$S=1+5+12+22+\cdots+t_n\,;$$

also $$S=0+1+\ 5+12+\cdots+t_{n-1}+t_n.$$

Hence, by subtraction,

$$0=1+4+7+10+\text{\&c.}+(t_n-t_{n-1})-t_n$$
$$=\{1+4+7+10+\text{\&c. to } n \text{ terms}\}-t_n.$$

$$\therefore\quad t_n=\frac{n}{2}\{2+(n-1)3\}=\frac{n(3n-1)}{2},$$

i.e., the nth term of the given series $=\frac{3}{2}.n^2-\frac{1}{2}n$.

Hence, the 1st term $=\frac{3}{2}.1^2-\frac{1}{2}.1$,

2nd " $=\frac{3}{2}.2^2-\frac{1}{2}.2$,

3rd " $=\frac{3}{2}.3^2-\frac{1}{2}.3$, and so on.

Hence, $$S=\frac{3}{2}(1^2+2^2+3^2+\text{\&c.}+n^2)-\frac{1}{2}(1+2+3+\text{\&c.}+n)$$
$$=\frac{3}{2}.\frac{n(n+1)(2n+1)}{6}-\frac{1}{2}.\frac{n(n+1)}{2}=\frac{n(n+1)}{4}.2n=\frac{n^2(n+1)}{2}.$$

Example 5. Sum the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$ &c. to n terms.

Let S denote the sum to n terms.

Now, we have

$$t_1=\frac{1}{1.2}=1-\frac{1}{2},$$
$$t_2=\frac{1}{2.3}=\frac{1}{2}-\frac{1}{3},$$
$$t_3=\frac{1}{3.4}=\frac{1}{3}-\frac{1}{4},$$

&c., &c., &c.,

$$t_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

Hence, $$S=1-\frac{1}{n+1}=\frac{n}{n+1}.$$

Example 6. Divide 15 into three parts which are in A. P. and whose product $=120$.

Let $\alpha-\beta$, α and $\alpha+\beta$ be the numbers ;

then, we have

$$\left.\begin{aligned}(\alpha-\beta).\alpha.(\alpha+\beta)&=120 \quad\cdots\quad (1)\\ \text{and}\quad (\alpha-\beta)+\alpha+(\alpha+\beta)&=\ 15. \quad\cdots\quad (2)\end{aligned}\right\}$$

From (2), $3a=15$. $\therefore\ a=5$.
From (1), $a(a^2-\beta^2)=120$.
$\therefore$ $5.(25-\beta^2)=120$. $\therefore\ 25-\beta^2=24$.
$\therefore\ \beta^2=1$. $\therefore\ \beta=\pm 1$.

Hence, the numbers are 4, 5, 6.

Example 7. If a^2, b^2, c^2 be in A. P., then

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.}$$

Evidently $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.,

$$\text{if } \frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a};$$

i.e., $\text{if } \frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$,

i.e., if $(b-a)(b+a)=(c-b)(c+b)$,

i.e., if $b^2-a^2=c^2-b^2$;

but, this is true by hypothesis.

$$\therefore\ \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.}$$

Example 8. If a, b and c be respectively the pth, qth and rth terms of an A. P., prove that $a(q-r)+b(r-p)+c(p-q)=0$.

Let α denote the first term and β the common difference of the A. P., of which a, b and c are the pth, qth and rth terms; then, we must have

$$\left.\begin{aligned} a&=\alpha+(p-1)\beta \quad \cdots \quad (1)\\ b&=\alpha+(q-1)\beta \quad \cdots \quad (2)\\ c&=\alpha+(r-1)\beta \quad \cdots \quad (3)\end{aligned}\right\}$$

Now, we have to eliminate α and β from these three equations.

Subtracting (2) from (1), and (3) from (2), we have

$$a-b=(p-q)\beta,$$
$$b-c=(q-r)\beta.$$

Hence, $(a-b)(q-r)=(b-c)(p-q)$,

or, $a(q-r)+b(r-p)+c(p-q)=0$.

Example 9. A person lends Rs. 1000 to a friend agreeing to charge no interest and also to recover the amount by monthly instalments decreasing successively by Rs. 2. In how many months will the loan be paid up, if the first instalment be Rs. 64 ? [C. U. 1920]

Let n = the number of months required,

the successive instalments are evidently in A. P.

whose 1st term = 64,

and whose common difference = − 2.

Since, the sum of the n instalments = Rs. 1000, the sum of the 1st n terms of this A. P. = 1000,

$$i.e., \quad \frac{n}{2}\{2\times 64+(n-1)(-2)\}=1000,$$

$$\text{or,} \quad (65n-n^2)=1000,$$

$$\text{or,} \quad n^2-65n+1000=0,$$

$$\text{or,} \quad (n-25)(n-40)=0.$$

Hence, $n=25$, or, 40

But n cannot be 40, since in that case the 40th instalment

= the 40th term of the A. P.

$=64+(-2)(40-1)=-14,$

which is inadmissible, as no instalment can be negative

∴ n must be 25.

EXERCISE 143

1. The $(n+1)$th term of a series in A. P. is $\dfrac{ma-nb}{a-b}$; required the sum of the series to $(2n+1)$ terms.

2. Find the first five terms of the series of which the sum to n terms is $2n^2+7n$.

3. The sum to n terms of an A. P. is $3n^2+10n$; find the first term and the common difference.

4. Find the 35th term of the series of which the sum to n terms is n^2+n.

5. Sum the series : $1+3+6+10+15+$ &c. to n terms.

6. Sum the series : $2+5+10+17+$ &c. to n terms.

7. Sum the series : $2+7+14+23+34+$ &c. to n terms.

8. Sum the series : (i) $\dfrac{1}{3.5}+\dfrac{1}{5.7}+\dfrac{1}{7.9}+$ &c. to n terms.

(ii) $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dfrac{1}{5.7}+$ &c. to n terms. [W. B. S. F. 1953]

(iii) $\dfrac{1}{a(a+b)}+\dfrac{1}{(a+b)(a+2b)}+\dfrac{1}{(a+2b)(a+3b)}+$ &c. to n terms.

9. Find 4 numbers in A. P., such that their sum shall be 56, and the sum of their squares 864.

[Let $\alpha-3\beta$, $\alpha-\beta$, $\alpha+\beta$ and $\alpha+3\beta$ be the numbers.]

10. Divide 36 into three parts which are in A. P., and whose product $=1536$.

11. The sum of three numbers in A. P. is 15, and the sum of the squares of the two extremes is 58. What are the numbers ?

12. There are four numbers in A. P., the sum of the two extremes is 8, and the product of the means is 15. What are the numbers ?

13. Find six numbers in A. P., such that the sum of the two extremes may be 16 and the product of the two middle terms 63.

[Let $\alpha-5\beta$, $\alpha-3\beta$, $\alpha-\beta$, $\alpha+\beta$, $\alpha+3\beta$, $\alpha+5\beta$ be the numbers.]

14. If $(b-c)^2$, $(c-a)^2$, $(a-b)^2$ are in A. P., show that

$$\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b} \text{ are in A. P.}$$

15. If a, b, c be in A. P., show that

(1) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A. P. (2) $b+c$, $c+a$, $a+b$ are in A. P.

(3) $a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are in A. P.

(4) $\frac{1}{a}\left(\frac{1}{b}+\frac{1}{c}\right), \frac{1}{b}\left(\frac{1}{c}+\frac{1}{a}\right), \frac{1}{c}\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A. P.

(5) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A. P.

(6) $(a+2b-c)(2b+c-a)(c+a-b)=4abc$. [D. B. 1931]

16. If a, b and c be respectively the sums of p, q and r terms of an A. P., prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$$

17. The pth term of an A. P. is a and the qth term, b. Show that the sum of the first $(p+q)$ terms is

$$\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}.$$ [M. U. 1887]

[See Example 2, Art. 275]

18. There are n Arithmetic means between 3 and 54, such that the 8th mean : $(n-2)$th mean $=3:5$; find n.

19. If S_1, S_2, S_3 be the sums of n terms of three Arithmetic series the first term of each being 1 and the respective common difference 1, 2, 3, prove that $S_1 + S_3 = 2S_2$.

20. If there be r Arithmetic Progressions, each beginning from unity, whose common differences are 1, 2, 3, &c., r, show that the sum of their nth terms is $= \frac{1}{2}\{(n-1).r^2 + (n+1).r\}$.

21. Sum the series : $n.1 + (n-1).2 + (n-2).3 + (n-3).4 + \&c. + 1.n$.

[The rth term of the series $= \{n - (r-1)\}.r = (n+1)\,r - r^2$. Hence, the required sum $= (n+1)(1+2+3+\cdots+n) - \{1^2+2^2+3^2+\cdots+n^2\} = \&c.$]

22. On the ground are placed n stones ; the distance between the first and second is one metre, between the 2nd and 3rd three metres, between the 3rd and 4th five metres, and so on. How far will a person have to travel who shall bring them, one by one, to a basket placed at the first stone ?

23. A class consists of a number of boys whose ages are in A. P., the common difference being four months. If the youngest boy is just eight years old, and if the sum of the ages is 168 years, find the number of boys in the class. [C. U. Entr. Paper, 1872]

24. The interior angles of a rectilineal figure are in A. P. If the least angle is 42° and the common difference is 33°, find the number of sides.

25. If sums of the first p, q and r terms of an A. P. are x, y and z respectively, prove that $xqr(q-r) + yrp(r-p) + zpq(p-q) = 0$.

CHAPTER XXXVIII

GEOMETRICAL PROGRESSION

282. Definition. Quantities are said to be in Geometrical Progression when each is equal to the product of the preceding and some **constant factor.**

The constant factor is called the **common ratio** of the series, and it is found by dividing *any* term by that which immediately *precedes it.*

Thus, each of the following series forms a Geometrical Progression :

$$\begin{array}{llllll} 1, & 2, & 4, & 8, & 16, & \&c. \\ 1, & \frac{1}{2}, & \frac{1}{4}, & \frac{1}{8}, & \frac{1}{16}, & \&c. \\ 1, & -\frac{1}{3}, & \frac{1}{9}, & -\frac{1}{27}, & \frac{1}{81}, & \&c. \\ a, & ar, & ar^2, & ar^3, & ar^4, & \&c. \end{array}$$

In the first example the common ratio is 2, in the second $\frac{1}{2}$, in the third $-\frac{1}{2}$, and in the fourth r.

N. B. 'Geometrical Progression' is briefly written as G. P.

283. To find the *n*th term of a G. P.

If a be the first term and r the common ratio of a Geometric series, we have the 2nd term $=a.r$, the 3rd term $=a.r^2$, the 4th term $=a.r^3$,……, the 10th term $=a.r^9$,……, the 21st term $=a.r^{20}$, and so on. Hence, the nth term $=a.r^{n-1}$.

Example. Find the 6th term of the series 2, 6, 18, 54, &c.

Here, $a=2$ and the common ratio $=\frac{6}{2}=3$;

$\therefore$ the 6th term $=2\times(3)^{6-1}=486$.

284. Given any two terms of a G. P., to find the series completely.

Example 1. Find the G. P. whose 5th term is 81 and whose 8th term is 2187.

Let $a=$ the 1st term, and $r=$ the common ratio.

$$\therefore \quad 81=a.r^{5-1}=ar^4, \quad \cdots \quad \cdots \quad (1)$$

$$\text{and} \quad 2187=ar^{8-1}=ar^7. \quad \cdots \quad \cdots \quad (2)$$

Dividing (2) by (1), $r^3=\frac{2187}{81}=27$. $\therefore$ $r=3$.

Hence, $ar^4=a.3^4=81$,

$$\text{or,} \quad a=\frac{81}{3^4}=1.$$

Thus, the series is 1, 3, 9, 27, &c.

Example 2. If c and d be the pth and qth terms respectively of a G. P., to determine it completely.

Let $a=$ the 1st term, and $r=$ the common ratio.

$$\therefore \quad c = \text{the } p\text{th term of the G. P.}$$

$$=ar^{p-1}. \quad \cdots \quad (1)$$

Similarly, $d=ar^{q-1}$. $\cdots$ (2)

By division, $r^{q-p}=\frac{d}{c}$; $\therefore$ $r=\left(\frac{d}{c}\right)^{\frac{1}{q-p}}$.

Substituting for r in (1), we have

$$a=\frac{c}{r^{p-1}}=\frac{c}{\left(\frac{d}{c}\right)^{\frac{p-1}{q-p}}}=\left(\frac{c^{q-1}}{d^{p-1}}\right)^{\frac{1}{q-p}}.$$

Hence, the 1st term and the common ratio being known, the complete series may be written down.

EXERCISE 144

1. Find the 8th term of the series 4, 12, 36, &c.

2. Find the 6th term of the series $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, &c.

3. Find the 9th term of the series 1, 4, 16, 64, &c.

4. Find the 6th term of the series 1, -3, 9, -27, &c.

5. Find the 5th term and the $(n-1)$th term of the series $\frac{2}{3}$, -1, $\frac{3}{2}$, &c.

6. Find the 7th term of the series -21, 14, $-9\frac{1}{3}$, &c.

7. Find the nth term of the series $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\cdots$

8. The first two terms of a series in G. P., are 125 and 25, what are the 6th and 7th terms ?

9. Find the series (i) whose 6th and 11th terms are respectively 192 and 6144 ; (ii) whose 2nd and 8th terms are 9 and $\frac{1}{81}$ respectively ; (iii) whose 5th and 8th terms are 8 and $-\frac{64}{27}$ respectively.

10. The pth and the qth terms of a G. P., are c and d respectively. Find the nth term.

11 *If every term of a G. P. is multiplied or divided by the same quantity, the resulting series is also a G. P.*

12 In a G. P., if the $(p+q)$th term $=m$ and the $(p-q)$th term $=n$, find the pth and qth terms. [B. U. 1888]

13. In a G. P., prove that the product of any pair of terms equidistant from the beginning and the end is constant.

14. There are $2n$ terms in a Geometric series. Prove that the product of the first and last terms is equal to the product of the two middle terms.

285. To find the sum of a number of terms in Geometrical Progression.

Let a be the first term, r the common ratio, n the number of terms and S the sum required ; then

$$S=a+ar+ar^2+ar^3+\&c.+ar^{n-1}.$$

$$\therefore\ Sr=\quad ar+ar^2+ar^3+\&c.+ar^{n-1}+ar^n.$$

Hence, by subtraction,

$$Sr-S=ar^n-a. \qquad \therefore\ S(r-1)=a(r^n-1).$$

$$\therefore\quad S=\frac{a(r^n-1)}{r-1} \quad \cdots \quad \cdots \quad (1)$$

$$\text{or,}\quad =\frac{a(1-r^n)}{1-r}. \quad \cdots \quad \cdots \quad (2)$$

Cor. If l denote the last (or the nth) term of the series, we have $l = ar^{n-1}$; hence, from (1), $S = \dfrac{rl-a}{r-1}$. ... (3)

Note. *The formula (2) may conveniently be used in all cases except when r is positive and greater than 1.*

Example 1. Find the sum of $\frac{16}{27} - \frac{8}{9} + \frac{4}{3} - \&c.$ to 7 terms.

The common ratio $= -\frac{8}{9} \div \frac{16}{27} = -\frac{8}{9} \times \frac{27}{16} = -\frac{3}{2}$.

Hence, by formula (2), the sum $= \dfrac{\frac{16}{27}\{1-(-\frac{3}{2})^7\}}{1+\frac{3}{2}} = \dfrac{\frac{16}{27}\{1+\frac{2187}{128}\}}{\frac{5}{2}}$

$= \frac{16}{27} \times \frac{2315}{128} \times \frac{2}{5} = \frac{463}{108} = 4\frac{31}{108}$.

Example 2. Find the sum of $3 + 4\frac{1}{2} + 6\frac{3}{4} + \&c.$ to 5 terms.

The common ratio $= 4\frac{1}{2} \div 3 = \frac{9}{2} \times \frac{1}{3} = \frac{3}{2}$.

Hence, if S denote the required sum, we have by formula (1),

$$S = \frac{3.\{(\frac{3}{2})^5 - 1\}}{\frac{3}{2} - 1} = \frac{3.\{\frac{243}{32} - 1\}}{\frac{1}{2}} = 3 \times \frac{211}{32} \times 2 = \frac{633}{16} = 39\frac{9}{16}.$$

Example 3. Find, without the help of any formula, the sum of the series $1 + 5 + 25 + \&c.$ to 10 terms.

The common ratio $= 5$.

$\therefore$ the 10th term $= 1.5^9 = 5^9$.

Suppose S is the required sum.

$\therefore \quad S = 1 + 5 + 5^2 + \cdots + 5^9$ ... ... (1)

$\therefore \quad 5S = \quad 5 + 5^2 + \cdots + 5^9 + 5^{10}$ ... ... (2)

Subtracting (1) from (2),

$4S = 5^{10} - 1$.

$\therefore \quad S = \frac{1}{4}(5^{10} - 1) = \frac{1}{4}(9765625 - 1) = \frac{1}{4} \times 9765624 = 2441406$.

EXERCISE 145

1. Sum $1 + 3 + 9 + 27 + \&c.$ to 12 terms.
2. Sum $1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \dfrac{1}{3^3} + \&c.$ to n terms. [C. U. 1939 (Suppl.)]
3. Sum $81 - 27 + 9 - \&c.$ to 8 terms.
4. Sum $2 - 4 + 8 - \&c.$ to 10 terms.
5. Sum $\frac{4}{9} - \frac{1}{3} + \frac{1}{4} - \&c.$ to 5 terms.
6. Sum $2 - 4 + 8 - \&c.$ to $2r$ terms.
7. Sum $2\frac{1}{4} - 1 + \frac{4}{9} - \&c.$ to n terms.

8. Find without applying any formula the sum of

(i) The series $1+\dfrac{1}{2}+\dfrac{1}{2^2}+$ &c. to n terms.

[C. U. 1938 ; D. B. 1940]

(ii) The series $5+15+45+$ &c. to 8 terms.

9. Show that the sum of n terms of a G. P. beginning with the pth term, is r^{p-q} times the sum of an equal number of terms of the same series beginning with the qth term.

236. If n be an integer and r a given proper fraction, to prove that r^n diminishes as n increases.

Let $r=\frac{3}{7}$. Now, since $\frac{3}{7}$ of any number is undoubtedly less than that number,

$(\frac{3}{7})^2$ is less than $\frac{3}{7}$, because $(\frac{3}{7})^2=\frac{3}{7}$ of $\frac{3}{7}$;
$(\frac{3}{7})^3$ is less than $(\frac{3}{7})^2$, because $(\frac{3}{7})^3=\frac{3}{7}$ of $(\frac{3}{7})^2$;
$(\frac{3}{7})^4$ is less than $(\frac{3}{7})^3$, because $(\frac{3}{7})^4=\frac{3}{7}$ of $(\frac{3}{7})^3$;
and so on

Hence, it is clear that in the series $\frac{3}{7}, (\frac{3}{7})^2, (\frac{3}{7})^3, (\frac{3}{7})^4, \ldots\ldots$ each term is less than the preceding one ; which is briefly expressed by saying that $(\frac{3}{7})^n$ diminishes as n increases.

Similarly, the proposition may be proved for any other value of r which is less than 1.

Hence, generally speaking, if r has a given value less than 1, r^n diminishes as n increases.

Note. *From the above it is quite clear that if r be a proper fraction, r^n is very small when n is infinitely large.*

237. The sum of a Geometrical series continued to infinity.

Let us consider the series a, ar, ar^2, ar^3, &c.

If S denote the sum to n terms, we have

$$S=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}$$

If then r be a *proper fraction*, the larger n is, the smaller will be (r^n and $\therefore$) $\dfrac{ar^n}{1-r}$; hence by sufficiently increasing the value of n we can make $\dfrac{ar^n}{1-r}$ less than any assigned quantity, however small ; and therefore by ***sufficiently increasing the value of n, the sum of n terms of the series can be made to differ from $\dfrac{a}{1-r}$ by as small a quantity as we please.***

This statement is usually put thus : *the sum of an infinite number of terms of the Geometrical Progression is* $\frac{a}{1-r}$, or more briefly, *the sum to infinity is* $\frac{a}{1-r}$.

Let us apply all these remarks to a particular example.

Consider the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.

Here, $a=1$, $r=\frac{1}{2}$; hence the sum to n terms

$$=\frac{1}{1-\frac{1}{2}}\left(1-\frac{1}{2^n}\right)=2\left(1-\frac{1}{2^n}\right)=2-\frac{1}{2^{n-1}}.$$

Now, by taking n large enough, 2^{n-1} can be made as large as we please, and therefore, $\frac{1}{2^{n-1}}$ as small as we please.

Hence, we may say that *by taking n large enough, the sum of n terms of the series can be made to differ from 2 by as small a quantity as we please* ; or briefly, *the sum of an infinite number of terms of this series is 2.*

N. B. It must be borne in mind that the sum of n terms of a Geometrical Progression approaches a fixed limit as n increases indefinitely only when r is less than unity. If r be greater than unity there is no such fixed limit.

Example 1. Prove that in a decreasing Geometrical Progression continued to infinity each term bears a constant ratio to the sum of all which follow it.

Let the series be a, ar, ar^2, ar^3, &c., where r is less than unity.

Then, the nth term $=ar^{n-1}$ and the sum of all the terms which follow this

$$=ar^n(1+r+r^2+r^3+\text{\&c. to infinity})$$

$$=ar^n.\frac{1}{1-r}.$$

Hence, the ratio of the nth term to the sum of all which follow it

$$=\left(ar^{n-1}\div\frac{ar^n}{1-r}\right)=\frac{1-r}{r}.$$

Now, this is constant *whatever value n may have*, which proves the proposition.

Example 2. Sum to infinity $\frac{3}{2}-\frac{2}{3}+\frac{8}{27}-$ &c.

Here, $a=\frac{3}{2}$, and $r=-\frac{2}{3}\div\frac{3}{2}=-\frac{4}{9}$.

Hence, the required sum $=\frac{\frac{3}{2}}{1+\frac{4}{9}}=\frac{3}{2}\times\frac{9}{13}=\frac{27}{26}$.

EXERCISE 146

Sum to infinity each of the following series :

1. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ &c. 2. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$ &c. 3. $\frac{2}{3}+\frac{1}{2}+\frac{3}{8}+\frac{9}{32}+$ &c.

4. $1-\frac{2}{3}+\frac{4}{9}-$ &c. 5. $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+$ &c.

6. $\frac{2}{5}+\frac{3}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+$ &c. [Split this up into two series.]

7. $\frac{4}{7}+\frac{5}{7^2}+\frac{4}{7^3}+\frac{5}{7^4}+$ &c. 8. $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+$ &c.

9. $(\sqrt{2}+1)+1+(\sqrt{2}-1)+$ &c.

10. Find the common ratio of a G. P., continued to infinity in which each term is ten times the sum of all the terms which follow it.

288. Recurring Decimals. Recurring decimals furnish a good illustration of infinite Geometrical Progressions.

Thus, for example, $\cdot 2\dot{3}\dot{4}=\cdot 234343434\cdots\cdots$

$$=\left.\begin{array}{l} \cdot 2 \\ +\cdot 034 \\ +\cdot 00034 \\ +\cdot 0000034 \\ +\text{\&c., \&c.} \end{array}\right\} =\frac{2}{10}+\frac{34}{10^3}+\frac{34}{10^5}+\frac{34}{10^7}+\text{\&c.}$$

Here the terms after $\frac{2}{10}$ constitute a G. P. of which the first term is $\frac{34}{10^3}$ and the common ratio $\frac{1}{10^2}$.

Hence, we may take $\cdot 2\dot{3}\dot{4}=\frac{2}{10}+\frac{34}{10^3}+\left\{1-\frac{1}{10^2}\right\}=\frac{2}{10}+\frac{34}{990}=\frac{232}{990}$, which agrees with the value found by the usual Arithmetical rule.

289. Geometric means. Definition 1. When three quantities are in Geometrical Progression the middle one is called the **Geometric mean** between the other two.

Definition 2. When any number of quantities x_1, x_2, x_3, &c., are such that a, x_1, x_2, x_3, &c., b are in G. P., then x_1, x_2, x_3, &c., are called **Geometric means** between a and b.

(i) To find the Geometric means between two given quantities.

Let a and b be the two given quantities ; G the Geometric mean.

Then since, a, G, b are in G. P., we must have, $\frac{G}{a}=\frac{b}{G}$, each being equal to the common ratio. $\therefore$ $G^2=ab$, and $\therefore$ $G=\sqrt{ab}$.

(ii) To insert a given number of Geometric means between two given quantities.

Let a and b be the two given quantities; and x_1, x_2, x_3, x_4, &c., x_n, the n means to be inserted.

Then a, x_1, x_2, x_3, &c., x_n, b are in G. P.

Let r denote the common ratio of the series;

then $b=$ the $(n+2)$th term $=a.r^{n+1}$.

$$\therefore \quad r^{n+1}=\frac{b}{a}, \text{ and } \quad \therefore \quad r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

$$\text{Hence, } x_1=a.\left(\frac{b}{a}\right)^{\frac{1}{n+1}}; \quad x_2=a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}}; \quad x_3=a.\left(\frac{b}{a}\right)^{\frac{3}{n+1}}; \text{ and so on.}$$

Example. Insert 3 Geometric means between $\frac{1}{2}$ and 128.

Let x_1, x_2, x_3 be the means.

Then, $\frac{1}{2}, x_1, x_2, x_3$, 128 are in G. P.

Hence, if r be the common ratio of the series,

we must have $128=$ the 5th term $=\frac{1}{2}.r^4$.

$$\therefore \quad r^4=256, \text{ whence } r=4.$$

$$\text{Hence,} \quad \left.\begin{aligned} x_1&=\tfrac{1}{2}.4=2\\ x_2&=\tfrac{1}{2}.4^2=8\\ x_3&=\tfrac{1}{2}.4^3=32 \end{aligned}\right\}$$

290. *The Arithmetic mean of any two positive quantities is greater than their Geometric mean.*

Let a and b be two positive quantities.

$\therefore$ their Arithmetic mean $=\frac{a+b}{2}$, and Geometric mean $=\sqrt{ab}$.

$$\text{Now, } \frac{a+b}{2}-\sqrt{ab}=\tfrac{1}{2}[a-2\sqrt{a}.\sqrt{b}+b]=\tfrac{1}{2}(\sqrt{a}-\sqrt{b})^2$$

$$=\text{a positive quantity.}$$

$$\therefore \quad \frac{a+b}{2}>\sqrt{ab}.$$

EXERCISE 147

1. Insert 2 Geometric means between 3 and 24.
2. Insert 3 Geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$.
3. Insert 4 Geometric means between $\frac{2}{3}$ and $-5\frac{1}{16}$.
4. Insert 5 Geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

5. What are the three Geometric means between 25 and 164025 ? [Pat. U. 1919]

6. If a, b and c be in G. P., and x, y be the Arithmetic means between a, b and b, c respectively, prove that

$$\frac{a}{x}+\frac{c}{y}=2 \text{ and } \frac{1}{x}+\frac{1}{y}=\frac{2}{b}.$$ [P. U. 1892]

7. The Arithmetic mean of a and b is to their Geometric mean as m to n; show that $a:b=m+\sqrt{m^2-n^2}:m-\sqrt{m^2-n^2}$. [A. U. 1889]

8. If the Arithmetic and Geometric means between two quantities be respectively A and B, prove that the quantities are

$$A+\sqrt{A^2-B^2} \text{ and } A-\sqrt{A^2-B^2}.$$

[Let the numbers be a and b. Suppose $a > b$.

$$\therefore\quad a+b=2A, \qquad \cdots \qquad \cdots \quad (1)$$

and $\sqrt{ab}=B$.

Now, $(a-b)^2=(a+b)^2-4ab=4(A^2-B^2)$,

or, $a-b=2\sqrt{A^2-B^2}$. ... ... (2)

(taking the positive root, since, $a > b$, i.e., $a-b$ is positive.)

Adding (1) and (2), $2a=2A+2\sqrt{A^2-B^2}$, or, $a=A+\sqrt{A^2-B^2}$.

Also, subtracting (2) from (1), $b=A-\sqrt{A^2-B^2}$.]

291. Miscellaneous Series and Examples.

Example 1. If $x < 1$, sum the series

$$1+2x+3x^2+4x^3+\&c., \text{ to infinity.}$$

Let S denote the required sum ; then

$$S=1+2x+3x^2+4x^3+\&c.$$

and $\therefore$ $Sx= \quad x+2x^2+3x^3+\&c.$

Hence, by subtraction,

$$S(1-x)=1+x+x^2+x^3+\&c., \text{ to infinity}$$

$$=\frac{1}{1-x}.$$

$$\therefore\quad S=\frac{1}{(1-x)^2}.$$

Example 2. Sum to n terms $5+55+555+\text{\&c.}$

Let S denote the required sum ; then

$$\begin{aligned}S&=5+55+555+\text{\&c.}\quad\text{to } n \text{ terms}\\&=5\{1+11+111+\text{\&c.}\quad\text{to } n \text{ terms}\}\\&=\tfrac{5}{9}\times 9\{1+11+111+\text{\&c.}\ \text{to } n \text{ terms}\}\\&=\tfrac{5}{9}\{9+99+999+\text{\&c.}\quad\text{to } n \text{ terms}\}\\&=\tfrac{5}{9}\{(10-1)+(10^2-1)+(10^3-1)+\text{\&c. to } n \text{ terms}\}\\&=\tfrac{5}{9}\{(10+10^2+10^3+\text{\&c.}\ \ \text{to } n \text{ terms})-n\}\\&=\frac{5}{9}\left\{\frac{10(10^n-1)}{10-1}-n\right\}=\frac{50}{81}\left(10^n-1\right)-\frac{5n}{9}.\end{aligned}$$

Example 3. Sum to n terms $1+5+13+29+\text{\&c.}$

Let t_n denote the nth term of the series, and S the required sum ; then

$$S=1+5+13+29+\cdots+t_n\,;$$

and $$S=0+1+\ 5+13+\cdots+t_{n-1}+t_n.$$

Therefore, by subtraction,

$$0=(1+4+8+16+\text{\&c. to } n \text{ terms})-t_n.$$

$$\begin{aligned}\therefore\quad t_n&=1+\{4+8+16+\text{\&c. to } (n-1) \text{ terms}\}\\&=1+\frac{4(2^{n-1}-1)}{2-1}\\&=1+2^2.(2^{n-1}-1)=2^{n+1}-3.\end{aligned}$$

Hence, the 1st term $=2^2-3$,
" 2nd " $=2^3-3$,
" 3rd " $=2^4-3$
and so on.

Hence,
$$\begin{aligned}S&=(2^2-3)+(2^3-3)+(2^4-3)+\text{\&c.}+(2^{n+1}-3)\\&=(2^2+2^3+2^4+\text{\&c. to } n \text{ terms})-3n\\&=\frac{2^2\,(2^n-1)}{2-1}-3n\\&=4(2^n-1)-3n.\end{aligned}$$

Example 4. If a, b, c, d be in G. P., show that
$$(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2.$$

We have $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$, each of them being equal to the common ratio ; $\therefore\ b^2=ac,\ c^2=bd$, and $bc=ad$. $\cdots$ (a)

Hence, $(b-c)^2+(c-a)^2+(d-b)^2$

$$=(b^2+c^2-2bc)+(c^2+a^2-2ca)+(d^2+b^2-2db)$$
$$=2(b^2-ac)+2(c^2-bd)+a^2+d^2-2bc$$
$$=2\times0+2\times0+a^2+d^2-2ad. \qquad [\text{ by } a\ ]$$
$$=(a-d)^2.$$

Example 5. If a, b, c, d be in G. P., shew that
$a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G. P.

Evidently $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G. P.,
if $(a^2-b^2)(c^2-d^2)=(b^2-c^2)^2$.

Now, since a, b, c, d are in G. P., we have $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$.

$\therefore$ $ac=b^2$, $bd=c^2$ and $ad=bc$.

Hence, $(a^2-b^2)(c^2-d^2)=a^2c^2-b^2c^2-a^2d^2+b^2d^2$

$$=b^4-b^2c^2-b^2c^2+c^4$$
$$=b^4-2b^2c^2+c^4=(b^2-c^2)^2.$$

$\therefore$ $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G. P.

Example 6. If p, q, r be in A. P., prove that the pth, qth, and rth terms of any Geometric series form a Geometric series.
[W. B. S. F. 1952]

Suppose the 1st term of the Geometric series $=a$ and the common ratio $=R$.

$\therefore$ the pth term $=aR^{p-1}$, qth term $=aR^{q-1}$ and rth term $=aR^{r-1}$, aR^{p-1}, aR^{q-1} and aR^{r-1} will be in G. P., if $(aR^{q-1})^2=aR^{p-1}.aR^{r-1}$, *i.e.*, $a^2R^{2q-2}=a^2R^{p+r-2}$, *i.e.*, if $2q-2=p+r-2$,

or, $2q=p+r$.

Now, $2q=p+r$, if p, q, r are in A. P.

Example 7. The continued product of three numbers in G. P. is 216, and the sum of the products of them in pairs is 156; find the numbers.

Let $\frac{a}{r}$, a, ar be the numbers;

then by the conditions given, we must have

$$\left.\begin{aligned} &\frac{a}{r}\cdot a.ar=216 \qquad \cdots \ (1)\\ \text{and}\quad &\frac{a}{r}\cdot a+\frac{a}{r}\cdot ar+a.ar=156 \qquad \cdots \ (2)\end{aligned}\right\}$$

From (1), $a^3=216$. $\therefore$ $a=6$.

Hence, from (2), $\frac{1}{r}+1+r=\frac{156}{36}=\frac{13}{3}$ $\therefore$ $3(1+r+r^2)=13r$,

or, $(3r^2-10r+3)=0$, or, $(r-3)(3r-1)=0$.

$\therefore$ $r=3$, or, $\frac{1}{3}$.

Hence, the numbers are 2, 6, 18.

EXERCISE 148

1. Find by the method of summation of infinite Geometric series the values of :

(i) $\cdot 02\dot{7}$; (ii) $1\cdot 1\dot{4}\dot{5}$; (iii) $\cdot 21\dot{5}0\dot{1}$; (iv) $\cdot\dot{1}4285\dot{7}$

2. Sum $1+3x+5x^2+7x^3+$ &c. to infinity.

3. Sum $1.2x+2.4x^2+3.8x^3+$ &c. to infinity.

4. Sum $1.3x+4.9x^2+7.27x^3+$ &c. to infinity.

5. Sum $a+2a^2+3a^3+4a^4+$ &c. to n terms.

6. Sum $1-3x+5x^2-7x^3+$ &c. to infinity.

7. Sum $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+$ &c. to infinity.

8. Sum $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+$ &c. to n terms.

9. Find the nth term, and the sum to n terms of the series .
1.1, 2.3, 4.5, 8.7, &c.

10. Sum $1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+$ &c. to n terms.

11. Sum to n terms $4+44+444+$ &c.

12. Sum the series $\cdot 9+\cdot 99+\cdot 999+$ &c. to n terms.

13. Sum the series $1+3+7+15+$ &c. to n terms.

14. Sum to n terms $-6-4+0+8+24+$ &c.

15. Find the sum of $6+9+21+69+261+$ &c. to n terms.

16. Find the sum of $(1)+(1+3)+(1+3+3^2)+(1+3+3^2+3^3)+\cdots$ to n terms. [C. U. 1931]

17. If a, b, c, d be in G. P., show that

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2.$$

[We have $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=k$ (say) ;

thus, $a=bk$, $b=ck$, $c=dk$,

hence $a^2+b^2+c^2=k^2(b^2+c^2+d^2)$,

and also $a^2+b^2+c^2=k(ab+bc+cd)$.]

18. If a, b, c, d are in G. P., prove that

(i) $(b+c)(b+d)=(c+a)(c+d)$;

(ii) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$;

(iii) $a^2+b^2,\ b^2+c^2,\ c^2+d^2$ are in G. P. [C. U. 1919]

19. Three numbers whose sum is 15 are in A. P.; if 1, 4 and 19 be added to them respectively, the results are in G. P. Determine the numbers.

[Let $\alpha-\beta,\ \alpha,\ \alpha+\beta$ be the numbers.]

20. Three numbers whose product is 512 are in G. P.; if 8 be added to the first and 6 to the second, the numbers are in A. P. Find the numbers.

21. The sum of three quantities in G. P. is $24\frac{1}{3}$, and their product is 64; find them.

22. If a, b, c be respectively the pth, qth and rth terms of a Geometric series, prove that $a^{q-r}b^{r-p}c^{p-q}=1$.

23. If a, b, c be in A. P. and x, y, z in G. P., prove that $x^{b-c}y^{c-a}z^{a-b}=1$.

24. If a, b, c be in G. P., prove that $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A. P.

[D B. 1946 ; G. U. 1948]

25. The 1st term and nth term of a Geometric series are a and l respectively and the product of the first n terms of the series is P. Prove that $P=(al)^{\frac{n}{2}}$. [C. U. 1918 ; D. B. 1943]

26. If S be the sum, P the product and R the sum of the reciprocals of n terms in G. P., prove that $P^2=\left(\frac{S}{R}\right)^n$.

27. Find the sum of n terms of the series, the rth term of which is $(2r+1)2^r$.

28. If $A=1+r^a+r^{2a}+\cdots$ to infinity and $B=1+r^b+r^{2b}+\cdots$ to infinity, prove that $r=\left(\frac{A-1}{A}\right)^{\frac{1}{a}}=\left(\frac{B-1}{B}\right)^{\frac{1}{b}}$.

29. If there be n terms in G. P., prove that the nth root of their product is equal to the square root of the product of the first and last terms.

30. If n Geometrical means be found between two quantities a and c, show that their product will be $(ac)^{\frac{n}{2}}$.

31. If a, b, c, d are in G. P., show that the reciprocals of a^2-b^2, b^2-c^2, c^2-d^2 are also in G. P.

32. If S_1, S_2, S_3, &c., S_n are the sums of infinite Geometric series, whose first terms are 1, 2, 3, &c., n, and whose common ratios are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., $\frac{1}{n+1}$ respectively, prove that

$$S_1+S_2+S_3+\&c.+S_n=\frac{n}{2}(n+3).$$

33. Find the sum of the infinite series—

$1+(1+a)r+(1+a+a^2)r^2+(1+a+a^2+a^3)r^3+$ &c., r and a being proper fractions.

CHAPTER XXXIX

VARIATION

292. Definition. One quantity is said to *vary directly as* another when the two quantities are so related that if one of them be changed, the other is changed *in the same ratio*; or, in other words, if a, a' be *any two* values of a quantity A, and b, b' the *corresponding* values of a second quantity B, then A is said to vary directly as B when $a : a' = b : b'$.

For instance, suppose the measure of the area of a triangle is a, when that of the base is b; now if the height remaining unchanged, the base is increased to $2b$, then as we know from Geometry the area will become $2a$; if the base becomes $3b$, the area will be $3a$; and so on. Thus, the height remaining the same if the base is doubled, trebled, quadrupled, &c., the area also becomes doubled, trebled, quadrupled, &c., (*i.e.*, the area changes *in the same ratio* as the base) and so we say that if the height of a triangle remains unaltered, the area *varies directly* as the base.

Note 1. *The word **directly** is often omitted, so that when we say A varies as B, it is implied that A varies directly as B.*

Note 2. *The symbol $\propto$ is used to express variation: thus, $A \propto B$ stands for "A varies as B".*

293. If A varies as B, then the numerical measure of *any* value of A and that of the *corresponding* value of B are in a constant ratio.

Let a_1, a_2, a_3, &c., be the measures of a series of values of A, and let b_1, b_2, b_3, &c., be the measures of the corresponding values of B.

Then, by definition, $\frac{a_1}{a_2}=\frac{b_1}{b_2}$; $\frac{a_2}{a_3}=\frac{b_2}{b_3}$; $\frac{a_3}{a_4}=\frac{b_3}{b_4}$; and so on.

Hence, $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\frac{a_4}{b_4}=$&c., which proves the proposition.

Note. *Putting m for each of the above ratios, we have* $a_1=mb_1$, $a_2=mb_2$, $a_3=mb_3$, *and so on. Thus, when A varies as B, the numerical measure of any value of A is equal to that of the corresponding value of B multiplied by a constant. This result is briefly expressed as follows: "If* $A \propto B$, *then* $A=mB$, *where m is a constant."*

294. Definition. (1) One quantity A is said to *vary inversely* as another quantity B, when A varies directly as the reciprocal of B.

Thus, if A varies inversely as B, $A=\frac{m}{B}$, where m is constant.

Illustration: If 20 men do a certain work in 4 hours, 10 men would do it in 8 hours; 40 men in 2 hours; and so on. Thus, when the number of men *diminishes*, the time *proportionally increases* and *vice versa*. This is expressed by saying that if the amount of work to be done remains constant, the number of men varies inversely as the time.

(2) One quantity is said to *vary jointly* as a number of others when it varies directly as their product. Thus, if A varies jointly as B and C, $A=m.BC$, where m is constant.

Illustration: The monthly income of a day labourer varies jointly as his daily earning and the number of days he works in a month.

(3) A is said to vary directly as B and inversely as C when A varies jointly as B and the reciprocal of C, that is, when $A=m.\frac{B}{C}$, where m is constant.

Illustration. The time of travelling a distance varies directly as the distance and inversely as the speed of travelling.

295. An Important Theorem.

If A varies as B when C is constant, and A varies as C when B is constant, then will A vary as BC when both B and C vary.

Suppose a_1 is the value of A when b_1 is that of B, and c_1 that of C. Suppose also that a_2 is the value of A when b_2 is that of B and c_2 that of C. Then the proposition will be proved if we can show that $a_1 : a_2=b_1c_1 : b_2c_2$.

Now, the change of A from a_1 to a_2 is due to *two* causes, namely,

(1) the change of B from b_1 to b_2 and (2) the change of C from c_1 to c_2.

Hence, it is clear that if *one* only of these causes be present (*i.e.*, if either B or C *alone* undergoes the supposed change), A will change from a_1 to some value which is *different* from a_2. Let, therefore, a' be the value of A when b_2 is that of B, and c_1 that of C.

Thus, we have the value of A

$=a_1$ when those of B and C are respectively b_1 and c_1 ... (1)

$=a'$ when those of B and C are respectively b_2 and c_1 ... (2)

$=a_2$ when those of B and C are respectively b_2 and c_2 ... (3)

Hence, from (1) and (2), we see that A changes from a_1 to a', when B changes from b_1 to b_2, C *remaining constant* (*i.e.*, retaining the value c_1), and, therefore, by hypothesis,

$$\frac{a_1}{a'}=\frac{b_1}{b_2}, \quad \ldots \quad \ldots \quad (\alpha)$$

and from (2) and (3), we see that A changes from a' to a_2 when C changes from c_1 to c_2, B *remaining constant* (*i.e.*, retaining the value b_2.) and, therefore, by hypothesis,

$$\frac{a'}{a_2}=\frac{c_1}{c_2}. \quad \ldots \quad \ldots \quad (\beta)$$

Hence, from (α) and (β),

$$\frac{a_1}{a'}\times\frac{a'}{a_2}=\frac{b_1}{b_2}\times\frac{c_1}{c_2}, \text{ or, } \frac{a_1}{a_2}=\frac{b_1c_1}{b_2c_2}$$ which proves the proposition.

Illustration : (1) Suppose that a number of plants have to be watered ; the quantity of water supplied for watering evidently varies directly as the number of men employed *if the time for watering remains unchanged* ; and also it varies directly as the number of hours for which the men can work, *if the number of men engaged remain the same* ; hence, if the number of men and the number of hours be both variable, the quantity of water will vary as the product of the number of men and the number of hours.

(2) The area of a triangle varies directly as the base when the height is constant, and it also varies directly as the height when the base is constant ; hence when both the base and the height are variable, the area varies as the product of the numbers which express the base and the height.

Cor. If there be any number of quantities B, C, D, &c., each of which varies as another quantity A, when the rest are constant ; then if they are all variable, A varies as their product.

296. Some results worth remembering.

(1) If $A \propto B$ and $B \propto C$, then $A \propto C$.

For, let $A=mB$, and $B=nC$, where m and n are constants ; then $A=mnC$; and $\therefore$ as mn is constant, $A \propto C$.

(2) If $A \propto C$, and $B \propto C$, then $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

For, let $A = mC$, and $B = nC$, where m and n are constants; then $A + B = (m+n)C$, and $A - B = (m-n)C$. $\therefore (A \pm B) \propto C$.

Also $\sqrt{AB} = \sqrt{mnC^2} = C\sqrt{mn}$. $\therefore \sqrt{AB} \propto C$.

(3) If $A \propto BC$, then $B \propto \dfrac{A}{C}$, and $C \propto \dfrac{A}{B}$.

For, let $A = mBC$, then $B = \dfrac{1}{m} \cdot \dfrac{A}{C}$. $\therefore B \propto \dfrac{A}{C}$.

Similarly, $C \propto \dfrac{A}{B}$.

(4) If $A \propto B$, and $C \propto D$, then $AC \propto BD$.

For, let $A = mB$, and $C = nD$, then $AC = mnBD$; $\therefore AC \propto BD$.

(5) If $A \propto B$, then $A^n \propto B^n$.

For, let $A = mB$, then $A^n = m^n B^n$. $\therefore A^n \propto B^n$.

(6) If $A \propto B$, then $AP \propto BP$, where P is *any* quantity *variable* or *constant*.

For, let $A = mB$, then $AP = mBP$. $\therefore AP \propto BP$.

297. Examples. Application of the principles explained in some of the preceding articles will be illustrated by the following examples.

Example 1. If y varies as x, and $y = 5$ when $x = 12$, find the value of y when $x = 18$.

By supposition, $y = mx$, where m is constant.

Putting $y = 5$, $x = 12$, we have $5 = m.12$. $\therefore m = \frac{5}{12}$.

Hence, x and y are connected by the relation $y = \frac{5}{12}x$.

Hence, when $x = 18$, we have $y = \frac{5}{12}.18 = \frac{15}{2} = 7\frac{1}{2}$.

Example 2. If z varies as $px + y$, and if $z = 3$ when $x = 1$, $y = 2$, and $z = 5$ when $x = 2$ and $y = 3$, find p.

By supposition, $z = m(px + y)$, where m is constant.

Putting $z = 3$, $x = 1$, $y = 2$, we have $3 = m(p+2)$. ... (1)

Again putting $z = 5$, $x = 2$, $y = 3$, we have $5 = m(2p+3)$. ... (2)

Hence, from (1) and (2), by division, $\dfrac{3}{5} = \dfrac{p+2}{2p+3}$, whence $p = 1$.

Example 3. If y = the sum of 3 quantities, of which the 1st $\propto x^2$, the 2nd $\propto x$, and the 3rd is constant; and when $x = 1, 2, 3$; $y = 6, 11, 18$ respectively, find the *equation* between x and y.

By supposition, $y = mx^2 + nx + p$, where m, n, p are constants.

Now, since $y=6$, when $x=1$, we have

$$\left.\begin{array}{lll} & 6=m+n+p. \quad \cdots \quad \cdots & (1) \\ \text{Similarly,} & 11=4m+2n+p, \quad \cdots \quad \cdots & (2) \\ \text{and} & 18=9m+3n+p. \quad \cdots \quad \cdots & (3) \end{array}\right\}$$

From (1) and (2), by subtraction, $3m+n=5$. ⋯ (4)

Similarly, from (2) and (3), $5m+n=7$. ⋯ (5)

Now, subtracting (4) from (5), we have

$$2m=2. \quad \therefore \quad m=1\,;$$

hence, from (4), $n=2$. $\therefore$ from (1), $p=3$.

Hence, the equation between x and y is $y=x^2+2x+3$.

Example 4. If (i) $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$;
and (ii) $a \propto b$, prove that $a^2-b^2 \propto ab$.

(i) By supposition, $a+b=m(a-b)$, where m is constant.

Hence, $(a+b)^2=m^2(a-b)^2$,

or, $a^2+b^2+2ab=m^2(a^2+b^2-2ab)$.

$\therefore \quad (m^2-1)(a^2+b^2)=2ab(1+m^2)$.

$$\therefore \quad a^2+b^2=\frac{2(m^2+1)}{m^2-1}\cdot ab.$$

But $\dfrac{2(m^2+1)}{m^2-1}$ is constant. $\therefore \quad a^2+b^2 \propto ab$.

(ii) Since $a=mb$,

multiplying both sides by a, we have $a^2=m.ab$ ⋯ (1)

and also multiplying both sides by $\dfrac{b}{m}$, we have $b^2=\dfrac{ab}{m}$. ⋯ (2)

Subtracting (2) from (1),

$$a^2-b^2=\left(m-\frac{1}{m}\right)\cdot ab, \text{ where } \left(m-\frac{1}{m}\right) \text{ is constant,}$$

$$\therefore \quad a^2-b^2 \propto ab.$$

Example 5. The wages of 5 men for 6 weeks being £14. 5*s*., how many weeks will 4 men work for £19 ?

Let x denote the wages (in pounds), earned by y men in z weeks.

Then, evidently $x \propto y$, when z is constant ;

and also $x \propto z$, when y is constant.

$\therefore$ when y and z are both variable,

$$x \propto yz,$$

i.e., $x=m.yz$, when m is constant.

Now, since $x=14\frac{1}{4}$, when $y=5$ and $z=6$.

$$\therefore \quad 14\frac{1}{4}=m\times 5\times 6. \quad \cdots \quad \cdots \quad (1)$$

Also, if z_1 denote the required number of weeks, then, since the corresponding values of x and y are respectively 19 and 4, we have

$$19=m\times 4\times z_1 \quad \cdots \quad \cdots \quad (2)$$

Hence, dividing (1) by (2),

$$\frac{3}{4}=\frac{5\times 6}{4\times z_1}, \text{ whence } z_1=10\,;$$

i.e., the required time $=10$ weeks.

Example 6. Assuming that the quantity of work done varies as the cube root of the number of agents when the time is the same, and varies as the square root of the time when the number of agents is the same; find how long 3 men would take to do one-fifth of the work which 24 men can do in 25 hours.

Let x denote the quantity of work done by y men in z hours.

Then by supposition,

$$x \propto y^{\frac{1}{3}} \text{ when } z \text{ and } \therefore z^{\frac{1}{2}} \text{ is constant,}$$

$$\text{and also, } x \propto z^{\frac{1}{2}} \text{ when } y \text{ and } \therefore y^{\frac{1}{3}} \text{ is constant.}$$

Hence, when both y and z and $\therefore$ $y^{\frac{1}{3}}$ and $z^{\frac{1}{2}}$ are variable,

$$x \propto y^{\frac{1}{3}}z^{\frac{1}{2}},$$

i.e., $x=k.y^{\frac{1}{3}}z^{\frac{1}{2}}$, when k is constant.

Now, since by the problem,

$$x=1, \text{ when } y=24 \text{ and } z=25$$

$$\therefore \quad 1=k.\sqrt[3]{24}.\sqrt{25}. \quad \cdots \quad (1)$$

Also, if z_1 be the required number of hours, since the corresponding values of x and y are respectively $\frac{1}{5}$ and 3, we have

$$\tfrac{1}{5}=k.\sqrt[3]{3}.\sqrt{z_1}. \quad \cdots \quad \cdots \quad (2)$$

Hence, dividing (1) by (2), $5=\dfrac{\sqrt[3]{24}\times 5}{\sqrt[3]{3}\times\sqrt{z_1}}=\dfrac{\sqrt[3]{8}\times 5}{\sqrt{z_1}}.$

$$\therefore \quad \sqrt{z_1}=2 \text{ and } \therefore z_1=4\,;$$

i.e., the required time $=4$ hours.

Example 7. A sphere of metal is known to have a hollow space about its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ of the weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weights of spheres of the same substance $\propto$ $(\text{radii})^3$.

Let R be the outer radius and W the weight of a solid sphere of the given metal of radius R; also let r be the inner radius (*i.e.*, radius of the spherical cavity), and w the weight of a solid sphere of the given metal of radius r.

Then, by hypothesis,

$$W = KR^3, \text{ and } w = Kr^3, \text{ where } K \text{ is constant.}$$

Now, since $(W-w)$ is the weight of the given sphere, we have, by the question, $W-w=\frac{7}{8}W$, hence, we must have

$$K(R^3-r^3)=\tfrac{7}{8}KR^3.$$

$$\therefore \tfrac{1}{8}R^3=r^3, \text{ whence } \frac{r}{R}=\frac{1}{2}.$$

Example 8. A point moves with a speed which is different in different kilometres, but invariable in the same kilometre, and its speed in any kilometre varies inversely as the number of kilometres travelled before it commences this kilometre. If the second kilometre be described in 2 hours, find the time occupied in describing the nth kilometre.

Evidently, the time of describing any kilometre varies *inversely* as the speed in that kilometre; hence, if v_n denote the speed in nth kilometre and t_n the number of hours required to describe the nth kilometre, we must have

$$t_n=\frac{m}{v_n}, \text{ where } m \text{ is constant.}$$

Also, by hypothesis, $v_n=\dfrac{K}{n-1}$, where K is constant;

$$\text{hence, } t_n=\frac{m}{K}(n-1).$$

Evidently, then t_n is known if $\dfrac{m}{K}$ is known; and since the time of describing the 2nd kilometre is two hours (*i.e.*, $t_n=2$, when $n=2$), we have

$$2=\frac{m}{K}\cdot 1. \qquad \therefore \frac{m}{K}=2.$$

Hence, $t_n=2(n-1)$,

i.e., the nth kilometre is described in $2(n-1)$ hours.

Example 9. A locomotive engine without a train can go 24 kilometres an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached. With four waggons its speed is 20 kilometres an hour. Find the greatest number of waggons with which the engine can move.

Let x = the number of waggons attached.

Then the number of kilometres travelled by the train per hour (*i.e.*, its speed) $= 24 - m\sqrt{x}$, where m is a constant.

Now, since the speed is 20 kilometres per hour when $x = 4$, we must have

$$20 = 24 - m\sqrt{4} = 24 - 2m. \qquad \therefore \quad m = 2$$

Hence, the speed of the engine with x waggons $= 24 - 2\sqrt{x}$; evidently, therefore, the speed diminishes as x increases.

Now, let us see for what value of x the speed is reduced to nothing. If x_1 be this value, we must have

$$0 = 24 - 2\sqrt{x_1}. \qquad \therefore \quad \sqrt{x_1} = 12, \text{ and } x_1 = 144.$$

Thus, when 144 waggons are attached, the engine *just* fails to move the train.

Hence, the *greatest number* of waggons with which the engine *can move* = 143.

Example 10. If x, y, z be variable quantities such that $y+z-x$ is constant, and that $(x+y-z)(x+z-y)$ varies as yz, prove that $x+y+z$ varies as yz.

By supposition, we have $y+z-x=k$ ... (1)

and $(x+y-z)(x+z-y)=myz$, ... (2)

where k and m are constants.

Now, from (2), we have $x^2-(y-z)^2=myz$.

$$\therefore \quad x^2-(y+z)^2=(m-4)yz,$$

$$\text{or,} \quad (x+y+z)(x-y-z)=(m-4)yz.$$

Hence, from (1),

$$(x+y+z)(-k)=(m-4)yz.$$

$$\therefore \quad x+y+z=\left(\frac{4-m}{k}\right)yz, \quad i.e., =(\text{a constant})\times yz.$$

Hence, $\quad x+y+z \propto yz.$

EXERCISE 149

1. If $y \propto x$, and $y=5$ when $x=15$, find the *equation* between x and y.

2. If $y \propto x$, and $y=10$ when $x=25$, find y when $x=35$.

3. If P varies inversely as Q, and $Q=10$ when $P=2$, what will P become when $Q=8$?

4. If $P \propto QR$, and the three corresponding values of P, Q, R be 6, 9, 10 respectively, find the value of P when $Q=5$ and $R=3$.

5. If the square of x vary as the cube of y, and $x=2$, when $y=3$, find the *equation* between x and y.

6. Given that y varies as the sum of two quantities, one of which varies as x directly, the other as x inversely and that $y=4$, when $x=1$, and $y=5$ when $x=2$, find the *equation* between x and y.

7. If $xy \propto x^2+y^2$, and $y=4$ when $x=3$, find the *equation* between x and y.

8. Given that y is equal to the sum of two quantities, one of which varies as x, and the other varies inversely as x^2, and when $x=1, 2$, $y=6, 5$ respectively, find the *equation* between x and y.

9. If $y =$ the sum of 3 quantities of which the 1st is constant, the 2nd $\propto x$, and the 3rd $\propto x^2$, also when $x=3, 5, 7$, $y=0, -12, -32$ respectively, find the *equation* between x and y.

10. Given that $y^2 \propto a^2-x^2$ and when $x=\sqrt{a^2-b^2}$, $y=\frac{b^2}{a^2}$, find the *equation* between x and y.

11. If $y=r+s$, whilst $r \propto x$, and $s \propto \sqrt{x}$; and if, when $x=4$, $y=5$, and when $x=9$, $y=10$, show that $6y=5(x+\sqrt{x})$.

12. Assuming that the time of oscillation of a pendulum varies as the square root of its length; if the length of a pendulum which oscillates once in a second be 39·2 inches, find the length of one which oscillates 56 times in a minute.

13. If 13 men earn £7 in 15 days of 8 hours each, what will be the wages of 52 men for 12½ days of 9 hours each?

14. Given that the volume of a sphere varies as the cube of its radius, prove that the volume of a sphere whose radius is 6 centimetres is equal to the sum of the volumes of three spheres whose radii are 3, 4, 5 centimetres.

15. The volume of a pyramid varies jointly as its height and the area of its base; and when the area of the base is 60 square metres and the height 14 metres, the volume is 280 cubic metres. What is the area of the base of a pyramid whose volume is 390 cubic metres and whose height is 26 metres?

16. Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 square centimetres, when the radius is 7 centimetres; find the area of a circle whose radius is 10·5 centimetres.

17. If the volume of a cone whose height is 12 centimetres and base 30 square centimetres be 120 cubic centimetres, find the volume of another whose height is 20 centimetres and base 144 square centimetres; the volume of a cone varying as the height and base jointly.

18. The volume of a circular cylinder varies as the square of the radius of the base when the height is the same and as the height when the base is the same. The volume is 88 cubic metres when the height is 7 metres, and the radius of the base is 2 metres; what will be the height of a cylinder on a base of a radius 9 metres, when the volume is 396 cubic metres?

19. Two circular gold plates, each one centimetre thick, the diameters of which are 6 centimetres and 8 centimetres respectively, are melted and formed into a single circular plate one centimetre thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

20. Given that the illumination from a source of light *varies inversely* as the *square* of the distance, how much farther from a candle must a book, which is now three inches off, be removed, so as to receive just half as much light?

21. A solid spherical mass of glass, 1 decimetre in diameter, is blown into a shell bounded by two concentric spheres, the diameter of the outer one being 3 decimetres. Calculate the thickness of the shell. (The volume of a sphere varies directly as the cube of its diameter.)

22. When a body falls from rest, its distance from the starting point varies as the square of the time it has been falling; if a body falls through 402½ feet in 5 seconds, how far does it fall in 10 seconds? Also how far does it fall in the 10th second?

23. If 10 men can reap a field of 7·5 hectares, in 3 days of 12 hours each, how long will it take 8 men to reap 9 hectares, working 16 hours a day?

24. The square of the time of a planet's revolution varies as the cube of its distance from the Sun; find the time of Venus's revolution assuming the distance of the Earth and Venus from the Sun to be 91½ and 16 millions of miles respectively.

[If P be the time of revolution measured in days, and D the distance in millions of miles, we have $P^2=KD^3$, where K is a constant, &c.]

25. The value of a silver coin varies directly as the square of its diameter while its thickness remains the same and directly as its thickness while its diameter remains the same. The silver coins have their diameters in the ratio of 4 : 3; find the ratio of their thickness if the value of the first be four times the value of the second.

[B. U. P. E. 1885]

26. The value of diamonds $\propto$ the square of their weights, and the square of the value of rubies $\propto$ the cube of their weights. A diamond of a carats is worth m times the value of a ruby of b carats, and both together are worth £c. Required the value of a diamond and of a ruby, each weighing n carats.

27. If $a \propto b$ and $b \propto c$, show that $(a^2+b^2)^{\frac{3}{2}} \propto c^3$.

28. If $x+y \propto x-y$, show that $x^2+y^2 \propto xy$ and $x^3+y^3 \propto xy(x \pm y)$.

29. Given that $x+y \propto z+\frac{1}{z}$, and that $x-y \propto z-\frac{1}{z}$, find the relation between x and z, provided that $z=2$, when $x=3$, and $y=1$.

[B. U. P. E. 1888]

30. If $x \propto \frac{1}{y}$, prove that $x+y$ is least when $x=y$.

[We have $xy=$ a constant.]

31. The consumption of coal by a locomotive varies as the square of the velocity; when the speed is 16 miles an hour the consumption of coal per hour is 2 tons; if the price of coal be 10*s.* per ton and the other expenses of the engine be 11*s.* 3*d.* an hour, find the least cost of a journey of 100 miles. [Apply the preceding example.]

32. If $z \propto y$, and $y \propto x$, show that

$$x+y+z \propto (yz)^{\frac{1}{2}}+(zx)^{\frac{1}{2}}+(xy)^{\frac{1}{2}}$$

ANSWERS

Exercise 1. [Pages 2-3]

1. 220.
2. 22.
3. 12 kilometres.
4. 8 kilometres.
5. 9.
6. 12.
7. 45 minutes.
8. 15 minutes.
9. 20.
10. $\frac{1}{64}$; $20\frac{1}{4}$.
11. 45 sq. metres.
12. 7*s*. 6*d*.
13. 20.
14. 9.
15. 28.
16. 4.
17. 4480
18. 900.
19. 1952.
20. 720.

Exercise 2. [Page 9]

1. 34.
2. 0.
3. 4.
4. 1.
5. $5\frac{1}{3}$.
6. $1\frac{1}{3}$.
7. 0.
8. 4.
9. 12.
10. 8.
11. 2.
12. $\frac{1}{2}$.
13. 5.
14. 80.
15. 29.
16. 325.
17. 0.
18. 14.
19. 114.
20. 4.
21. 89.
22. 19.
23. 0.
24. 325.
25. 9.

Exercise 3. [Pages 11-12]

1. 24.
2. $37\frac{1}{3}$.
3. 4.
4. 720.
5. $3\frac{1}{3}$.
6. $\frac{2}{3}$.
7. 1.
8. 40.
9. $\frac{1}{3}$.
10. $\frac{8}{5}$.
11. 0.
12. 50.
13. 1.
14. 75.
15. 100.
16. 200.
17. 1520.
18. 41·625.
19. 22680.
20. 845000.

Exercise 4. [Page 14]

1. 4.
2. 2.
3. 6.
4. 18.
5. 8.
6. 16.
7. 32.
8. 256.
9. 11.
10. 21.
11. 11.
12. 9.
13. 3.
14. 162.
15. 18.
16. 9.
17. 0.
18. 21.
19. 23.
20. 1.
21. 98.
22. 50.
23. 9.
24. 42.
25. 51.
26. 2805.
27. 7.
28. 171.
29. 2401.
30. 192.
31. 1029.
32. 1218.
33. 48.
34. 143.
35. 18750.
36. 16.
37. 160.
38. 78.
39. 7.
40. 2.

Exercise 5. [Pages 17-18]

1. A's loss = £100. **2.** -70. **3.** -25. **4.** -100. **5.** -30.
6. $4, -3, 5$. **7.** $15, -10, -20, 30$. **8.** $-15, 10, 20, -30$.

Exercise 6. [Page 20]

1. -22. **2.** -18. **3.** $-31, -41$. **4.** -19. **5.** -1180.
6. -222. **7.** -2034. **8.** 658. **9.** -7128. **10.** -220416417.

Exercise 7. [Pages 21-22]

1. 3. **2.** -5. **3.** -4. **4.** -47. **5.** -14.
6. -51. **7.** 16. **8.** -8. **9.** -32. **10.** 1.

Exercise 8. [Pages 24-25]

1. $-x+y$. **2.** $m^2+n^2+p^2$. **3.** $c^2+a^2b-a^2$.
4. $2abc-3mnp^2$. **5.** $2a^3b-9b^2c^2-2df$.
6. $-6x^4y-11xyz-10x^2y^2$. **7.** $4(a^2bc-b^2ca+c^2ab)$.
8. $-25x^3mn+16m^3nx$. **9.** -14. **10.** -234.
11. 92. **12.** 5. **13.** 177. **14.** -4653
15. -12015. **16.** $-6a+b-3c$. **17.** $2x-z$.
18. $2x^3+9x^2+7$. **19.** $-a+2b-8d$. **20.** $2x^2-3y^2$.
21. 153. **22.** -125. **23.** 200. **24.** 120. **25.** 400.

Exercise 9. [Page 27]

1. -10. **2.** 12. **3.** -6. **4.** -22. **5.** 0.
6. -291. **7.** -77. **8.** 83. **9.** 17. **10.** 177.

Exercise 10. [Page 29]

1. $2a+3b-2c$. **2.** $-3a+3b+4c$. **3.** $3x+2y-3z$.
4. $2m^2-2m-4$. **5.** $2x^2+y^2-z^2$. **6.** $3x^2-2y^2-7xy$.
7. $4a^2-7ab-b^2$. **8.** $7bc-7c^2+10xy$. **9.** $-x^3+x^2-x+2$.
10. $-(x+2y)$. **11.** $3x-4y+5z$. **12.** $6-2m^3-5m$.
13. $-(3a^2b+3ab^2)$. **14.** $2a^2b^2$. **15.** $3ab^2-3a^2b$.

Exercise 11. [Pages 31-32]

1. $-4a+8b$. **2.** $7x-4y$. **3.** $-2x$. **4.** $-4a+2b$. **5.** $5a+2b$
6. $2b$. **7.** 6. **8.** 8. **9.** $-2a+7b$. **10.** 0.

11. $-2x+5y+7z$. 12. $-2c$ 13. $15x-15y$. 14. $8a-8b$
15. $11m-7n$. 16. $6a-6b-18c$. 17. $6x-6y-20z$. 18. $x-y-13z$.
19. $-3x-y-z$. 20. $a-11b+17c$. 21. $2x-12y+20z$.
22. $5a-b+11c$. 23. $x-3y+2z$. 24. $11a-2b-16c$.
25. $a-(b+c-d)+(-m+n-x)+y-z$.
26. $a-\{b+c-d+m+(-n+x-y+z)\}$.
27 $\{a-b-(c-d+m)\}-\{-n-(-x+y-z)\}$.
28. $-\{-a-(-b-c)\}-\{-d-(-m+n)\}-\{x-(y-z)\}$.

Exercise 12. [Page 33]

1. 15. 2. 18. 3. 36. 4. -32. 5. -45. 6. -78.
7. -24. 8. -35. 9. -45. 10. 36. 11. 60. 12. 64.

Exercise 13. [Pages 34-35]

1. 54. 2. 47. 3. -8. 4. -393. 5. -111.
6. 30. 7. 0 8. 1136. 9. -280.

Exercise 15. [Pages 39-40]

14. $-6x^7y^5$. 15. $21a^8b^4c^8$. 16. $40x^{17}y^{16}$.
17. $-156x^{10}y^9z^6$. 18. $140x^6y^7z^{20}$. 19. $-4x^{11}y^7$.
20. $-70a^{15}b^{12}$. 21. $48x^{16}y^{10}z^7$. 22. $24x^7y^8z^7$.

Exercise 16. [Page 40]

1. $-10x^7$. 2. $-20a^4b^6$. 3. $21m^7n^8$. 4. $-18x^4y^7$.
5. $3a^7b^{10}$. 6. $-40m^8n^7$. 7. $50x^4y^8z^8$. 8. $-24x^4y^4z^4$.
9. $48x^5y^5z^5$. 10. $25a^5b^9c^{18}$. 11. $-24x^8y^8z^5$.
12. $32a^8b^2x^2y^8$. 13. $35a^8b^8z^4$. 14. $-60a^6x^7y^5$.
15. $70x^5y^5z$. 16. $-18a^8b^6c^6$. 17. $63a^9x^8y^2$.
18. $160x^{14}y^7z^7$. 19. $65a^{10}b^{14}c^{20}$. 20. $112a^{18}x^{10}y^9z^7$.

Exercise 17. [Page 43]

1. $xy-2x^2$. 2. $-5a^2+10ab-15ac$. 3. $8x^2y-12xy^4$
4. $2a^3bc-3ab^3c-abc^3$. 5. $-3x^3y^2+6x^2y^3+3xy^4$.
6. $21a^2b^4-7ab^4-35a^3b^2+7a^2b^3$. 7. $-6a^4x+8a^3x^2-10a^3x$.
8. $-8m^4n+12m^3n^2-20m^2n^3$. 9. $-a^3b^2c^2+a^2b^3c^2-a^2b^2c^3$.
10. $x^3yz+xy^3z+xyz^3-xy^2z^2-x^2yz^2-x^2y^2z$.

11. $12c^4d^5 - 18c^5d^7 + 30c^3d^6 + 24c^4d^6$.

12. $-16a^7b^3 + 12a^6b^4 - 10a^5b^5 + 8a^4b^6$. 13. $7x^4 - 2x^2$. 14. 0.

15. $9x^6 - 25y^4$. 16. $x^6 + 4x^3$. 17. $a^{12}b^6 + 4a^4b^2$.

18. $4a^{18}b^{12} + 81a^6b^4$. 19. $3a^2y$. 20. (i) $x^3 + y^3 + z^3 - 3xyz$; (ii) 0

Exercise 18. [Page 46]

1. $-4x^3$. 2. $-3x^4$. 3. $4a^4x^3$. 4. $3x^5y^5$.

5. $2a^2b^2$. 6. $-2p^2q^2$. 7. $5x^6y^4z$. 8. $-8a^3c^2$.

9. $-3m^5n^6p$ 10. $3a^2c^2$. 11. $-5x^4y^2$. 12. $3a^6x^5y^4z^2$.

13. a^{44}. 14. $-7x^{48}$. 15. $-7m^{18}$. 16. $-7a^{41}b^{120}$

Exercise 19. [Page 47]

1. $3a - 2b$. 2. $3b^2 - 2a^2$. 3. $2a^2 - 3b^2$. 4. $3x^2 - 4xy$.

5. $3y^2 - 2x^2$. 6. $n^2 - 3mn + 4m^2$. 7. $ax - 2x^2 + 3a^2$.

8. $-3x^2 + 2a^2 - 5ax$. 9. $2m^2n^2 - 3m^4 - 4n^4$. 10. $-p^2 + \frac{5}{8}pq + \frac{3}{8}q^2$

11. $-2xy^2 + 3x^3 - 4y^3$. 12. $\frac{3}{4}x^3 - \frac{3}{2}a^3 - \frac{9}{4}a^2x$. 13. $3xa + \frac{15}{4}a^2 - 4x^2$.

14. $5m^4n^2 - 7m^2n^4 - 8p^6$. 15. $b^2c^2x^2y^2 - 2a^2c^2y^2z^2 + 3a^2b^2x^2z^2$.

Miscellaneous Exercises I

[Pages 47-52]

I

1. 10 ; $\frac{1}{2}$. 2. 8. 3. 15 ; $2a$; $7ab^2$; $16m^2pq$.

4 6. 5. $-\frac{1}{2}$. 8. 9, 7, 5, 2, −1, −3, −4, −8, −12.

II

1. 0, 25, 46, 45. 2. 16. 3. $(\sqrt[3]{a}) \times (\sqrt[3]{a}) \times (\sqrt[3]{a}) = a$, &c. ; 35.

5. $-7x^2y$; −560. 6. $16x^4 - 8xy^3 + 24x^2y^2 + y^4 - 32x^3y$; 81.

7. $30b + 13c - 23a$. 8. $x - 2y + z$.

III

1. (i) $c(a+b) = x + yz$; (iii) $(x+y)^2 = x^2 + y^2 + 2xy$;
(ii) $\sqrt[3]{m-n} + m^3n^3 < \sqrt{x} + \sqrt{y}$; (iv) $\because a > b, \quad \therefore 3a > 3b$

2. 5, −4, $-\frac{13}{2}$, $\frac{13}{2}$, −10. 3. −1000. 5. 66.

7. $-6a^2 + bc - 9x^2 + 16$ 8. a.

IV

1. -11 ; 1. 3. $4\frac{13}{18}$. 5. 9 6. $3x+2a+b$.
7. $a^2+b^2+c^2$. 8. $7x^2-y^2-2xy$.

V

3. 2, a, b, $a+b$. 4. $7\frac{1}{2}$. 5. 505. 7. y. 8. 32

VI

2. 536. 6. 60. 8. 3808.

VII

2. 2 ; 0. 4. $1+x$; $3a+b-5c$. 5. (i) $(a+b)(a-b)=a^2-b^2$.
(ii) $(a+b)^2-(a^2+b^2)=2ab$. 6. 0. 7. $a+b+c$.
8. $2a-\frac{3}{4}b+\frac{7}{8}c-\frac{8}{10}d$.

VIII

2. $2m$; $2n$. 3. $ma+mb+na+nb$; $a^2+2ab+b^2$.
4. 0 ; 0. 6. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.
7. $36a^8b^{15}c^{22}x^{10}y^{10}z^{10}+90a^{18}b^{20}c^{12}x^8y^8z^8+10a^{18}b^{10}c^{17}x^6y^6z^6$.
8. $2b^5c^{10}x^4y^2+5a^5b^{10}x^2z^4+3a^{10}c^5y^4z^2$.

Exercise 20 [Pages 54-56]

1. $x^2+8x+16$. 2. $9a^2+12a+4$. 3. $x^2+4xy+4y^2$.
4. $4x^2+28xy+49y^2$. 5. $9a^2+24ab+16b^2$. 6. $25a^2+70ab+49b^2$.
7. $a^2y^2+6abxy+9b^2x^2$. 8. $a^4+4a^2bc+4b^2c^2$.
9. $9x^4+12x^2y^2+4y^4$. 10. $16x^4+8x^2y^3+y^6$.
11. $\frac{9}{16}x^2+\frac{xy}{3}+\frac{4y^2}{81}$. 12. $\frac{1}{a^2}+\frac{2}{ab}+\frac{1}{b^2}$.
13. $\frac{a^2}{b^2}+2+\frac{b^2}{a^2}$. 14. 822649. 15. $x^2+1+\frac{1}{4x^2}$.
16. $a^2+4b^2+9c^2+4ab+6ac+12bc$. 17. $a^2b^2+b^2c^2+c^2a^2+2ab^2c$ $+2a^2bc+2abc^2$. 18. $4p^2+9q^2+16r^2+12pq+16pr+24qr$.
19. $x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2$. 20. $4x^2+9y^2+16z^2$ $+12xy+16xz+24yz$. 21. $x^4+y^6+z^8+2x^2y^3+2x^2z^4+2y^3z^4$.
22. $x^2+y^2+4a^2+9b^2+2xy+4xa+6xb+4ya+6yb+12ab$.
23. $9a^2+16b^2+c^2+4d^2+24ab+6ac+12ad+8bc+16bd+4cd$.
24. $4a^2+x^2+16y^2+9z^2+4ax+16ay+12az+8xy+6xz+24yz$.

25. $16m^2+9n^2+9p^2+4q^2+24mn+24mp+16mq+18np+12nq+12pq$.

26. $(x+2)^2$. **27.** $(3a+4b)^2$. **28.** $\left(5x+\frac{y}{2}\right)^2$.

29. $\left(x+\frac{1}{x}\right)^2$. **30.** $\left(\frac{4a}{3b}+\frac{3b}{4a}\right)^2$. **31.** $4x^2$.

32. $4z^2$. **33.** $16a^2$. **34.** $a^2+4ab+4b^2$. **35.** $x^2+2xy+y^2$.

36. 2500. **37.** 1000000. **38.** 25·8064. **39.** 1.

40. 0. **41.** 4. **42.** 9. **43.** 1.

44. 16. **45.** 25. **47.** a^2-2. **49.** 14. **50.** 3.

Exercise 21. [Pages 58-59]

1. x^2-6x+9. **2.** $4x^2-20x+25$. **3.** $9x^2-30xy+25y^2$.

4. $a^2x^2-2abxy+b^2y^2$. **5.** $64m^2-48mn+9n^2$.

6. $p^2m^2-2pqmn+q^2n^2$. **7.** $p^4-2p^2mn+m^2n^2$.

8. $x^4y^2-2x^3y^3+x^2y^4$. **9.** $x^6-4x^4z+4x^2z^2$.

10. $9a^6-30a^3b^3+25b^6$. **11.** $x^2y^2z^2+2abcxyz+a^2b^2c^2$.

12. $x^4y^2z^2-2x^3y^3z^2+x^2y^4z^2$. **13.** $a^4x^8-2a^2b^2x^4y^4+b^4y^8$.

14. $\frac{9}{49}x^2-\frac{xy}{2}+\frac{49}{144}y^2$. **15.** $\frac{1}{4x^2}-\frac{1}{xy}+\frac{1}{y^2}$.

16. $\frac{4x^2}{y^2}-2+\frac{y^2}{4x^2}$. **17.** $4x^2-\frac{x}{y}+\frac{1}{16y^2}$.

18. $a^2+4b^2+4c^2-4ab-4ac+8bc$. **19.** $25x^2+9y^2+36z^2-30xy-60xz$
$+36yz$. **20.** $9m^2+16n^2+25q^2-24mn-30mq+40nq$.
21. $a^4+9b^4+25c^4-6a^2b^2-10a^2c^2+30b^2c^2$. **22.** $x^2+y^2+a^2+b^2$
$-2xy-2xa-2xb+2ya+2yb+2ab$. **23.** $a^2+4x^2+9b^2$
$+16y^2-4ax-6ab-8ay+12xb+16xy+24by$. **24.** 7921.

25. 13689. **26.** 248004. **27.** 986049. **28.** $(4a-b)^2$. **29.** $(2x-5y)^2$.

30. $\left(2x-\frac{1}{2x}\right)^2$. **31.** $\left(\frac{1}{a}-\frac{1}{2b}\right)^2$. **32.** $\left(3x-\frac{1}{11x}\right)^2$. **33.** $(\cdot71)^2$.

34. $36b^2$. **35.** $64b^2$. **36.** $49a^2$. **37.** $121s^4$.

38. $25b^2c^2+10bc^2a+c^2a^2$. **39.** 25. **40.** 7. **41.** 4.

42. 81. **43.** 16. **44.** 25. **45.** 144.

50. (i) 5. (ii) 15. (iii) $a^2+b^2+c^2-ab-ac-bc$.

Exercise 22. [Pages 61-62]

1. x^2-9. 2. $25x^2-169$. 3. x^2-4a^2. 4. $a^2x^2-b^2y^2$.

5. $9a^2-\frac{1}{9a^2}$. 6. $\frac{9a^2}{16}-\frac{16b^2}{25}$. 7. $\frac{x^2}{y^2}-\frac{y^2}{x^2}$.

8. 39936. 9. 999744. 10. $a^2m^2-n^4$.

11. $x^2y^2-y^2z^2$. 12. $x^4-4y^2z^2$. 13. $x^2y^4-x^4y^2$.

14. x^4-1. 15. a^8-b^8. 16. $a^2+2ab+b^2-c^2$.

17. $a^2-b^2-2bc-c^2$. 18. $m^4+m^2n^2+n^4$. 19. x^4+4y^4.

20. $a^2x^2-b^2y^2+2bcyz-c^2z^2$. 21. $b^2y^2+c^2z^2-a^2x^2+2bcyz$.

22. $b^4m^2-c^4n^2-a^4p^2+2c^2a^2np$. 23. $a^6-64b^6-729c^6+432b^3c^3$.

24. a^4x^4+4. 25. $a^8x^8+a^4x^4+1$. 26. m^4+n^4. 27. x^8-1.

28. $4a(b-c)$. 29. $4a(3c-2b)$. 30. $4xy(x^2+y^2)$. 31. $4x(y-a+b)$.

32. $8a(3b-5c+7d)$. 33. 9376. 34. 1069840. 35. 4985645.

36. $(5x+6)(5x-6)$. 37. $(3a+4c)(3a-4c)$. 38. $(4m+7n)(4m-7n)$.

39. $(2p+9q)(2p-9q)$. 40. $(ax+8b)(ax-8b)$.

41. $(6x^2+11y^2)(6x^2-11y^2)$. 42. $(7+8d)(7-8d)$.

43. $(12c+5d)(12c-5d)$. 44. $\left(\frac{3x}{4y}+\frac{5y}{6x}\right)\left(\frac{3x}{4y}-\frac{5y}{6x}\right)$. 45. $\left(\frac{x}{3}+\frac{7}{y}\right)\left(\frac{x}{3}-\frac{7}{y}\right)$.

46. $\left(\frac{a}{11}+\frac{12}{5b}\right)\left(\frac{a}{11}-\frac{12}{5b}\right)$. 47. $(a+b+c)(a+b-c)$.

48. $(a+2b+5c)(a+2b-5c)$. 49. $(2x+3a-4b)(2x-3a+4b)$.

50. $(2a^2+b^2-3c^2)(2a^2-5b^2+3c^2)$. 51. $x\left(\frac{2}{x}-7x\right)$.

52. $\left(\frac{9}{2}x-\frac{3}{5y}+\frac{4}{9}y\right)\left(\frac{7}{2}x-\frac{3}{5y}-\frac{4}{9}y\right)$. 53. $(a+2b-3c)(a-2b+3c)$.

54. $(a^2+9b^2)(a+3b)(a-3b)$. 55. $(x-y+a-b)(x-y-a+b)$.

56. $(9x^2+25y^2)(3x+5y)(3x-5y)$. 57. $(7a-b)(a+15b)$.

58. $(5x-2y)(x+12y)$. 59. $(2a+3b-4c)(b-2c)$.

60. $(2m+5n-2p)(2m+n-8p)$. 61. $(5x-7y+12z)(x-y+2z)$.

Exercise 23. [Pages 64-65]

1. $x^3+9x^2+27x+27$. 2. $8x^3+12x^2+6x+1$.

3. $27a^3+27a^2b+9ab^2+b^3$. 4. $64x^3+144x^2y+108xy^2+27y^3$.

5. $x^6+6x^4y+12x^2y^2+8y^3$. 6. $x^3y^3+3x^2y^3z+3xy^3z^2+y^3z^3$.

7. $a^6b^3 + 3a^4b^2c^2d + 3a^2bc^4d^2 + c^6d^3$. 8. $a^3 + b^3 + 8c^3 + 3a^2b + 3ab^2 + 6a^2c + 12ac^2 + 6b^2c + 12bc^2 + 12abc$. 9. $8x^3 + 27y^3 + z^3 + 36x^2y + 54xy^2 + 12x^2z + 6xz^2 + 27y^2z + 9yz^2 + 36xyz$.

10. $x^9 + 3x^6y^3 + 3x^3y^6 + y^9$. 11. $\frac{a^3}{8} + \frac{a^2b}{2} + \frac{2ab^2}{3} + \frac{8b^3}{27}$.

12. $\frac{x^3}{y^3} + \frac{3x}{y} + \frac{3y}{x} + \frac{y^3}{x^3}$. 13. $\frac{1}{x^3} + \frac{3}{x^2y} + \frac{3}{xy^2} + \frac{1}{y^3}$.

14. $\frac{8}{27a^3} + \frac{4}{5a^2b} + \frac{18}{25ab^2} + \frac{27}{125b^3}$. 15. 1157625 16. 614125.

17. $125m^3$. 18. $x^3 + 3x^2y + 3xy^2 + y^3$. 19. $27b^3$.

20. $x^3 + 3x^2 + 3x + 1$. 21. $x^3 + 6x^2 + 12x + 8$. 22 $8a^3$.

23. 8. 24. 125. 25. 90. 26. 175. 28. 62.

29. 110. 33. 0. 34. -1 35. 0 36. 10.

Exercise 24. [Pages 66-67]

1. $x^3 - 6x^2 + 12x - 8$. 2. $8x^3 - 12x^2 + 6x - 1$.

3. $8 - 36a + 54a^2 - 27a^3$. 4. $27 - 108a + 144a^2 - 64a^3$.

5. $8a^3 - 36a^2b + 54ab^2 - 27b^3$. 6. $125m^3 - 300m^2n + 240mn^2 - 64n^3$.

7. $8x^3 - 60x^2y + 150xy^2 - 125y^3$. 8. $\frac{1}{27}a^3 - \frac{2}{9}a^2b + \frac{4}{9}ab^2 - \frac{8}{27}b^3$.

9. $27x^3 - 9x + \frac{1}{x} - \frac{1}{27x^3}$. 10. $\frac{1}{x^3} - \frac{3}{x^2y} + \frac{3}{xy^2} - \frac{1}{y^3}$.

11. 7762392. 12. 120553784.

13. $8a^3 - b^3 - c^3 - 12a^2b + 6ab^2 - 12a^2c + 6ac^2 - 3b^2c - 3bc^2 + 12abc$.

14. $8x^3 - 27y^3 - z^3 - 36x^2y + 54xy^2 - 12x^2z + 6xz^2 - 27y^2z - 9yz^2 + 36xyz$.

15. $p^6 - q^6 - r^6 - 3p^4q^2 + 3p^2q^4 - 3p^4r^2 + 3p^2r^4 - 3q^4r^2 - 3q^2r^4 + 6p^2q^2r^2$.

16. $64b^3$. 17. $x^3 - 3x^2y + 3xy^2 - y^3$. 18. $8x^3$. 19. 0

20. 343. 21. -505 22. 27 23. 0. 24. 36

25. 140. 27. 4

Exercise 25. [Page 68]

1. $x^3 + 1$. 2. $1 + 8x^3$. 3. $125p^3 + 1$ 4. $343a^3 + 64b^3$.

5. $512x^3 + 27y^3$. 6. $a^3b^3 + 64c^3$. 7. $a^3x^3 + 125b^3$.

8. $125a^3 + 729b^3$. 9. $(a+1)(a^2 - a + 1)$. 10. $(a+2)(a^2 - 2a + 4)$.

11. $(2x+1)(4x^2 - 2x + 1)$. 12. $(3a+2)(9a^2 - 6a + 4)$.

13. $(2m+4)(4m^2 - 8m + 16)$. 14. $(4p+5)(16p^2 - 20p + 25)$.

15. $(2x+5y)(4x^2-10xy+25y^2)$. 16. $(2x+6y)(4x^2-12xy+36y^2)$.
17. $(3a+7y)(9a^2-21ay+49y^2)$. 18. $(3x^2+8y^2)(9x^4-24x^2y^2+64y^4)$
19. $(6ax+y)(36a^2x^2-6axy+y^2)$.
20. $(3ab+4xy)(9a^2b^2-12abxy+16x^2y^2)$.
21. $(9abc+10xyz)(81a^2b^2c^2-90abcxyz+100x^2y^2z^2)$.
22 $(11ab^2x^3+9cy^2z^3)(121a^2b^4x^6-99ab^2cx^3y^2z^3+81c^2y^4z^6)$.

Exercise 26. [Page 69]

1. $1-8x^3$. 2. x^3-27. 3. $64a^3-1$. 4. $x^6-8y^3z^3$.
5. $27m^3-8n^3q^3$. 6. $64a^3b^3-b^3c^3$. 7. $(5a-1)(25a^2+5a+1)$.
8. $(7x-2y^2)(49x^2+14xy^2+4y^4)$. 9. $(6k-5l)(36k^2+30kl+25l^2)$.
10 $(1-8k)(1+8k+64k^2)$. 11. $(9m-4an^2)(81m^2+36man^2+16a^2n^4)$.
12. $(3xy-11y^2b^2)(9x^2y^2+33xy^3b^2+121y^4b^4)$.

Exercise 27. [Page 70]

1. x^2+3x+2. 2. $x^2+11x+18$. 3. x^2+x-30
4. $x^2-14x+33$. 5. $a^2+5a-176$. 6. $m^2+12m-133$.
7. $p^2+2p-143$. 8. $p^2-5p-204$. 9. $x^2+5x-36$.
10. $x^2-15x+50$. 11. $x^2-7x-60$. 12. $k^2-11k-26$.
13. $a^2+19a+70$. 14. $m^2-8m-84$. 15. $x^2-18x+65$
16. $x^2+19x+84$. 17. $a^2-14a+33$. 18. $x^2-9x-52$.
19. $m^2-11m-80$. 20. $x^2-18x+80$. 21. $a^2-6a-72$.
22. $m^2+6m-91$. 23. $x^2-26x+160$. 24. $x^2-13x-90$.
25. $x^2-6x-160$.

Exercise 28. [Page 75]

1. 4. 2. −5. 3. −4. 4. −5. 5. −5. 6. −60.
7. 3. 8. 4. 9. −3. 10. 3. 11. 13. 12. 5.
13. −2. 14. −2. 15. 1. 16. 2. 17. 3. 18. −4
19. 0. 20. 7. 21. −2. 22. −1. 23. 7. 24. 3. 25. 5
26. 7. 27. −6. 28. 0. 29. −8. 30. 9. 31. −2. 32. $\frac{1}{3}$.
33. 1. 34. −1. 35. 12. 36. 30. 37. 12.

Exercise 29. [Pages 77-78]

1. $15-x$. 2. $x-20$. 3. $x+25$. 4. $25-y$.
5. $y-2x$. 6. $\frac{21}{x}$. 7. $100-3x$. 8. $4x-3y$
9. xy kilometres 10. $\frac{x}{y}$ hours. 11. $(x+20)$ years ; $(x-3)$ years.

12. $\frac{60}{x}$ kilometres. 13. $\frac{40}{3x}$ metres. 14. $\left(5x-2\frac{3}{5}\right)$ rupees. 15. $x-2, x-1,$
$x, x+1, x+2.$ 16. $3x.$ 17. $2m+3.$ 18. $2x-2.$
19. $\frac{10x}{y}$ days. 20. $10ab.$ 21. $\frac{5ab}{8}.$ **22.** $\frac{x}{3y}$ kilometres.
23. $\frac{16a}{x}$ hours. 24. $(x+15)$ years ; $(x+45)$ years. **25.** $10y+x.$
26. $100x+10y+z.$ **27.** $100z+10y+x.$

Exercise 30. [Pages 80-81]

1. 9. **2.** 15. **3.** 9.
4. 6 metres and 3 metres. 5. 20. **6.** 40 and 10. 7. 80
8. 12. 9. 60. **10.** 40. **11.** 96.
12. 42, 43, 44. 13. 33. 14. 25, 65. **15.** 15 and 24.
16. 36. 17. 72. 18. 10, 11. 19. Rs. 600, Rs. 250.
20. Rs. 120, Rs. 300. 21. Rs. 35. **22.** 35, 25. **23.** 30, 10.

Exercise 31. [Page 84]

2. Take BE equal to AD; by guess let F be the middle point of DE. Then F is very approximately the middle point of AB, the error, if any, being indefinitely small.

7. $AB=5{\cdot}3$ cm., $BC=3{\cdot}7$ cm., $CA=7{\cdot}9$ cm., $AD=4{\cdot}7$ cm., $DC=3{\cdot}2$ cm.

Exercise 32. [Pages 87-88]

1. $6\frac{2}{3}$ units of length. **2.** $7\frac{1}{3}$ metres. **3.** $7\frac{1}{7}$ metres
4. $3{\cdot}5$ centimetres. **5.** $3{\cdot}6$ metres. **6.** $\cdot 7$ metre
7. 5 metres. **8.** 65 metres. **9.** 17 metres. **10.** $9{\cdot}5$ metres.

Exercise 34. [Pages 94-96]

1. (i) (33, 24) ; (−27, 33) ; (−15, −18) ; (27, −30).
(ii) (6·6, 4·8) ; (−5·4, 6·6) ; (−3, −3·6) ; (5·4, −6).
2. $(3\frac{2}{3}, 2\frac{2}{3})$; $(-3, 3\frac{2}{3})$; $(-1\frac{2}{3}, -2)$; $(3, -3\frac{1}{3})$. 5. 20.
6. 13. 7. 50. 8. 11 ; −13. **9.** 17·5 ; 36.
10. 12 ; 8. **11.** 12·5 units of area. **12.** 16 units of area.
13. 1 unit of area. **14.** 40 units of area ; 7, 4·5. **15.** (i) 83 ;
(ii) 78 ; (iii) 420 ; (iv) 72. **16.** 30 sq. cm. ; 5 cm. ; 90°.
17. 2·5 cm. **18.** 6, 7. **19.** 5. **20.** 32 units of area ; 7, 5.

Miscellaneous Exercises II

[Pages 96-98]

I

1. $x^2+y^2+z^2-2xy+2yz-2zx$. 7. $m^2+n^2+9p^2+2mn+3np+3pm$

8. $27x^2-93xy-66y^2$. 9. 217.

II

1. -4. 2. -1. 3. 3. 4. $\frac{b^2}{a}$. 5. $\frac{m^2+n^2+p^2}{mnp}$

6. 7. 7. 11. 8. 5. 9. $\frac{2}{3}$. 10. $1\frac{1}{17}$

III

1. 7. 2. 126. 3. Rs. 1500.

4. Rs. 2250 ; Rs. 900 ; Rs. 750 ; Rs. 300. 5. 40 ; 20 ; 36.

6. Rs. 50 ; Rs. 2 50 P.

Exercise 35. [Pages 103-105]

1. $-5x^2-2xy-y^2-2x-y-2$. 2. $4a^2b$. 3. $-m^3n^3-mnp-m^2n^2$.

4. $a^2b^2x^2$. 5. $a^4b^4c^4$. 6. $a^3b^3-b^3c^3+c^3a^3-a^2b^2c^2$.

7. $a^3+b^3+c^3-3abc$. 8. $2(x+y+z)$. 9. $2(x+y+z)$.

10. $x^2y+y^2z+z^2x$. 11. $a^2b+b^2c+c^2a+ab^2+bc^2+ca^2$.

12. $abc^2+bca^2+cab^2+a^2d+b^2d+c^2d$. 13. $-\frac{3}{2}(x+y+z)$

14. $-\frac{5}{12}(x+y+z)$. 15. 0. 16. 0.

17. 0. 18. 1280. 19. 1280. 20. 0.

21. $(a^2+b^2)(m+n+p+q+l)+(a^2-b^2)(m+n+p+q+k)+c^2(l+m+n)$.

22. $4(ax^2+by^2+cz^2)$. 23. 0. 24. $3(a^2+b^2+c^2-ab-bc-ca)$.

25. $(a+b+c)(x^2+y^2+z^2)$. 26. $-y-z$. 27. $9x+12y-3$.

28. $(y+7z)$ kilometres. 29. $(-x-3y+3a+b)$ rupees.

30. $(14a+16b-26c)$ rupees.

Exercise 36. [Pages 106-108]

1. $10x^5-11x^4y+10x^3y^2+6x^2y^3-3y^4$. 2. $2m^3nx-7n^3xm+12x^2mn+7m^2n^2x+8n^2x^2m$. 3. $11x^6-3x^5y-50x^4y^2+15x^3y^3+26x^2y^4-19xy^5+40y^6$. 4. $5ax^4-8a^2x^3+8yzbc^2+2y^2zbc+4yz^2bc$. 5. $-2-x^3y^5z+2xy^3z^5+2x^3z^5y+6x^2y^2z^2-3xyz^4$. 6. $4x^4y^3z^2-80x^3y^4z^2+28x^2y^3z^4-22x^3y^2z^4-102x^4y^2z^3+155x^2y^4z^3$.

7. $-12x^3y^4z^5 - 100x^3y^5z^4 + 58x^4y^5z^3 + 92x^4y^3z^5 + 39x^5y^3z^4 - 38x^5y^4z^3$. 8. $-4x^2 + 5xy - 7y^2 - 8yz$. 9. $6x^3 - 12x^2y^2 + 2a^2bx - 7bxy^2 - 9xyab$. 10. $-2x^4 + 6x^3y - 2x^2y^2 + 8xy^3 + 7y^4$.
11. $-2x^5 + 3x^4y - 10x^3y^2 - 4x^2y^3 - 13xy^4 + 50y^5$. 12. $a^2 + 5ab - 8b^2$
13. $4x^2 - 8xy + y^2 - 12x - 15y + 9$. 14. $2a^3 - 4a^2b + 7ab^2 - 15b^3$.
15. $-12x^3y + 7x^2y^2 - 8x^2 + 17y - 29$. 16. $5a^2 - 4ab - 5bc + 11b^2$.
17. $-2x^3 - 3y^2 - 5xy - 3x - 2$. 18. $-3a^3 - 11b^2c - 6ac^2 - 5b^3$
19. $-4x^3 - 22xy^2 - 45y^3 - 11x^2 - 24xy - 15$. 20. $\frac{11}{24}x + \frac{19}{168}y + \frac{496}{57}z$.
21 $\frac{3}{10}ax + \frac{11}{18}y + \frac{89}{25}mz$. 22. $30{\cdot}08c^2by + 1{\cdot}2a^2cx + {\cdot}45c^3z$. 23. $-\frac{129}{98}a^{\frac{1}{2}}c^{\frac{3}{5}}x - \frac{15}{41}a^{\frac{3}{2}}b^{\frac{5}{4}}y - b^{\frac{3}{2}}c^{\frac{5}{12}}z + lx - 6my - 5nz$
24. (i) $1{\cdot}2x + 2{\cdot}3y - {\cdot}6z$; (ii) $35x - 24{\cdot}7y - \frac{1}{21}z$, (iii) $3{\cdot}4a + 19{\cdot}04l^2 + 20m^2 + 30p$. 25. $2(bc^2 + ca^2 + ab^2)$. 26. 0.
27 0. 28. $ax + by + cz$. 29. $2ax + 12by - cz$. 30. $14x + 44y + 7z$
31 $2(a + b + c)$. 32. $x + y + z$. 33. $(2x + 4y - 2z)$ rupees.

Exercise 37. [Page 110]

1 $2a^2 + 5ab + 3b^2$. 2. $2m^2 - 5mn + 3n^2$. 3. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. 4. $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$. 5. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$. 6. $2a^2 + 2b^2 + 3c^2 - 5ab - 7ac + 5bc$. 7. $2x^2 + 3y^2 + 4z^2 - 5xy - 6xz + 7yz$. 8. $5x^2 - 2a^2 - 3b^2 + 3xa - 2xb + 5ab$. 9. $x^3 - y^3 - z^3 - x^2y - x^2z + xy^2 - y^2z + xz^2 - z^2y$. 10. $x^2y^2 - y^2z^2 - z^2x^2 - 2yz^2x$.

Exercise 38. [Pages 114-116]

1. $27a^3 + 45a^2b - 75ab^2 - 125b^3$. 2. $4a^2 - 9b^2 + 24bc - 16c^2$
3 $x^4 + 3x^2 + 4$. 4. $a^4 - 2a^2b^2 + b^4$. 5. $x^8 + x^4 + 1$
6. $x^6 - x^4y^4 + 2x^3y^3 + y^6$. 7. $m^6 + n^6$. 8. $p^6 - q^6$.
9. $a^5 - 26a^3b^2 + 25ab^4$. 10. $x^5 - 5x^3 + 5x^2 - 1$. 11. $x^6 - 2a^3x^3 + a^6$.
12. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$. 13. $x^6 + 10x - 33$. 14. $x^6 - 2x^3 + 1$.
15. $a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8$. 16. $x^3 + y^3 + z^3 - 3xyz$.
17. $a^3 + b^3 + c^3 - 3abc$. 18. $2a^6 - a^5b - 14a^4b^2 + 13a^3b^3 - 43a^2b^4 + 23ab^5 - 20b^6$. 19. $apx^3 + (bp - aq)x^2 - (cp + bq)x + cq$.
20. $mnx^3 - (n^2 + mr)x^2 + r^2$. 21. $ax^4 - (1 + a)bx^3 + (c + b^2 - ac)x^2 - c^2$.
22. $abx^5 - (b^2 + ac)x^4 + (2bc + ad)x^3 - (2bd + c^2)x^2 + 2cdx - d^2$.
23. $mpx^4 - (mq - mr + np)x^3 + (ms + nq - nr - ps)x^2 + (q - r - n)sx - s^2$.
24. $alx^3 + (2hl + am)x^2y + (bl + 2hm)xy^2 + bmy^3 + anx^2 + 2hnxy + bny^2$.

25. $l^2px^4 + m^2px^3y + n^2px^2y^2 + (l^2q + 2g^2p)x^3 + (m^2q + 2f^2p)x^2y + n^2qxy^2 + (c^2p + 2g^2q + l^2r)x^2 + (m^2r + 2f^2q)xy + n^2ry^2 + (2g^2r + c^2q)x + 2f^2ry + c^2r$. 26. $x^5 + 1\frac{766}{525}x^4y + 3\frac{289}{525}x^3y^2 + 2\frac{656}{735}x^2y^3 + 2\frac{4}{35}xy^4 + y^5$.
27. $x^6 + 1\frac{2}{3}x^5y + 3\frac{5}{11}x^4y^2 + \frac{86}{33}x^3y^3 + 4\frac{16}{33}x^2y^4 + \frac{1}{11}xy^5 + y^6$.
28. $\cdot621x^{12} + 3\cdot197x^{10} + 20\cdot7x^9 + \cdot405x^8 + \cdot3321x^7 + 2\cdot085x^6 + 16\cdot0872x^5 + 11\cdot07x^4 + 5\cdot8675x^3 + 29\cdot25x^2 + 6\cdot95x + 45$. 29. $\cdot399a^5 + 7\cdot289a^4b + 16\cdot71a^3b^2 + 32\cdot867a^2b^3 + 23\cdot789ab^4 + 25\cdot2b^5$. 30. $2\cdot3lx^5 + (3\cdot15l + 2\cdot3m)x^4y + (1\cdot17l + 3\cdot15m + 2\cdot3n)x^3y^2 + (2\cdot07l + 1\cdot17m + 3\cdot15n)x^2y^3 + (2\cdot07m + 1\cdot17n)xy^4 + 2\cdot07ny^5$. 31. $a^2x^5 - \frac{20}{21}abx^4y + (\frac{181}{90}ac - b^2)x^3y^2 + (\frac{55}{756}bc + \frac{4}{3}ad)x^2y^3 + (c^2 - \frac{6}{7}bd)xy^4 + \frac{8}{5}cdy^5$
32. $2\cdot25a^2m^6 + (3\cdot9ac - 1\cdot44b^2)m^4n^2 - (3\cdot84bd - 1\cdot69c^2)m^2n^4 - 2\cdot56d^2n^6$
33. $16a^4 - 81b^4$. 34. $625a^4x^4 - 1296b^4y^4$. 35. $x^{12} - y^{12}$.
36. $x^8 + 49x^4y^4 + 625y^8$. 37. $a^{16}x^{16} - b^{16}y^{16}$. 51. $-6x^2$
52. $-2y^4$. 53. $6x^2y$. 54. $15x^{\frac{5}{6}}y^{\frac{5}{4}}$. 55. $-3ab^{-2}$. 56. $-ay^{-1}$.
57. $12a^2b^2c^2$. 58. $15xyz$. 59. $-30a^2bc^{-1}$. 60. $76a^{\frac{4}{3}}x^{-\frac{7}{6}}y^{-2}$.
61 $a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$. 62. $a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$. 63. $9x^{\frac{4}{3}} - 16y^{\frac{2}{3}}$. 64. $a + b$.
65 $x - y$. 66. $a^3 + a^{\frac{3}{2}}b^{\frac{3}{2}} + b^3$. 67. $4x^{\frac{8}{3}} - 37x^{\frac{4}{3}}y^{\frac{4}{3}} + 9y^{\frac{8}{3}}$
68 $a^3 - b^2$. 69. $x^2 - y^2$. 70. $a - b^2$. 71. $x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$
72. $a^{3n} + x^{3n}$. 73. $a^{-5} - 6a^{-4}b + 13a^{-3}b^2 - 13a^{-2}b^3 + 6a^{-1}b^4 - b^5$.
74. $x^{-6} - 5x^{-3}y^3 + 4y^6$. 75. $4a^{-10} + 12a^{-\frac{15}{2}}b^{-\frac{3}{2}} + 9a^{-5}b^{-3} - 25b^{-6}$.
76. $6x^3 + 19x^2 + 42x + 45$. 77. $2x^3 - 7x^2 - 24x + 45$.
78. $3x^5 + 9x^4 + 11x^3 + 21x^2 + 28x + 12$. 79. $px^4 + qx^3 + p^2x^2 + p(q + r)x + qr$. 80. $\frac{1}{2}x^6 + \frac{1}{3}x^5 + 7\frac{5}{12}x^4 + 4\frac{1}{2}x^3 + 16\frac{1}{2}x^2 + 5x + 10$.
81. 1 82. -9. 83. $-6, 20$.

Exercise 39. [Pages 121-122]

1. $x - 2$. 2. $x - 5$. 3. $3x + 4$. 4. $5x - 7$.
5. $2a - 3b$. 6. $x^2 - xy + y^2$. 7. $2x - 3a$. 8. $x^2 - ax + a^2$.
9. $a^2 + 2ab - b^2$. 10. $x + 3$. 11. $2x - 1$. 12. $2ay - b$.
13. $am + 3n$. 14. $2x^2 + 3xy - 4y^2$. 15. $3y^3 - x^2y + 2x^3$.
16. $4m^2 - 6mn + 8n^2$. 17. $a^3 - 3a^2y - y^3$. 18. $27(z + a)$. 19. $z - x$.
20. $x^4 + 2ax^3 + 3a^2x^2 + 2a^3x + a^4$. 21. $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4$.

22. $x^2+(a+b)x+ab$. **23.** $x-c$. **24.** $a+b+c$. **25.** $ab+ac+bc$

26. $ab+ac-bc$. **27.** $x^2-(a-b)x-ab$.

28. $a^2+b^2+c^2-ab-ac-bc$. **29.** $x^2+y^2+1-xy+x+y$.

30. $x^2+4y^2+9z^2+2xy+3xz-6yz$. **31.** $x^2+y^2+z^2+xy-xz+yz$

32. $2x-3y-z$. **33.** $ab-ac-bc+c^2$. **34.** $x+c$.

35. $x+a$. **36.** $a^2+ab-bc-c^2$. **37.** $ab-ac+bc-b^2$.

38. $y^2x+2y^2z+yx^2-2yz^2-x^2z-xz^2$. **39.** x^2-ax+a^2.

40. $c+a-b$. **41.** $2(a+b)x$. **42.** $x+y+z+xyz$.

43. $16x^4-8x^2(2y^2+a^2)+(4y^2-a^2)^2$. **48.** a^3b. **49.** $ab^{-1}c^{\frac{1}{2}}$.

50. $-3x^{\frac{1}{3}}y^{\frac{2}{3}}z^{-\frac{1}{3}}$. **51.** $3x^{\frac{2}{3}}-4y^{\frac{1}{3}}$. **52.** $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$

53. $a^{\frac{3}{2}}-a^{\frac{3}{4}}b^{\frac{3}{4}}+b^{\frac{3}{2}}$. **54.** $2x^{\frac{4}{3}}-5x^{\frac{2}{3}}y^{\frac{2}{3}}-3y^{\frac{4}{3}}$.

55. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}}$. **56.** $2a^{-6}+3a^{-\frac{5}{2}}b^{-\frac{3}{2}}+5b^{-3}$.

57. $3x^{-\frac{5}{4}}-5x^{-\frac{5}{8}}y^{-\frac{3}{8}}+7y^{-\frac{3}{4}}$. **58.** $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$.

59. $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}-z^{\frac{1}{3}}x^{\frac{1}{3}}$.

Exercise 40. [Page 124]

1. $m^2-3mn+2n^2$. **2.** $a^2-3ab+b^2$. **3.** $2x^2-3xy-2y^2$.

4. $a^2-4ax-2x^2$. **5.** $3+2x-2x^2+x^3$. **6.** x^2-2x+3.

7. $2a^2+3ab-4b^2$. **8.** $a^2-2ax+4x^2$. **9.** $a^2-2ab+2b^2$.

10. $2x^2-3x-8$. **11.** $x^3+3x^2+9x+27$.

12. $a^4+2a^3+4a^2+8a+16$. **13.** $3-x^2+2x^3$.

14. $3x^2-4x+5$. **15.** $32+16x+8x^2+4x^3+2x^4+x^5$.

16. $x^4+2x^3+3x^2+2x+1$. **17.** $2a^2-3ab+4b^2$.

18. $a^2+3ab-5b^2$. **19.** $x^3+2x^2a+2xa^2+a^3$

20. $a^3-3a^2b-b^3$. **21.** $x^4+2yx^3+3y^2x^2+2y^3x+y^4$

22. $x+6+\dfrac{5}{x+5}$. **23.** $x^2+\frac{1}{3}xy+\frac{1}{9}y^2+\dfrac{\frac{1}{27}y^3}{x-\frac{1}{3}y}$. **24.** r.

25. $\frac{1}{3}+\frac{7}{9}x+\frac{16}{27}x^2+\frac{16}{81}x^3$ is the quotient and $\frac{16}{81}x^4$ is the remainder.

Exercise 41. [Page 126]

13. x^3+x^2+x+1. **14.** $x^3-x^2y+xy^2-y^3$.

15. $x^4+x^3+x^2+x+1$. **16.** $x^4-x^3y+x^2y^2-xy^3+y^4$.

17. $x^5+x^4+x^3+x^2+x+1$. **18.** $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$.

19. $x^6+x^5+x^4+x^3+x^2+x+1$.

20. $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.

Exercise 42. [Pages 129-130]

1. $25x^2+90xy+81y^2$. **2.** $256a^2-416ab+169b^2$.
3. $x^2+200x+10000$. **4.** $y^2+1000y+250000$.
5. $a^2+1998a+998001$. **6.** $y^2+20002y+100020001$.
7. 976144. **8.** 1024144. **9.** 10100·25. **10.** 9920·16.
11. $8x^3+60x^2+150x+125$. **12.** 1157625. **13.** 985074·875.
14. 513152864·216. **15.** (i) 50000032 ; (ii) 2000288.
16. (i) $(3x+3y)^2-(x-y)^2$; (ii) $(5x+8y)^2-(x+2y)^2$;
(iii) $(x+100)^2-2^2$; (iv) $(500)^2-5^2$; (v) $(2x+100)^2-(\cdot 4)^2$.
17. a^4-x^4. **18.** $16a^4-81$. **19.** $a^8+a^4b^4+b^8$.
20. 99999984. **21.** 99999744. **22.** $8a^3+12a^2x+6ax^2+x^3$.
23. $a^6-12a^4+48a^2-64$. **24.** x^6+64. **25.** $8y^3-27$.
26. x^6-64. **27.** $4x^2+240x+1575$. **28.** $36x^2+108x-1075$
29 $36x^2-408x+1075$. **30.** $100a^2$. **31.** x^2+y^2+2xy.
32. $8000a^3$. **33.** $125a^3$. **34.** $1331a^3$.
35. $5x^3$. **36.** $(9a+8b+8)(a+8b-4)$. **37.** $(2x+5y)(4x^2-10xy+25y^2)$.
38. $(8a+13x-4)\{(8a+13x)^2+4(8a+13x)+16\}$.
39. $(15a+3b+2)(15a+3b-2)$. **40.** 225. **41.** 512
42 10000. **43.** 8099·999996. **44.** 92355. **45.** 1.
46. $(2+5x+3x^2)(2+5x-x^2)$. **47.** $a^2b^2(x^2+1)^2-(a^2+b^2)^2x^2$.
48 $(11x^2+28x+10)^2-(x^2+x+5)^2$. **49.** $(49x^2+98ax+39a^2)^2+(5a^2)^2$.

Exercise 43. [Pages 137-138]

1. (i) 121 ; (ii) 49 ; (iii) 4. **3.** 144 sq. cm. **4.** 15 sq. metres.
5. 55 sq. metres. **6.** 84 sq. metres. **7.** 500 sq. metres. **8.** 50.
9. 76 sq. metres. **10.** 171 sq. metres.

Exercise 44. [Page 139]

1 $a(b+c)$. **2.** $a^{x+2}(b+c)$. **3.** $x^3y^3(y-2x)$.
4. $2xyz(x+2y-3z)$. **5.** $2a^3b(2a^2-3ab-4b^2)$.
6. $ax^2(y-5axy^2+3x)$. **7.** $3x^2y^2z^2(x^2y-4y^2z+7xz^2)$.
8. $14a^5b^5(2a^3-3b^3)$. **9.** $36x^3y^3(2x^2+3y^2)$.
10. $13a^5b^5c^5(3b^2c^2-5c^2a^2-7a^2b^2)$

Exercise 45. [Page 139]

1. $(3a+4b)(3a-4b)$. 2. $a(2a+5x)(2a-5x)$.
3. $(6x^2+1)(6x^2-1)$. 4. $(4x^2+1)(2x+1)(2x-1)$.
5. $x(4x^2+3)(4x^2-3)$. 6. $x(4x^2+9)(2x+3)(2x-3)$.
7. $(1+4a^2)(1+2a)(1-2a)$. 8. $x^2(1+9x^2)(1+3x)(1-3x)$.
9. $\left(6+\frac{x^2}{a}\right)\left(6-\frac{x^2}{a}\right)$. 10. $(8a^2+7x^3)(8a^2-7x^3)$.
11. $(11+m^3)(11-m^3)$. 12. $(7x^3a^5+9)(7x^3a^5-9)$.
13. $(ab+5cd)(ab-5cd)$. 14. $(9x^6+8a^5)(9x^6-8a^5)$.
15. $p^2(q^2+10)(q^2-10)$. 16. $x^3(12x^2+5a^2)(12x^2-5a^2)$.
17. $3a^5(8a^2+9x^2)(8a^2-9x^2)$. 18. $2ax(7ax^2+8)(7ax^2-8)$.
19. $4x^5a^3(9x^6a^3+11)(9x^6a^3-11)$. 20. $5m^{15}n^7(7m^4n^3+11)(7m^4n^3-11)$
21. $(a+3b+5c)(a+3b-5c)$. 22. $(a+3b-5c)(a-3b+5c)$. 23. $4xy$.
24. $(5a+3x)(a+x)$. 25. $(2a-2b+3c-3d)(2a-2b-3c+3d)$.
26. $(7x+5y-3z)(7x-5y+3z)$. 27. $12(5x-1)(x+2)$. 28. $4a(b-c)$.
29. $(3a+b-c)(a-7b+9c)$. 30. $(14a+21x-23y)(2a+27x-41y)$
31. $-9(x+a)(x-a)(x^2+a^2)$ 32. $28a(5a-3)$.
33. $(x+y)(a+b)(x+y+a+b)(x+y-a-b)$.

Exercise 46. [Pages 140-141]

1. $(x^2+x+1)(x^2-x+1)$. 2. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.
3. $(a^2+ax+x^2)(a^2-ax+x^2)$.
4. $(a^2+ax+x^2)(a^2-ax+x^2)(a^4-a^2x^2+x^4)$.
5. $(x^2+4x+8)(x^2-4x+8)$. 6. $(2x^2+6x+9)(2x^2-6x+9)$.
7. $9(x^2+2x+2)(x^2-2x+2)$. 8. $(a^2+2a+3)(a^2-2a+3)$.
9. $(x^2+x-3)(x^2-x-3)$. 10. $(2x^2+2x+3)(2x^2-2x+3)$.
11. $(2x^2+2x-3)(2x^2-2x-3)$. 12. $(2x^2+3x+3)(2x^2-3x+3)$.
13. $(2a^2+5a-3)(2a^2-5a-3)$. 14. $(2a^2+10a+25)(2a^2-10a+25)$.
15. $(3x^2+x+4)(3x^2-x+4)$. 16. $(3a^2+a-4)(3a^2-a-4)$.
17. $(3x^2+3x-4)(3x^2-3x-4)$. 18. $(3a^2+5a+4)(3a^2-5a+4)$.
19. $(4x^2+6xa+5a^2)(4x^2-6xa+5a^2)$.
20. $(3a^2+7ax+5x^2)(3a^2-7ax+5x^2)$. 21. $(x^2+4x+12)(x^2-4x+12)$.
22. $(a^2+5ab-5b^2)(a^2-5ab-5b^2)$. 23. $(6a^2+2ab-b^2)(6a^2-2ab-b^2)$.
24. $(7m^2+2mn-4n^2)(7m^2-2mn-4n^2)$.
25. $(8a^2+12ax+9x^2)(8a^2-12ax+9x^2)$.

26. $(2x^2+14xa+49a^2)(2x^2-14xa+49a^2)$. **27.** $(x+y-z)(x-y+z)$.
28. $(2a+b-3c)(2a-b+3c)$. **29.** $(3x+2y-3z)(3x-2y+3z)$.
30. $(a+2b-5c)(a-2b+5c)$. **31.** $(4y+3x-5z)(4y-3x+5z)$.
32. $(a-2b+3c-2d)(a-2b-3c+2d)$. **33.** $(x-2y+z)(x-z)$.
34. $(2x+3a+5b+1)(2x+3a-5b-1)$.
35. $(3x+2y-7z-5)(3x-2y+7z-5)$.
36. $(4a+3b-4c-3)(4a-3b+4c-3)$.
37. $(x-7y+5z-2)(x-7y-5z+2)$.
38. $(4x+5a+3y-7b)(4x+5a-3y+7b)$.
39. $(7x-4y+8z-1)(7x-4y-8z+1)$.
40. $(a+b-c-d)(a-b+c-d)$.

Exercise 47. [Page 142]

1. $(a-2b)(a^2+2ab+4b^2)$. **2.** $a(a-3x)(a^2+3ax+9x^2)$.
3. $(2x+1)(4x^2-2x+1)(64x^6-8x^3+1)$. **4.** $(a-2b)(a^2+2ab+4b^2)\times(a^6+8a^3b^3+64b^6)$. **5.** $(3a^2+5x^2)(9a^4-15a^2x^2+25x^4)$.
6. $(m+n)(m-n)(m^2-mn+n^2)(m^2+mn+n^2)$. **7.** $(7x+8y)\times(49x^2-56xy+64y^2)$. **8.** $(2x^2-1)(2x^2+1)(4x^4+2x^2+1)(4x^4-2x^2+1)$.
9. $(a-2x^2)(a+2x^2)(a^2+2ax^2+4x^4)(a^2-2ax^2+4x^4)$. **10.** $(5x^3-6a^3)\times(25x^6+30x^3a^3+36a^6)$. **11.** $ab(4a^4+7b^4)(16a^8-28a^4b^4+49b^8)$.
12. $x^2y^2(3x^3+2y^3)(3x^3-2y^3)(9x^6-6x^3y^3+4y^6)(9x^6+6x^3y^3+4y^6)$.
13. $(a+b)^2(a^4-2a^3b+6a^2b^2-2ab^3+b^4)$. **14.** $2(x+y)(x-y)\times(4x^4-14x^2y^2+13y^4)$. **15.** $2(a-b)(a^2+ab+b^2)(4a^6-2a^3b^3+b^6)$.
16. $\left(a^2+\frac{b^2}{3}\right)\left(a^4-\frac{a^2b^2}{3}+\frac{b^4}{9}\right)$. **17.** $\left(a-\frac{2}{b}\right)\left(a^2+\frac{2a}{b}+\frac{4}{b^2}\right)$.
18. $\left(\frac{1}{2x}+\frac{2}{y}\right)\left(\frac{1}{4x^2}-\frac{1}{xy}+\frac{4}{y^2}\right)$.

Exercise 48. [Pages 147 148]

1. $(x+1)(x+2)$. **2.** $(x+2)(x+3)$. **3.** $(a+1)(a+3)$.
4. $(x-4)(x-1)$. **5.** $(x+2)(x+6)$. **6.** $(x-3)(x-4)$.
7. $(x+5)(x+3)$. **8.** $(x-5)(x+3)$. **9.** $(x-4)(x-9)$.
10. $(x+4)(x-9)$. **11.** $(x-2)(x-12)$. **12.** $(x-2)(x-20)$
13. $(x+10)(x-3)$. **14.** $(x+8)(x-6)$. **15.** $(x+18)(x-2)$
16. $(x+12)(x-3)$. **17.** $(x+14)(x-3)$. **18.** $(x+18)(x-4)$.

19. $(x-8)(x+5)$: **20.** $(x-16)(x+5)$. **21.** $(x-32)(x+3)$.
22. $(x-14)(x+4)$. **23.** $(x-7)(x+6)$. **24.** $(x-9)(x+8)$.
25. $(x+10)(x+12)$. **26.** $(x+20)(x-4)$ **27.** $(x-24)(x+3)$
28. $(x+12)(x-7)$. **29.** $(x-12)(x-8)$. **30.** $(x+26)(x-3)$
31. $(x-12)(x+6)$. **32.** $(x-21)(x-4)$. **33.** $(x-22)(x-4)$
34. $(x+15)(x-8)$. **35.** $(x-10)(x+8)$. **36.** $(x+14)(x-6)$
37. $(a-8)(a+7)$ **38.** $(m-15)(m+6)$. **39.** $(a+20)(a-3)$
40. $(a-9)(a-6)$. **41.** $(p-24)(p+2)$. **42.** $(m+9)(m-8)$
43. $(m+30)(m-2)$. **44.** $(a-24)(a-5)$. **45.** $(x+13)(x-6)$.
46. $(a-51)(a+2)$ **47.** $(a-15)(a-4)$. **48.** $(x+16)(x-4)$.
49. $(a-30)(a+4)$ **50.** $(x+15)(x-7)$. **51.** $(x-7y)(x+6y)$.
52. $(a-8b)(a-4b)$. **53.** $(m+6n)(m-5n)$. **54.** $(a+4b)(a-3b)$
55. $(a-5b)(a+3b)$ **56.** $(x-8y)(x+y)$. **57.** $(x+6y)(x-5y)$.
58. $(p-6q)(p-8q)$ **59.** $(p+10q)(p-8q)$. **60.** $3(x+24y)(x-4y)$.
61. $(a+1)(a-1)(a^2+5)$ **62.** $(x^2+5)(x^2-3)$.
63. $2(x+2)(x-2)(x^2+7)$. **64.** $(x-1)(x^2+x+1)(x^3+3)$.
65. $(a-2)(a^2+2$ [illegible] $-2)$. **66.** $(x-1)(x+3)(x^2+x+1)(x^2-3x+9)$.
67. $(a-1)(a+2)(a^2+a+1)(a^2-2a+4)$.
68. $(x^2+2)(x^2-2)(x-3)(x-2)(x^2+4)$. **69.** $(a+2)(a-2)(a^2+4)(a^4+5)$.
70. $(x^2+1)(x^2-2)(x^4-x^2+1)(x^4+2x^2+4)$. **71.** $(a+1)^2(a^2+2a-2)$.
72. $(x+1)(x+2)(x^2+3x+1)$. **73.** $(x-1)^2(x+1)(x-3)$
74. $(a+1)(a-4)(a^2-3a+1)$. **75.** $(x+1)(x-5)(x^2-4x+1)$
76. $(x+1)(x-2)(x+2)(x-3)$. **77.** $(x-2)(x-3)(x-1)(x-4)$.
78. $(a-2)(a+9)(a+2)(a+5)$ **79.** $(a-4)(a+10)(a+2)(a+4)$
80. $(x+1)(x-9)(x+2)(x-10)$. **81.** $(2x-5)(x+3)$.
82. $(3a-5)(2a+3)$ **83.** $(4m+3)(2m-3)$. **84.** $3x(2x-3y)(3x+8y)$.
85. $(5a-3b)(2a-7b)$. **86.** $(3m-4n)(4m+5n)$. **87.** $(2x+5y)(6x-y)$.
88. $(4a+5b)(5a-6b)$. **89.** $(3x-5y)(6x-7y)$. **90.** $5xy(4x-3y)(3x+8y)$

Exercise 49. [Pages 150-151]

1. $(x+3)(x+1)$. **2.** $(x+5)(x+1)$. **3.** $(x+5)(x+3)$. **4.** $(x-7)(x-3)$.
5. $(x-8)(x+6)$. **6.** $(x-9)(x+5)$ **7.** $(x-8)(x-4)$.
8. $(x-11)(x+5)$. **9.** $3x(x-7)(x+5)$. **10.** $(2x+1)(x-3)$.
11. $(3x+1)(x-2)$. **12.** $(3x+2)(x+4)$. **13.** $(4x-1)(x+2)$.
14. $(2x-1)(3x+2)$. **15.** $(2x+1)(3x-4)$. **16.** $(3x-1)(2x+3)$.
17. $(2x+3)(4x-5)$. **18.** $(2x-5)(2x+7)$. **19.** $(2x-3)(3x+4)$.
20. $(3x+2)(x-6)$. **21.** $(2x+5)(x-7)$. **22.** $(2x-7)(x+6)$.

23. $(3x-5)(x+6)$. 24. $(3x-2)(4x+3)$. 25. $(a+5b)(2a-3b)$.
26. $(2x-3y)(3x-2y)$. 27. $(3m+2n)(2m-5n)$. 28. $(3p-4q)(p+3q)$.
29. $(2a-5b)(4a+3b)$. 30. $(5m-2n)(2m+3n)$. 31. $(4x-y)(3x+4y)$.
32. $(3a-4b)(5a+3b)$. 33. $(2a-b)(a-2b)$. 34. $(a-3b)(3a+b)$.
35. $(x+3y)(3x-y)$. 36. $(a+4)(4a-1)$. 37. $(a-4b)(4a-b)$.
38. $(x-5)(5x+1)$. 39. $(x-5y)(5x-y)$. 40. $(x+6)(6x+1)$.
41. $(a+6b)(6a-b)$. 42. $(a-6b)(6a+b)$. 43. $(a-7b)(7a-b)$.
44. $(a+7b)(7a-b)$. 45. $(a-7b)(7a+b)$. 46. $(8x-y)(x+8y)$.
47. $(9x-y)(x-9y)$. 48. $(10x-y)(x+10y)$. 49. $(2a+2b-1)(a+b+2)$.
50. $(x-y)^2(2x^2+2y^2+xy)$. 51. $(a+b)^2(2a^2+2b^2+ab)$.
52. $(x-4y)(4x-y)(x^2+y^2)$. 53. $(x+2)(x-2)(2x^2+3)$.
54. $(2a+3b)(2a-3b)(2a^2+b^2)$. 55. $(3a+4b)(3a-4b)(a^2+2b^2)$.
56. $(x-2)(2x-1)(x^2+2x+4)(4x^2+2x+1)$.
57. $(2a^2+b^2)(2a^2-b^2)(a^2+2b^2)(a^2-2b^2)$.

Miscellaneous Exercises III

[Pages 157-162]

I

1. (i) $y^3(x-2z)-y^2xz+y(xz^2-x^3-2z^3)+(x^3z-xz^3)$;
(ii) $(xy^3-x^3y)+z(x^3-xy^2-2y^3)+z^2xy-z^3(x+2y)$.
2. 94. 4. $x^4+x^3y+x^2y^2+xy^3+y^4$. 5. $8ab$; 128.
6. $2(x^2+y^2+z^2-yz-zx-xy)$. 7. $(a+c)^2-(b-d)^2$
8. $(2x+3y)(2x+3y-4)$.

II

1. $\frac{3}{4}[3ax^3+ax^2y+3axy^2+7dy^3]$. 2. $m(am+b)(m^2+2)$.
3. $x-2x^{\frac{1}{2}}+1$. 5. $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$.
6. $x^4-(p-1)x^3+(q-p+1)x^2-(p-1)x+1$. 7. $a^4+a^2b^2+b^4$.
8. $(a-b)(b-c)$; $(b+2a+3c)(b-2a-3c)$.

III

1. px^3+qx^2+rx+s. 2. $-3b^3$. 3. 392. 4. 16.
5. $x^2-xy-xz+yz$. 6. $(4a-1)(16a^2+8a+3)$.

IV

1. $-l^4r^2+6l^3mnr-2l^3n^3-3l^2m^2n^2-4l^2m^3r+6lm^4n-2m^6$.

2. $3a^3x^4+6a^2bx^3+3ab^2x^2+4b^3x+12$. 3. a^3-64b^3.

4. (i) $a^3-3abc+(b^3+c^3)$; (ii) $a^2(b-c)-a(b^2-c^2)+(b^2c-bc^2)$;
(iii) $a^4(b-c)-a(b^4-c^4)+(b^4c-bc^4)$.

5. $x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$; -11; -68.

8. $a+2b+3c$.

V

1. 121. 2. $\frac{1}{2}(ax^5+bx^4y-cx^3y^2-dx^2y^3+exy^4-fy^5)$.

4. $8a^3+12a^2c+6ac^2+c^3$. 5. $8x^6+4x^5+12x^4-8x^2+24x-32$.

7. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$. 8. (i) $a^2x(a-x)(6ax+1)$; (ii) $(x+yz)(y+zx)$.

VI

1. 87659405. 2. -125. 4. x^2+1. 5. $(x^2-7x+9)^2+(5)^2$.

6. $(a+b+c+d)(a-b-c+d)(a+b-c-d)(a-b+c-d)$.

7. (i) $(a-b)(a-b+2)$; (ii) $(2a-b)(3a+b+3)$;
(iii) $(5x+2y)(3x-2y+2)$. 8. $(2x-y)a^2+(x+y)ax-x^3$.

VII

1. 7. 3. $x^{\frac{3}{2}}+y^{\frac{3}{2}}+z^{\frac{3}{2}}-3x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}$. 4. $x+a$.

6. $2x^2y^2+2y^2z^2+2z^2x^2-x^4-y^4-z^4$. 7. (i) $(2x-3)(3x+5)$;
(ii) $(5x-5y-3)(7x-7y-4)$; (iii) $(x-3y^2)(11x-21y^2)$.

VIII

1. $1+(a+b)x+\frac{(a+b)(a+b-1)}{2}x^2$. 2. 55. 4. $2(a+b)x$.

5. a^2-2; a^3-3a; a^4-4a^2+2. 8. $(x^2-3xy-y^2)(x^2+3xy-y$

IX

1. (i) $a^4+a^2x^2+x^4$. (ii) 4. 4. $x^3+6x+\frac{12}{x}+\frac{8}{x^3}$.

6. $a^2(b-c)-a(b^2-c^2)+bc(b-c)$. 8. 3.

X

1. $2(a+m)(c+n)+2bd$. 2. 16. 5. $a^{\frac{3}{2}}+b+c^{\frac{3}{4}}-3a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$.

6. $(3x-7)(5x-2)$. 7. 17. 8. Rs. 125 each.

Exercise 51. [Pages 164-165]

1. a^2b^2. 2. $4a^2$. 3. $3xy^2$. 4. $5a^2y^3$.
5. $9m^2n^3$. 6. $4ax$. 7. $12mp$. 8. $15x^2y^2z^2$.
9. $18a^2c^2$. 10. $24c^3$. 11. $12z^2$. 12. $15m^3n^3p^3q^3$.
13. $18a^2b^2c^2d^2$. 14. 6. 15. $8x^2y^2$.

Exercise 52. [Page 166]

1. $a(a+b)$. 2. $x^3y^3(x+y)$. 3. $3(x+3)$.
4. $4a^2(a^2+bc)$. 5. $m^3n^3(m-n)^2$. 6. $ax(2a+3x)$.
7. $2a^2b^2(3a+4b)$. 8. $3x^2y^2(x-2y)$. 9. $2ab(a+2b)$.
10. $16x^3a^3(x^2-a^2)$. 11. $8(x^2+ax+a^2)$. 12. $8xa^2(x^2+a^2)$.
13. $6(a+3b)$. 14. $4x(x-5)$. 15. $xy(x+6y)$. 16. $a^2x^2(a+2x)$.
17. $2x+3$. 18. $a-2b$. 19. $a-b$. 20. $x+2$.
21. $4ab(3a+b)$.

Exercise 53. [Page 169]

1. a^2b^2. 2. a^3b^2c. 3. $30x^2y^4$. 4. $28m^4n^3p$.
5. $24x^3y^3z^2$. 6. $140a^2b^2c^2$. 7. $120a^3b^3c^3$. 8. $180x^4y^3z^2a^2$.
9. $a^2b^2(a^2-b^2)$. 10. $24(x^2-y^2)^2$. 11. $(x-1)(x-2)(x-3)$.
12. $a^2(a-x)(a+3x)(a-2x)$. 13. $a^2(a-2)(a+2)(a+4)$.
14. $12a^3x^2(x^2-a^2)(x^2-ax+a^2)$. 15. $48(x-2)(x+5)(x+6)$.
16. $(x-3)(x+5)(x+4)(x+7)$. 17. $3a^2(4a^2-9b^2)(a^2-b^2)$.
18. $(8a^3+27b^3)(8a^3-27b^3)$. 19. $12x^2(4x^2-25y^2)(2x-y)$.
20. $(2x-3a)^2(9x^2-a^2)$. 21. $2x(4x^4+81)$.
22. $6(3a-x)^2(a^2-4x^2)$. 23. $(2x-1)^2(4x^2-1)(x+3)$.
24. $(x-2y)(x-4y)(x-3y)(x+5y)$. 25. $(x+2)(2x-1)(3x+1)$.
26. $(1-4x^2)(1+2x+4x^2)(1+2x-4x^2)$. 27. $(x^2-3)^2(9x^2-1)(9x^4-1)$.

Exercise 54. [Pages 171-172]

1. $\frac{1}{2b}$. 2. $\frac{3x}{4y}$. 3. $\frac{2a}{5x}$. 4. $\frac{3xz}{5y^2}$. 5. $\frac{2d}{3ab}$.
6. $\frac{2az}{5xy}$. 7. $\frac{2d^5}{3a}$. 8. $\frac{3npq}{5m}$. 9. $\frac{x-a}{x}$. 10. $-\frac{1}{x+3}$.
11. $\frac{2x-3a}{2x}$. 12. $-\frac{a}{a+4b}$. 13. $\frac{3a}{x+4a}$. 14. $\frac{2x^2}{x^2-2a^2}$. 15. $\frac{4x}{x+3}$.
16. $\frac{x-2}{x-3}$. 17. $\frac{x-3}{x+4}$. 18. $\frac{a-4b}{a-5b}$. 19. $\frac{a^2}{a+b}$. 20. $\frac{1-4x}{1-5x}$.

21. $\frac{x-y}{x+7y}$. 22. $\frac{1-2a^2}{1+3a^2}$. 23. $\frac{x^2-13}{x^2-4}$. 24. $\frac{3ax}{a-3x}$. 25. $\frac{2x+3}{3x+4}$.

26. $\frac{x-a}{x+a}$. 27. $\frac{x+5a}{x+7a}$. 28. $\frac{2x-5}{3x-2}$. 29. $\frac{2x-5a}{3x-7a}$. 30. $\frac{2-3ax}{1-5ax}$.

31. $\frac{x-a}{x^2+a}$. 32. $\frac{3a+5b}{3c-1}$. 33. $\frac{2x+3a}{3x+2a}$. 34. $\frac{2a-b}{a^2-1}$. 35. $\frac{a-b-c}{a+b-c}$.

Exercise 55. [Pages 173-174]

1. $\frac{2adf}{4bdf}, \frac{3bcf}{4bdf}, \frac{4bde}{4bdf}$.

2. $\frac{6ax^2}{12abc}, \frac{4by^2}{12abc}, \frac{3cz^2}{12abc}$.

3. $\frac{15x^2ab}{60x^3y^2}, \frac{10xybc}{60x^3y^2}, \frac{6y^2ca}{60x^3y^2}$.

4. $\frac{a^2(a+b)}{a(a^2-b^2)}, \frac{ab(a-b)}{a(a^2-b^2)}, \frac{c(a-b)}{a(a^2-b^2)}$.

5. $\frac{x^2(a-2b)}{a(a^2-4b^2)}, \frac{ay^2(a+2b)}{a(a^2-4b^2)}$.

6. $\frac{2a^2}{a(a-b)}, \frac{c-a}{a(a-b)}$.

7. $\frac{2a(a+b)}{a^2-b^2}, \frac{-3b(a+b)}{a^2-b^2}, \frac{4\,[\text{illegible}](a-b)}{a^2-b^2}$.

8. $\frac{2b^2c^2x(a-x)}{a^2b^2c^2(a^2-x^2)}, \frac{3c^2a^2y(a+x)}{a^2b^2c^2(a^2-x^2)}, \frac{4a^2b^2z}{a^2b^2c^3(a^2-x^2)}$.

9. $\frac{a^2x(2x+3y)}{xy(4x^2-9y^2)}, \frac{b^2y(2x-3y)}{xy(4x^2-9y^2)}, \frac{c^2}{xy(4x^2-9y^2)}$.

10. $\frac{a^2(x^2-x+1)}{x^4+x^2+1}, \frac{b^2(x^2+x+1)}{x^4+x^2+1}$.

11. $\frac{3(x+3)}{x^3+2x^2-5x-6}, \frac{4(x+1)}{x^3+2x^2-5x-6}$.

12. $\frac{a^2-4b^2}{a^4+8ab^3}, \frac{abc}{a^4+8ab^3}$.

13. $\frac{a(a^2+3ab+9b^2)}{a^3-27b^3}, \frac{b(a-3b)}{a^3-27b^3}, \frac{c}{a^3-27b^3}$.

14. $\frac{a^2(a-b+c)}{ab(a^2+b^2-c^2-2ab)}, \frac{b^2(a-b-c)}{ab(a^2+b^2-c^2-2ab)}, \frac{abc}{ab(a^2+b^2-c^2-2ab)}$.

15. $\frac{(c-a)^2}{(a-b)(b-c)(c-a)}, \frac{(a-b)^2}{(a-b)(b-c)(c-a)}, \frac{(b-c)^2}{(a-b)(b-c)(c-a)}$.

Exercise 56. [Pages 176-177]

1. $\frac{a^2+b^2}{ab}$. 2. 0. 3. 1. 4. $\frac{4ab}{a^2-b^2}$.

5. $\frac{a+b}{2(a-b)}$. 6. $\frac{12xy}{4x^2-9y^2}$. 7. $\frac{a^2-2ab-b^2}{(a+b)^2(a-b)}$.

8. $\frac{2a^3}{a^2-b^2}$. 9. $\frac{1}{(a-b)(b-c)}$. 10. $\frac{2}{x^2-4x+3}$.

11. $\frac{2}{x^2+10x+16}$. 12. $\frac{6xy}{8x^3+27y^3}$. 13. $\frac{2ab}{a^2-b^2}$.

14. 0. 15. $\frac{8x^2y^2}{x^4-y^4}$. 16. $\frac{-64ax^3}{a^4-16x^4}$.

17. $\frac{x^2}{6(x^2-9)}$. **18.** $\frac{2b}{1-16a^2b^2}$. **19.** $\frac{4x^4}{x^4-16a^4}$.

20. $\frac{8a^7b}{a^8-b^8}$. **21.** $\frac{108x^4}{81x^4-y^4}$. **22.** $\frac{9ax(x+a)}{x^4-81a^4}$.

23. $\frac{4ab}{(a-b)^2}$. **24.** $\frac{6x^2-12}{x^4-5x^2+4}$. **25.** $\frac{6a^2x}{4x^4-5a^2x^2+a^4}$.

26. $\frac{48a^3}{x^4-10a^2x^2+9a^4}$. **27.** $\frac{2x}{x^4-1}$. **28.** $\frac{x-c}{(x-a)(x-b)}$.

29. $\frac{3x-7}{x^2-5x+6}$. **30.** $\frac{3(x+6a)}{x^2+11ax+28a^2}$. **31.** $\frac{2}{x+3}$.

32. 0. **33.** $\frac{4x^3}{1+x^4+x^8}$. **34.** $\frac{12x^4}{x^6-64}$. **35.** $\frac{66ax^5}{16x^8-6561a^8}$

Exercise 57. [Page 179]

1. $\frac{1}{3}$. **2.** $\frac{a^2}{9}$. **3.** xyz. **4.** $\frac{x^2y^2z^2}{9a^2b^2c^2}$. **5.** $\frac{5n^2x^2}{8my}$

6. $\frac{x+2}{x}$. **7.** 3. **8.** $\frac{a^2-b^2}{a}$. **9.** $\frac{a^2-4x^2}{a^2}$. **10.** $\frac{x^2-1}{x^2-x-6}$

11. 1. **12.** 1. **13.** $\frac{a^2+b^2}{a}$. **14.** $\frac{x}{2}$. **15.** $\frac{x+2}{x+3}$

16. $\frac{a^2(a-b)}{x}$. **17.** $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$. **18.** $\frac{8ab}{9x^2}+2+\frac{9x^2}{8ab}$. **19.** $2\left(\frac{bc}{ad}+\frac{ad}{bc}\right)$.

20. 1. **21.** $\frac{a+b-c}{a+b+c}$. **22.** 1.

Exercise 58. [Pages 181-182]

1. $\frac{5ax}{6by}$. **2.** $\frac{(a+b)^2}{b}$. **3.** $\frac{x-7}{x-5}$. **4.** $a-b$.

5. $\frac{m-n}{m+2n}$. **6.** $(m-n)^2$. **7.** 1. **8.** 1.

9. $\frac{x^2+y^2}{2xy}$. **10.** $\frac{x^2-8x+12}{x^2-10x+21}$. **11.** $\frac{1}{p^2+q^2}$ **12.** a^2-b^2.

13. xy. **14.** ab. **15.** $2x$. **16.** 1. **17.** $\frac{xy}{x^2+y^2}$

18. $-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$. **19.** $\frac{a}{a-b}$. **20.** a^2-b^2 **21.** $a-b$

Exercise 59. [Pages 184-186]

1. 2. **2.** 2. **3.** -5. **4.** 1. **5.** 2.
6. $\frac{1}{2}$. **7.** 3. **8.** -5. **9.** 6. **10.** $a+b$.
11. $2a$. **12.** $\frac{1}{2}(a+b)$. **13.** $a+b$. **14.** $m-n$.
15. $a+b$. **16.** $\frac{1}{4}(a+b)$. **17.** $\frac{2ab}{a+b}$. **18.** $\frac{12ab}{a+3b}$.
19. $c+d$. **20.** $\frac{1}{3}(a+b+c)$. **21.** $-\frac{1}{3}(a+b+c)$. **22.** ab.
23. $\frac{1}{ab}$. **24.** 13. **25.** 16. **26.** 20. **27.** -3
28. 8. **29.** 10. **30.** 9. **31.** 9. **32.** 5.
33. 8. **34.** 5. **35.** 7. **36.** $\frac{8a}{25}$. **37.** 24.
38. 18. **39.** $\frac{6a}{7}$. **40.** 56. **41.** $4\frac{1}{2}$. **42.** 6.
43. $10\frac{3}{8}$. **44.** $\frac{ac+b^2}{b^2+c^2}$. **45.** $-2\frac{88}{73}$. **46.** 8. **47.** 11.
48. 2. **49.** $25a+24b$. **50.** $\frac{2ab}{a+b}$. **51.** 72. **52.** $7\frac{1}{2}$.

Exercise 60. [Page 187]

1. 27. **2.** 5. **3.** 20. **4.** 2. **5.** 10.
6. 5. **7.** 5. **8.** 5. **9.** 7. **10.** 5.

Exercise 61. [Page 189]

1. $\frac{15}{16}$. **2.** $\frac{5}{2}$. **3.** $2\frac{4}{7}$. **4.** 4·05. **5.** 3.
6. 3. **7.** $10\frac{7}{8}$. **8.** $a^2+b^2+c^2$. **9.** $\frac{1}{3}(ab+bc+ca)$.
10. 0. **11.** $a^3+b^3+c^3-3abc$. **12.** 0.

Exercise 62. [Pages 193-195]

1. 90 × 180 ; 100 × 230. **2.** 15 metres, 12 metres. **3.** Rs. 33
4. 50, 30. **5.** 20 men, 16 women. **6.** 119, 121, 123, 125.
7. 222, 224, 226, 228, 230, 232.
8. *A*, 84 kilometres and *B*, 70 kilometres in 56 hours.
9. 28 days. **10.** Rs. 55. **11.** 24 metres. **12.** 14, 6.
13. Worked for 22 days. **14.** 4 days. **15.** Rs. 52, 52 ten paise
16. *A*, Rs. 162 ; *B*, Rs. 118 ; *C*, Rs. 104. **17.** 34 sheep ; Rs. 1400.
18. 176 kilometres from London ; 16 hours. **19.** 44. **20.** 32.
21. 72. **22.** 23. **23.** 5, 8, 2, 24. **24.** 19, 5, 4, 32.
25. 22, 31, 9, 54.

Exercise 63. [Page 197]

1. $\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$ 2. $\left.\begin{matrix}x=5\\y=2\end{matrix}\right\}$ 3. $\left.\begin{matrix}x=7\\y=6\end{matrix}\right\}$ 4. $\left.\begin{matrix}x=4\\y=7\end{matrix}\right\}$

5. $x=\frac{ac+b^2}{a^2+b}$, $y=\frac{ab-c}{a^2+b}$. 6. $\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$ 7. $\left.\begin{matrix}x=40\\y=16\end{matrix}\right\}$

8. $\left.\begin{matrix}x=8\\y=5\end{matrix}\right\}$ 9. $\left.\begin{matrix}x=6\\y=4\end{matrix}\right\}$ 10. $\left.\begin{matrix}x=6\\y=8\end{matrix}\right\}$

Exercise 64. [Page 199]

1. $\left.\begin{matrix}x=3\\y=2\end{matrix}\right\}$ 2. $\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$ 3. $\left.\begin{matrix}x=7\\y=2\end{matrix}\right\}$ 4. $\left.\begin{matrix}x=3\\y=7\end{matrix}\right\}$ 5. $\left.\begin{matrix}x=4\\y=4\end{matrix}\right\}$

6. $\left.\begin{matrix}x=13\\y=\ \ 3\end{matrix}\right\}$ 7. $\left.\begin{matrix}x=6\\y=5\end{matrix}\right\}$ 8. $\left.\begin{matrix}x=5\\y=5\end{matrix}\right\}$ 9. $\left.\begin{matrix}x=2\frac{1}{2}\\y=1\frac{1}{2}\end{matrix}\right\}$ 10. $\left.\begin{matrix}x=6\\y=2\end{matrix}\right\}$

Exercise 65. [Pages 203-204]

1. $\left.\begin{matrix}x=3\\y=2\end{matrix}\right\}$ 2. $\left.\begin{matrix}x=4\\y=1\end{matrix}\right\}$ 3. $\left.\begin{matrix}x=7\\y=4\end{matrix}\right\}$ 4. $\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$

5. $\left.\begin{matrix}x=4\\y=2\end{matrix}\right\}$ 6. $\left.\begin{matrix}x=6\\y=4\end{matrix}\right\}$ 7. $\left.\begin{matrix}x=2\\y=1\end{matrix}\right\}$ 8. $\left.\begin{matrix}x=-2\\y=\ \ 3\end{matrix}\right\}$

9. $\left.\begin{matrix}x=5\\y=2\end{matrix}\right\}$ 10. $\left.\begin{matrix}x=1\\y=3\end{matrix}\right\}$ 11. $\left.\begin{matrix}x=1\\y=2\end{matrix}\right\}$ 12. $\left.\begin{matrix}x=\ \ 3\\y=-1\end{matrix}\right\}$

13. $\left.\begin{matrix}x=1\\y=4\end{matrix}\right\}$ 14. $\left.\begin{matrix}x=-5\\y=\ \ 2\end{matrix}\right\}$ 15. $\left.\begin{matrix}x=-2\\y=\ \ 1\end{matrix}\right\}$ 16. $\left.\begin{matrix}x=\ \ 5\\y=11\end{matrix}\right\}$

17. $x=\frac{bc-c^2}{ba-a^2}$, $y=\frac{ac-c^2}{ab-b^2}$. 18. $\left.\begin{matrix}x=7\\y=9\end{matrix}\right\}$ 19. $\left.\begin{matrix}x=\frac{1}{2}\\y=\frac{1}{5}\end{matrix}\right\}$

20. $\left.\begin{matrix}x=3\\y=2\end{matrix}\right\}$ 21. $\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$ 22. $\left.\begin{matrix}x=7\\y=4\end{matrix}\right\}$ 23. $\left.\begin{matrix}x=10\\y=\ \ 5\end{matrix}\right\}$

24. $\left.\begin{matrix}x=\ \ 4\\y=10\end{matrix}\right\}$ 25. $\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$ 26. $x=\frac{a^2-b^2}{am-bn}$, $y=\frac{a^2-b^2}{an-bm}$.

27. $x=\frac{3}{5}$, $y=\frac{2}{3}$. 28. $x=\frac{1}{5}$, $y=\frac{1}{2}$. 29. $x=4$, $y=2$.

30. $x=\frac{1}{15}$, $y=18$. 31. $x=4$, $y=3$. 32. $x=4$, $y=4$.

Exercise 66. [Pages 208-211]

1. $\frac{4}{5}$. 2. 7, 9. 3. 6, 2. 4. 60, 15.

5. Father's age 70 years, son's age 30 years.

6. Father's age 38 years, son's age 14 years. 7. 24, 15.

8. $\frac{6}{13}$. 9. $\frac{4}{5}$. 10. $\frac{3}{8}$. 12. Rs. 15 ; Rs. 24. 13. 3, 5.

14. 6 kilometres and 3 kilometres per hour. 15. 50 ; 100.

16. 20 days. 17. 480 sq. metres.

18. Tea Rs. 5·40 P. and coffee Rs. 6 per kilogram.
19. 3 kilometres, $4\frac{1}{2}$ kilometres per hour. 20. 22 and 26. 21. 75
22. 65. 23. 21, 40. 24. A horse, Rs. 240 ; a cow, Rs. 120
25. 5s., 3s. 26. *A*, 24 days ; *B*, 48 days. 27. $\frac{8}{15}$
28. 15 kilometres. 29. 72. 30. 75s., 35s.
31. 34 sheep ; Rs. 1400. 32. 27.

Exercise 67. [Page 221]

7. (1) $6x-5y=0$; (2) $5x+7y=35$; (3) $x+y+2=0$;
(4) $21x-5y+124=0$; (5) $5x+9y+55=0$. 8. $y=1$. 9. 5 units

Exercise 68. [Pages 223-224]

1. 3, −3. 2. $a, -a$. 3. 14, −14. 4. $2\frac{1}{3}, -2\frac{1}{3}$. 5. 5, −5
6. 3, −3. 7. 5, −5. 8. $2a, -2a$. 9. $a, -a$. 10. 6, −6
11. 1, −1. 12. 2, −2. 13. 6 metres. 14. 9 metres.
15. 5 cm. each.

Exercise 69. [Page 225]

1. 2, 4. 2. 5, −4. 3. 1, −4. 4. 4, −13. 5. $3, -\frac{4}{3}$.
6. $\frac{1}{2}, -\frac{4}{3}$. 7. $7, -\frac{1}{3}$. 8. $2, \frac{1}{2}$. 9. $2, -\frac{8}{13}$. 10. a, b.
11. 19, 21. 12. 8. 13. 16, −6. 14. 3, 13. 15. 40 years
16. Rs. 75.

Miscellaneous Exercises IV

[Pages 226-229]

I

1. $12a^2$; $720a^6b^4c^5x^2y^3$. 2. $(x-3)^2, (x-3)(4x+1)$; $x-3$.
3. $(a-b)(b-c)(b-2a-3c)(2a+b+3c)$.
4. 4. 6. x^4+2.
7. $x=\frac{b}{a^2-ab+b^2}$, $y=\frac{a}{a^2-ab+b^2}$. 8. $\frac{ab}{a+b}$ hrs. ; $\frac{abc}{ab-bc-ac}$ hrs

II

1. $x-3$. 2. $(x-a)(x+b)(x^2+a)$. 3. (i) $\frac{2}{3}x^2y^2$; (ii) 3.
4. $\frac{a^4+b^4}{a^2b^2}$. 5. $\frac{5(x-3)}{3(x-5)}$. 6. 2. 7. 9.
8. $x=\frac{b^2+c^2-a^2}{2a}$, $y=\frac{c^2+a^2-b^2}{2b}$.

III

1. $(a-b)(x+a)$. 2. x^6-1. 3. $x+3$. 4. $(x+a)(x-b)(x+b)$.
5. $\frac{8}{x^4-16}$. 6. $x=\frac{2(b-1)}{2ab-(a+b)}$, $y=\frac{2(a-1)}{2ab-(a+b)}$.
7. Rs. 2800 and Rs. 1200. 8. $3, -3$.

IV

2. $x-y$. 3. $\frac{1}{x^2-3x+2}$. 4. $\frac{1}{m^2-m+1}$
6. 345. 7. $\frac{13}{21}$. 8. $8, -8$.

V

1. $x^2-(a+b)x+ab$. 2. $280x^3-123x^2-37x+6$. 3. 1.
4. $\frac{1}{a+c}$. 8. $4, -4$.

VI

1. $x-2$. 2. $abc(x-a)(x-b)(x-c)$. 4. 0. 6. 2.
7. Rs. 144. 8. $4, -4$.

VII

1. $ax-(a-1)$. 2. $(a-b)(b-c)(c-a)$. 4. $\frac{3}{x^2-4x+3}$.
5. $\frac{a^2+b^2}{2ab}$. 7. $x=10, y=15$. 8. $4, -4$.

Exercise 70. [Page 231]

1. $x^3+6x^2+11x+6$. 2. $x^3+14x^2+59x+70$. 3. $x^3-x^2-24x-36$.
4. $x^3-x^2-70x-200$. 5. $x^3-4x^2-29x-24$. 6. $x^3+x^2-46x+80$
7. $x^3-37x+84$. 8. $x^3-6x^2-37x+210$. 9. $x^3-23x^2+167x-385$.
10. $x^3-18x^2+99x-162$. 11. $x^3-13x^2-8x+240$. 12. x^3+25x^2
$+199x+495$. 13. $x^3-52x+96$. 14. $x^3-23x^2+151x-273$.
15. x^3+13x^2-144. 16. $x^3-7x^2-138x+1080$.
17. $x^3-3x^2-73x+315$. 18. $x^3+35x^2+396x+1440$.
19. $x^3-148x-672$. 20. $x^3-31x^2+290x-800$.

Exercise 71. [Pages 232-233]

1. $x^2+y^2+z^2+2xy-2xz-2yz$. 2. $x^2+y^2+z^2-2xy+2xz-2yz$
3. $x^2+y^2+z^2-2xy-2xz+2yz$. 4. $x^2+y^2+z^2+2xy-2xz-2yz$.

5. $x^2+y^2+z^2-2xy-2xz+2yz$. 6. $a^2+x^2+y^2+z^2-2ax+2ay-2az-2xy+2xz-2yz$. 7. $a^2+x^2+y^2+z^2-2ax-2ay-2az+2xy+2xz+2yz$. 8. $m^2+n^2+p^2+q^2+r^2+2mn+2mp+2mq+2mr+2np+2nq+2nr+2pq+2pr+2qr$. 9. $p^2+q^2+r^2+x^2+y^2-2pq+2pr-2px-2py-2qr+2qx+2qy-2rx-2ry+2xy$. 10. $a^2+b^2+c^2+x^2+y^2+z^2-2ab+2ac-2ax+2ay+2az-2bc+2bx-2by-2bz-2cx+2cy+2cz-2xy-2xz+2yz$. 11. $a^2+4x^2+9y^2+16z^2-4ax-6ay-8az+12xy+16xz+24yz$. 12. $4a^2+b^2+4c^2+d^2-4ab+8ac-4ad-4bc+2bd-4cd$. 13. 49. 14. 9. 15. 0. 16. 144. 17. 1635 18. 1. 19. 63. 20. 0. 21. 47. 22. 69.

Exercise 72. [Pages 235-236]

1. $x^5+5x^4+10x^3+10x^2+5x+1$. 2. $x^6+6x^5+15x^4+20x^3+15x^2+6x+1$. 3. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$. 4. $a^9+9a^8b+36a^7b^2+84a^6b^3+126a^5b^4+126a^4b^5+84a^3b^6+36a^2b^7+9ab^8+b^9$. 5. $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$. 6. $m^7-7m^6n+21m^5n^2-35m^4n^3+35m^3n^4-21m^2n^5+7mn^6-n^7$. 7. $x^4+8x^3+24x^2+32x+16$. 8. $x^5+10x^4+40x^3+80x^2+80x+32$ 9. $x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x+1$. 10. $x^4+12x^3+54x^2+108x+81$. 11. $x^5-5x^4+10x^3-10x^2+5x-1$. 12. $64-192z+240z^2-160z^3+60z^4-12z^5+z^6$. 13. $16x^4-32x^3+24x^2-8x+1$. 14. $x^9-9x^8y+36x^7y^2-84x^6y^3+126x^5y^4-126x^4y^5+84x^3y^6-36x^2y^7+9xy^8-y^9$. 15. $243x^5-810x^4+1080x^3-720x^2+240x-32$ 16. $1-8a+28a^2-56a^3+70a^4-56a^5+28a^6-8a^7+a^8$. 17. $1-7c+21c^2-35c^3+35c^4-21c^5+7c^6-c^7$. 18. $1-18x+135x^2-540x^3+1215x^4-1458x^5+729x^6$. 19. $1-14x+84x^2-280x^3+560x^4-672x^5+448x^6-128x^7$. 20. $256x^8-1024x^7a+1792x^6a^2-1792x^5a^3+1120x^4a^4-448x^3a^5+112x^2a^6-16xa^7+a^8$. 21. $x^{10}-10x^9a+45a^8a^2-120x^7a^3+210x^6a^4-252x^5a^5+210x^4a^6-120x^3a^7+45x^2a^8-10xa^9+a^{10}$. 22. $243x^5-810x^4a+1080x^3a^2-720x^2a^3+240xa^4-32a^5$. 23. $10x^4+20x^2+2$. 24. $2x^6+30x^4+30x^2+2$. 25. $14x^6a+70x^4a^3+42x^2a^5+2a^7$.
26. 16. 27. 32. 28. 64. 29. 128. 30. 256.
31. 30. 32. 3. 33. 0. 34. 16. 35. 0.

Exercise 73. [Page 237]

1. $x^3+y^3-z^3+3xyz$. 2. $p^3-8q^3-r^3-6pqr$.
3. $8x^3-27y^3-z^3-18xyz$. 4. $a^3-8b^3+27+18ab$.

5. $27a^3-125b^3-64-180ab$. 6. $(x-y-1)(x^2+y^2+1+xy+x-y)$.
7. $(x-y+2)(x^2+y^2+4+xy-2x+2y)$.
8. $(x-2y-3z)(x^2+4y^2+9z^2+2xy+3xz-6yz)$. 9. 0.
10. 0. 11. 0. 16. 0. 17. 392000

Exercise 74. [Page 238]

3. $8(c-b)(a-b)(a-c)$. 4. $(y-z)(x-z)(y-x)$. 5. 0. 6. 0.

Exercise 75. [Page 240]

1. $2x^2y+3x^2z+12y^2z+4y^2x+9z^2x+18z^2y+12xyz$.
2. $64x^2y+320x^2z+5y^2z+8y^2x+200z^2x+25yz^2+80xyz$.
3. $2a^2b+3a^2c+12b^2c+4ab^2+9ac^2+18bc^2+12abc$.
4. $9x^2y+90x^2z+10y^2z+3xy^2+300z^2x+100z^2y+90xyz$.
5. $2(x^3+y^3+z^3)+7(x^2y+x^2z+y^2z+y^2x+z^2x+z^2y)+16xyz$.
6. $2a^2b-3a^2c-12b^2c-4ab^2-9ac^2+18bc^2+12abc$.
7. $4abc$. 8. $4abc$. 9. 0. 10. $27abc$.

Exercise 76. [Pages 241-242]

4. 0. 8. $84abc$. 9. $6xyz$. 10. $3(y+z-x)(2x-2y+z)(x+y-2z)$.
11. $3(2x+3y+3z)(3x+2y+3z)(3x+3y+2z)$. 12. 2567. 13. 16800.
14. 1280. 15. 1331.

Exercise 77. [Page 245]

1. $(x^2+y^2+z^2+yz+zx-xy)(x+y-z)$.
2. $(p-2q-r)(p^2+4q^2+r^2+2pq+pr-2qr)$.
3. $(2x-3y-z)(4x^2+9y^2+z^2-3yz+2zx+6xy)$.
4. $(a+2b+1)(a^2+4b^2+1-2b-a-2ab)$.
5. $(2a+3b-4)(4a^2+9b^2+16+12b+8a-6ab)$.
6. $x^2+y^2+4+2y-2x+xy$. 7. $(x^2-x+2)(x^4+x^3-x^2+2x+4)$
8. $2(z-y)(3x^2+y^2+z^2+yz-3zx-3xy)$.
9. $(a^2+3a+5)(a^4-3a^3+4a^2-15a+25)$. 10. $x-5y+3$.
11. $a^2+b^2+c^2-ab+ac+bc$. 12. $x^2+y^2+1+xy+x-y$.
13. $x^2+4y^2+9z^2+2xy-3zx+6yz$. 14. $2a-3b-c$.

Exercise 78. [Page 249]

1. $(b-c)(a-c)(a-b)(a^2+b^2+c^2+bc+ac+ab)$. 2. $(b+c)(b-c)\times$ $(a+c)(a-c)(a+b)(a-b)$. 3. $(b-c)(a-c)(a-b)(a^3+b^3+c^3+a^2b+a^2c$ $+b^2a+b^2c+c^2a+c^2b+abc)$. 4. $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc$

$+ca+ab)$. 5. $(b-c)(a-c)(a-b)(bc+ca+ab)$. 6. $(y+z)(z+x)(x+y)$.
7. $(y-z)(z-x)(x-y)(yz+zx+xy)$. 8. $5(y-z)(z-x)(x-y)\times$
$(x^2+y^2+z^2-yz-zx-xy)$. 9. $-(y-z)(z-x)(x-y)$
10. $-(b-c)(c-a)(a-b)$. 11. $-(b-c)(c-a)(a-b)$.
12. $(a+b+c)(bc+ca+ab)$. 13. $3(2x-y+z)(x-z)(x+y)$
14. $-(b-c)(c-a)(a-b)(b^2+bc+c^2)(c^2+ca+a^2)(a^2+ab+b^2)$.
15. $-(y-z)(z-x)(x-y)(y+z)(z+x)(x+y)(y^2z^2+z^2x^2+x^2y^2)$.
16. $3(2a+b+c)(a+2b+c)(a+b+2c)$. 17. $(y+z-x)(z+x-y)(x+y-z)$
18. $-(y-z)(z-x)(x-y)$. 19. $-(y-z)(z-x)(x-y)(x+y+z+3)$.
20. $(y-z)(z-x)(x-y)(x+y+z)$.
21. $(ax+by+cz)(by+cz-ax)(cz+ax-by)(ax+by-cz)$.
22. $(x+2y+3z)(2y+3z-x)(3z+x-2y)(x+2y-3z)$.
23. 4200. 24. 249. 25. 1950.

Exercise 79. [Pages 255-256]

1. $(x+1)(x^2+1)$. 2. $(x+1)^2(x-1)$. 3. $(x+1)(x-1)^2$.
4. $(ab+c)(ac+b)$. 5. $(x-a)(x+b)(x^2-bx+b^2)$. 6. $(ax+by)(bx+ay)$
7. $(x-z)(x+y+z)$. 8. $(x+a)(b-c)$. 9. $(x^2-ab)(2a-3b)$.
10. $(a-b)(a+b+c)$. 11. $(2a-3b)(2a+3b+4c)$.
12. $(ax-by)(ax+by+cz)$. 13. $(x^2-yz)(x^2+yz+y^2)$.
14. $(4x-5a)(4x+5a+3b)$. 15. $(a+b)(a^2+ab+b^2)$.
16. $(m-n)(m^2-mn+n^2)$. 17. $(a-b)(a+b)^3$.
18. $(x+y)(x-y)^3$. 19. $(a+2)(a^2+3a+4)$.
20. $(x-5)(x^2-12x+25)$. 21. $(2a-3b)(4a+3b)(a+3b)$.
22. $(x-y)(x-y-1)$. 23. $(2a-b)(2a-b-3)$.
24. $(x^2+a^2)(x-a)^2$. 25. $(a^2+2b^2)(a-b)(a-2b)$.
26. $(a+2b)(a+b+c)$. 27. $(x-3y)(x-y+z)$.
28. $(m-2n)(m-3n+2p)$. 29. $(a-3b)(a-7b+5c)$.
30. $(x+4a)(2x-3a+4b)$. 31. $(a-4b)(a-2b+3)$.
32. $(3x+y)(x-3y+2)$. 33. $(a-b-c)(a+b+c+1)$.
34. $(x-2y+3z)(x+2y-3z+4)$. 35. $(3x-4y-2z)(3x-4y+2z-5)$
36. $(a+b)(a-b)(x+a)(x-a)(2x^2-3a^2)$. 37. $(2x-3b)(x^2+ax-b)$.
38. $(x+a)(x-a)(a+b)^2$. 39. $(a-1)^2(2a^2-a+2)$.
40. $(a-1)(a^2-6a+1)(a^2+3a+1)$. 41. $(2x+2y+z)(x+2y+2z)$.
42. $(x-y-z)(2x+3y+z)$. 43. $(a^2+1)^2(a^4-7a^2+1)$.
44. $(2x+y-3z)(2x-3y+3z)$. 45. $(x-a)(x^2+ax+a^2)(x^2-ax+a^2)$.
46. $(x+1)(x+2)(x+4)$.

Exercise 80. [Pages 258-259]

1. $(x+1)(x+3)(x+4)$. 2. $(x+2)(x+3)(x+4)$. 3. $(x-1)(x-2)(x-3)$.

4. $(x-2)(x+3)(x+4)$. 5. $(x-1)(x^2-3x-2)$. 6. $(x+1)(x^2+4x-6)$.

7. $(x-2)(x^2-4x+5)$. 8. $(x-2)(x+2)(x^2-3x-5)$.

9. $(x-1)(x+2)(x^2-4x+5)$. 10. $(x+1)(x-3)(x^2-3x-2)$.

11. $(x-2)(x+3)(x^2+4x-6)$. 12. $(x+2)(x-4)(x^2-5x+7)$.

13. $(x-5)(x^2-2x+3)$. 14. $(x+3)(x^2-3x+4)$. 15. $(x+2)(x-4)^2$.

16. $(x-2)(2x^2+x+2)$. 17. $(x+2y)(x^2-2xy-5y^2)$.

18. $(a+3b)(a^2+ab-3b^2)$. 19. $(a-2b)(5a^2+7ab+14b^2)$.

20. $(2x-1)(4x^2+2x+3)$. 21. $(x-1)(x+3)(2x+1)$.

22. $(x+1)^2(x-2)$. 23. $(a-b)(2a^2+ab+b^2)$.

24. $(x-1)(3x^2+11x+3)$. 25. $(x+3y)(x^2-3xy+3y^2)$.

26. $(x+a-b)(x-a+2b)$. 27. $\{x^2+(a+b)^2y^2\}\{x+(a-b)y\}\{x-(a-b)y\}$.

28. $\{a^2+(x+y)^2b^2\}\{a^2+(x-y)^2b^2\}$. 29. $(a+2x-y)(a-x+2y)$.

30. $(x+2a+b)(x-a+2b)$. 31. $(x+3y-z)(x+y+z)$.

32. $(x+4y-2z)(x-2y+2z)$. 33. $(a+2b-c)(a+c)$.

34. $(x+y)(x-y+2)$. 35. $(x+y+1)(x-y+5)$.

36. $(a+5b-c)(a-b+c)$. 37. $(x-y-z)(x-5y+z)$.

38. $(x-2y-2z)(x-8y+2z)$. 39. $(a+b-3c)(a-13b+3c)$.

40. $(x+12y-3z)(x+3z)$. 41. $(x+y-5z)(x-15y+5z)$.

42. $(2a+b-3c)(2a-3b+3c)$. 43. $(x^2+4x-3)(x^2+2x+3)$.

44. $(a^2+ab-b^2)(a^2-5ab+b^2)$. 45. $(2x^2-4x-3)(2x^2-6x+3)$.

46. $(x-1)^2(x^2+1)$. 47. $(a^2+3a-5)(a^2-3a+5)$.

48. $(a-bx)(a-bx-cx^2)$. 49. $(x^2y^2+xy-z+1)(x^2y^2-xy+z+1)$.

50. $\{(y+z)x-y+z\}\{(y-z)x+y+z\}$.

51. $\{(a+b)x+(a-b)y\}\{(a-b)x+(a+b)y\}$.

52. $(x^2-2x-1)(x^2-2x-4)$. 53. $(a^2-3a+5)(a^2-3a+1)$.

54. $(2x^2+3x-3)(2x^2+3x-4)$. 55. $(x^2-xy+y^2)(x^2-4xy+y^2)$.

56. $(x^2-2x+4)(x^2-3x+4)$. 57. $(a^2-2ab+2b^2)(a^2-5ab+2b^2)$.

58. $(x^2-3x+5)(x^2+7x+5)$. 59. $(a-b)^2(a^2+6ab+b^2)$.

60. $(x^2+4x+10)(x^2+4x-2)$. 61. $(x^2-3x-5)(x^2-3x-17)$.

62. $(x-1)(x+8)(x^2+7x+30)$. 63. $(x-3)(2x+3)(2x^2-3x+7)$.

64. 0. 65. 0. 66. 0. 67. 300. 68. 5.

Exercise 82. [Pages 278-280]

1. 179. 2. 3. 3. $6\frac{1}{5}$. 4. $-8\frac{3}{4}$. 5. $\frac{1}{a}(ad-bc)$.

16. (i) $5a^2-11a+15=0$. (ii) $(b+c)^2(ab+ac+1)=d$. 17. $a=0$, or, 1

18. $2rn$, where r is any positive integer.
29. $(b-c)(a-c)(a-b)(ab+bc+ca)$. 36. $1+x+x^2+\cdots+x^{30}+x^{31}$.
38. 1111111. 41. $x^4-x^3y+x^2y^2-xy^3+y^4$. 42. $x^5-x^4y+x^3y^2$
$-x^2y^3+xy^4-y^5$. 43. $x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6$.
44. $x^{14}-x^{12}y^2+x^{10}y^4-x^8y^6+x^6y^8-x^4y^{10}+x^2y^{12}-y^{14}$.
45. $x^{15}+x^{14}y+\cdots+xy^{14}+y^{15}$. 46. $p=12$; $a=\frac{1}{3}$. 49. 0. 50. 1.

Exercise 83. [Pages 286-287]

1. $2x-1$. 2. $3x-2$. 3. $2x+5a$. 4. $x(3x+4)$.
5. $3a-1$. 6. $2a-3b$. 7. x^2+x+1. 8. x^2-xy+y^2.
9. $x(2x^2+x+1)$. 10. x^2+3x+1. 11. x^2+4x+1.
12. $x^2+2ax+3a^2$. 13. $x^2+3ax+5a^2$. 14. $2a^2-3ax+7x^2$.
15. $2-3x+5x^2$. 16. $1+4x-7x^2$. 17. $x^2(2x^2+3xa+4a^2)$.
18. $2(a^2+5a+2)$. 19. x^2+3x-2. 20. x^2-3x+5.
21. x^2+5x+1. 22. x^2+2x+4. 23. x^2+3x+5.
24. $2(x^2-2ax+2a^2)$. 25. $3x^2+2xy+4y^2$. 26. x^2+2x+3.
27. $4a^2+2a-5$. 28. x^2+2x+3.

Exercise 84. [Page 289]

1. x^2-5x+6. 2. $2x^3-17x+12$. 3. x^2+3x+4.
4. $3x^3-5x^2+7$. 5. $6x^2-11x+4$. 6. $2x^2+15x-8$.
7. $3x^2+5x-1$. 8. $5x^2-3x-1$. 9. $2x^2+3x-1$.
10. $3x^3-2x+1$. 11. x^2+x+2. 12. x^2+3x-2.

Exercise 85. [Pages 291-292]

1. $x+4$. 2. $2x-1$. 3. $2x-3$. 4. $2x^2+1$.
5. $3a-2b$. 6. $3a-5b$. 7. $3x-4$. 8. $2x^2-3$.

Exercise 86. [Page 293]

1. $9x^4+30x^3-17x^2-76x+32$. 2. $18x^4+3x^3-109x^2-84x+32$.
3. $48x^5-64x^4-120x^3+160x^2+27x-36$. 4. $45x^4-24x^3-123x^2$
$+40x+80$. 5. $12x^4-14x^3-94x^2+63x+180$. 6. $12x^6+8x^5$
$+25x^4+34x^3+15x^2+18x+8$. 7. $32x^6-24x^5-8x^4+18x^3-48x^2$
$+27x-18$. 8. $12x^6+24x^5+95x^4+118x^3+249x^2+144x+216$.
9. $x^2-12x+35$. 10. (x^2-3x+2), (x^2+2x-3).

Exercise 87. [Page 295]

1. $12x^4+4x^3-21x^2-16x-3$. 2. $12x^6+8x^5-101x^4+47x^3-62x^2$
$-81x-35$. 3. $x^6+5x^5-33x^4-149x^3+212x^2+684x-720$.
4. $16x^8+40x^7+20x^6+38x^5-20x^4-39x^3-15x^2-9x+9$.

Exercise 88. [Page 297]

1. $x+3$.
2. $\frac{x+3}{x+5}$.
3. $\frac{a+3b}{a-4b}$.
4. $\frac{x^2-ax+b^2}{x^2+ax-b^2}$.
5. $\frac{3x-2y}{2x+5y}$.
6. $\frac{1+2x-3x^2}{1-2x+3x^2}$.
7. $\frac{(x-1)^2}{x^2-3x+1}$.
8. $\frac{x^2+3x+5}{x^2+3x-5}$.
9. $\frac{x^2+3ax+7a^2}{2x^2-3ax+5a^2}$.
10. $\frac{2x+3}{3x+4}$.
11. $\frac{3x^2-ax-2a^2}{3x^2+ax-2a^2}$.
12. $\frac{2(a^2-5ab+7b^2)}{3(a^2+5ab+7b^2)}$.
13. $\frac{3(3x^2+4x+5)}{4(2x^2+3x+4)}$.
14. $\frac{a(3a^2-b^2)}{2a^2-b^2}$
15. $\frac{4x(2x^2-3y^2)}{5y(3x^2-2y^2)}$.
16. $2(a+b+c)$.
17. $1+xyz$.
18. $\frac{z+x-2y}{4(y+z)}$.
19. 2.
20. $\frac{7x-2y}{5x^2-3xy+2y^2}$
21. $\frac{(a+b)(b+c)(c+a)}{abc}$.

Exercise 89. [Pages 300-301]

1. $\frac{96ax^5}{16x^8-6561a^8}$.
2. $\frac{3}{(x+a)(x+4a)}$.
3. $\frac{a-d}{(x+a)(x+d)}$
4. $\frac{6a-11}{(a-1)(a-2)(a-3)}$.
5. $\frac{1}{(x+2)^2}$.
6. $\frac{5}{(x-3)(x-4)(x-5)}$
7. $\frac{a^2+ac+c^2}{ac(a+c)}$.
8. $\frac{1}{a-1}$.
9. 1.
10. 0.
11. 0
12. $\frac{x^4}{(x-a)(x-b)(x-c)(x-d)}$.
13. 1.
14. 1.
15. $\frac{1}{(x-1)(x-2)(x-3)}$
16. $\frac{2(a+x)}{(a^2+ax+x^2)}$.

Exercise 90. [Pages 303-304]

1. $-x^2y^2z^2$.
2. $\frac{1}{2}$.
3. 1.
4. $\frac{(b+c+a)^2}{2bc}$.
5. $\frac{1+a^2}{1+a}$
6. $\frac{4a^4}{a^4-x^4}$.
7. x.
8. $a-b$.
9. $\frac{1}{3(x-2)}$.
10. $-\frac{x^4+x^2y^2+y^4}{xy(x-y)^2}$.
11. $\frac{adf+ae}{bdf+be+cf}$.
12. x^2.
13. a^2+b^2.
14. m.
15. $4xy$.
16. 1.
17. x.
18. 1.
19. $(a+b+c)$.
20. $\frac{x^3}{x^4-x^2+1}$.
21. $\frac{1}{8}$.
22. [illegible].
23. 1.
24. a.

Exercise 91. [Pages 312-316]

21. 0. **24.** 3. **25.** $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$. **26.** 1. **27.** $\frac{a-b}{a+b}$.

28. 0. **29.** x^6+2. **30.** 1. **31.** $\frac{3(a+b)}{a-b}$. **32.** 1. **34.** 0.

35. 3. **36.** $x+y+z$. **37.** $a^2+b^2+c^2$. **38.** $a+b+c$. **39.** 0.

42. 0. **43.** 1. **44.** 2. **45.** 1. **46.** 0.

47. $\frac{1}{xyz}$. **48.** $\frac{1}{(x-a)(x-b)(x-c)}$. **49.** $\frac{x}{(x-a)(x-b)(x-c)}$.

50. $\frac{x^2}{(x-a)(x-b)(x-c)}$. **51.** $\frac{x^2+hx+k}{(x-a)(x-b)(x-c)}$. **55.** x^2.

Miscellaneous Exercises V

[Pages 316-320]

I

1. (i) $(x^2+20x+95)^2-16$; (ii) $(x^2+5x+5)^2-16$.
2. (i) $3(z+x)(y+2z+x)(z+2x-y)$; (ii) $(2a-b)(7a^2+8ab+4b^2)$.
3. $18(a^2b^2+b^2c^2+c^2a^2)$. 7. 0.

II

1. 0. 4. 0. 5. $\frac{(a-b)^2}{ab}$. 6. $2a+3b+c$.
7. $18x^4-45x^3+37x^2-19x+6$. 8. (i) $3x(x-1)(1-2x)$.
(ii) $(a-b)(b-c)(a-c)$.

III

1. $x^5+10x^3+40x+\frac{80}{x}+\frac{80}{x^3}+\frac{32}{x^5}$. 3. 0. 5. $2x-1$;
$(x-3)(2x-1)(3x-2)$. 6. x^2-2x+3. 7. $ab+bc+ca$.

IV

1. 242. 2. $(a-b)(2a-b)(a+b)(a+2b)(a^2+b^2)$. 3. 2528000.
5. $\frac{a^2+b^2}{a^2-2ab-b^2}$. 7. $(x-5y)(x-3y)(x+2y)(x+7y)$;
$(x-3y)(x+2y)(x+7y)$. 8. 0.

V

3. 1. 5. (i) $(3a^2-4ab+3b^2)(2a^2+17ab+2b^2)$;
(ii) $(3x^2-7x+3)(4x^2-3x+4)$; (iii) $(ax^2+bx+a)(bx^2+cx+b)$.
7. (i) $x-a$; (ii) $x-y-z$.

VI

4. $(x+y)^4+z^4$. 5. $3(x^2+y^2+z^2)$. 6. (i) $x^2+(2m-3)x-6m$;
(ii) $x-3$. 7. $2x^2-3x+1$; $2x^6-3x^5-7x^4+28x^3-36x^2+20x-4$.

VII

1. $x^2+2xy+y^2-1$. 2. x. 4. (i) $a+b$; (ii) $\frac{1-x}{(1+x)(1+2x^2)}$.
5. $(5x+2y)^2+4(2x-5y)^2$; $p=5$, $q=2$. 6. $(2x-1)(3x-1)$;
$(x-2)(2x-1)(3x-1)(2x+1)$. 7. -42.

VIII

4. $(a+b+c+d)(a+b-c-d)(a-b-c+d)(a-b+c-d)$. 5. -1.
6. 0. 8. x^2+1; $(x^2+1)^2(x^2-1)$.

Exercise 92. [Pages 323-324]

1. $-5\frac{1}{2}$. 2. $\frac{a^2+ab+b^2}{a+b}$. 3. $\frac{1}{3}(a+b+c)$. 4. $\frac{a^2+b^2+c^2}{2(a+b+c)}$.
5. 2. 6. 3. 7. 4. 8. 5. 9. $1\frac{1}{2}$. 10. 4. 11. 4.
12. 7. 13 1. 14. 4. 15. 2. 16. $\frac{17}{73}$. 17. 3. 18. 2.
19 $\frac{5}{8}$. 20. $2\frac{3}{4}$. 21. $26\frac{1}{3}$. 22. 13. 23. $55\frac{1}{2}$.
24. $-\frac{19}{28}$. 25. $-\frac{49}{114}$. 26. $-\frac{24}{7}$. 27. 15. 28. $\frac{21}{47}$.
29. $\frac{4}{7}$. 30. $\frac{ab(a+b-2c)}{a^2+b^2-ac-bc}$. 31. $\frac{(a^2-ab+b^2)c+ab}{ab(c+1)}$.
32. $\frac{nb-am}{m-n}$. 33. $\frac{a^2-bc}{b+c-2a}$. 34. $\frac{3ab-a^2-b^2}{a+b}$. 35. $3a$.

Exercise 93. [Pages 326-327]

1. $\frac{8}{9}$. 2. -3. 3. $-\frac{46}{83}$. 4. 2. 5. 1. 6. 2.
7. $\frac{7}{5}$. 8. $-\frac{11}{7}$. 9. 2. 10. $\frac{1}{4}$. 11. $\frac{5}{8}$. 12. $1\frac{8}{11}$.
13. $\frac{1}{2}$. 14. $\frac{7}{8}$. 15. $4\frac{1}{2}$. 16. 6. 17. 7. 18. $4\frac{1}{2}$.
19. $-\frac{4}{5}$. 20. 1. 21. $-\frac{5}{8}$. 22. $3\frac{1}{2}$.

Exercise 94. [Page 329]

1. 24. 2. $-b$. 3. $-\frac{3}{4}$. 4. 2. 5. 7.
6. $\frac{a^2+b^2}{a+b}$. 7. $\frac{ab}{a+b}$. 8. $\frac{ab(c+d)-cd(a+b)}{ab-cd}$. 9. 2. 10. $\frac{3}{2a^2}$.

11. 4. **12.** $\frac{a^2+b^2}{a+b}$. **13.** 3. **14.** 2. **15.** $\frac{ab}{a-b}$.
16. 25. **17.** 3. **18.** $\frac{2(a^2+b^2)}{a-b}$. **19.** 6. **20.** $\frac{1}{2}(a-b)$.

Exercise 95. [Pages 337-340]

1. $16\frac{4}{11}$ minutes past 3. **2.** $27\frac{3}{11}$ minutes past 5. **3.** (i) $5\frac{5}{11}$ minutes past 7 ; (ii) $21\frac{9}{11}$ and $54\frac{6}{11}$ minutes past 7 ; (iii) $38\frac{2}{11}$ minutes past 7. **4.** (i) at $5\frac{4}{11}$ minutes past 7 ; (ii) at $16\frac{4}{11}$ minutes past 6. **5.** $\frac{pa}{p+q}$ kilometres. **6.** 8 kilometres from the starting place of the faster walker ; 6 hours. **8.** 36 minutes. **9.** $3\frac{1}{2}$ and $4\frac{1}{2}$ kilometres per hour. **10.** 160. **11.** Rupees 180. **12.** 300. **13.** Greyhound, 960 ; hare, 1200. **14.** 180. **15.** Rupees 20 ; Rs. 5. **16.** 40. **17.** 76 kgs. of gold and 30 kgs. of silver. **18.** 4 hours and 6 hours. **19.** 42 years. **20.** $6\frac{2}{3}$ grams from the 1st bar, $13\frac{1}{3}$ grams from the 2nd. **21.** Rs. 7000, Rs. 4000. **22.** 11 paise : each man of the 1st set 6 paise, of the 2nd set 5 paise, of the 3rd set 4 paise, and of the 4th set 4 paise ; 63 p. **23.** 189. **24.** 25 oz. ; 8s. per oz. **25.** Rs. 84, Rs. 140. **26.** 2080. **27.** $\frac{3}{4}$*d.* each ; 512. **28.** $1\frac{4}{11}$ minutes past 3 p.m. **29.** $26\frac{28}{43}$ minutes past 4 p.m. **30.** 12. **31.** 654. **32.** 1504. **33.** 80. **34.** 736. **35.** 4550.

Exercise 96. [Pages 344-345]

1. $x=1$, $y=2$. **2.** $x=2$, $y=3$. **3.** $x=3$, $y=4$.
4. $x=4$, $y=5$. **5.** $x=5$, $y=6$. **6.** $x=6$, $y=7$.
7. $x=7$, $y=8$. **8.** $x=8$, $y=9$. **9.** $x=4$, $y=2$.
10. $x=5$, $y=3$. **11.** $x=7$, $y=4$. **12.** $x=5$, $y=8$.
13. $x=8$, $y=12$. **14.** $x=6$, $y=14$. **15.** $x=8$, $y=18$.
16. $x=8$, $y=9$. **17.** $x=12$, $y=16$. **18.** $x=21$, $y=12$.
19. $x=21$, $y=24$. **20.** $x=18$, $y=28$. **21.** $x=99$, $y=15$.
22. $x=10$, $y=8$. **23.** $x=3$, $y=7$. **24.** $x=4$, $y=7$.
25. $x=3$, $y=5$. **26.** $x=1$, $y=2$, $z=3$.
27. $x=2$, $y=-3$, $z=1$. **28.** $x=3$, $y=4$, $z=2$.
29. $x=2$, $y=6$, $z=4$. **30.** $x=1$, $y=3$, $z=5$.
31. $x=2$, $y=3$, $z=4$. **32.** $x=3$, $y=6$, $z=9$.
33. $x=4$, $y=10$, $z=14$. **34.** $x=8$, $y=12$, $z=20$.
35. $x=3$, $y=4$, $z=5$.

Exercise 97. [Pages 348-349]

1. $x=1,\ y=2,\ z=3.$

2. $x=2,\ y=3,\ z=4.$

3. $x=2,\ y=3,\ z=4.$

4. $x=2,\ y=3,\ z=4.$

5. $x=3,\ y=2,\ z=1.$

6. $x=3,\ y=2,\ z=1.$

7. $x=4,\ y=3,\ z=2.$

8. $x=4,\ y=5,\ z=6.$

9. $x=7,\ y=5,\ z=3.$

10. $x=1,\ y=-2,\ z=3.$

11. $x=3,\ y=2,\ z=5.$

12. $x=3,\ y=\frac{1}{2},\ z=\frac{2}{3}.$

13. $x=10,\ y=20,\ z=5.$

14. $x=2,\ y=-3,\ z=4.$

15. $x=5,\ y=6,\ z=7.$

16. $x=2,\ y=4,\ z=6.$

17. $x=2,\ y=5,\ z=10.$

18. $x=y=z=12.$

19. $x=6,\ y=12,\ z=8.$

20. $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}.$

21. $x=7,\ y=10,\ z=9.$

22. $x=1,\ y=-2,\ z=3.$

23. $x=\frac{b^2+c^2-a^2}{2bc},\ y=\frac{c^2+a^2-b^2}{2ac},\ z=\frac{a^2+b^2-c^2}{2ab}.$

24. $x=1,\ y=2,\ z=3.$

25. $x=-28,\ y=10,\ z=9.$

Exercise 98. [Pages 351-352]

1. $x=\frac{a}{2},\ y=\frac{b}{2},\ z=\frac{c}{2}.$

2. $x=\frac{2}{a+b-c},\ y=\frac{2}{a-b+c},\ z=\frac{-2}{b+c-a}.$

3. $x=\frac{2abc}{ab+ac-bc},\ y=\frac{2abc}{bc+ab-ac},\ z=\frac{2abc}{ac+bc-ab}.$

4. $x=\frac{c(a^2+b^2)}{a^2-b^2},\ y=\frac{c(a^2+b^2)}{2ab}.$

5. $x=2,\ y=4,\ z=6.$

6. $x=5,\ y=3,\ z=1.$

7. $x=12,\ y=10,\ z=8.$

8. $x=13,\ y=8,\ z=9.$

9. $x=4,\ y=5,\ z=7.$

10. $x=\frac{b(2a-b)}{a-b},\ y=\frac{a(2b-a)}{b-a}.$

11. $x=\frac{Abc}{(a-b)(a-c)},\ y=\frac{Aac}{(b-a)(b-c)},\ z=\frac{Aab}{(c-b)(c-a)}.$

12. $x=\frac{1}{(b-c)(a-c)},\ y=\frac{1}{(a-b)(c-b)},\ z=\frac{1}{(c-a)(b-a)}.$

13. $x=\frac{a^2bc}{(b-a)(c-a)},\ y=\frac{ab^2c}{(b-a)(b-c)},\ z=\frac{abc^2}{(c-b)(c-a)}.$

14. $x=abc,\ y=ab+bc+ca,\ z=a+b+c.$

15. $x=b-c,\ y=c-a,\ z=a-b.$

16. $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0.$

17. $a=6.$

18. $w=4,\ x=12,\ y=5,\ z=7.$

19. $x=5,\ y=4,\ z=3,\ w=2,\ t=1.$

20. $x=ab,\ y=bc,\ z=ac.$

Exercise 99. [Pages 359-362]

1. 375. **2.** 50 kgs. ; Rs. 28 per kg. **3.** *A*, Rs. 14 ; *B*, Rs. 19. **4.** 20, 30, 60. **5.** 3*s*. 6*d*., 4*s*. 2*d*. **6.** *A*, $\frac{pm}{p+n-m}$; *B*, $\frac{pm}{m-n}$ days. **7.** *A*, Rs. 980 ; *B*, Rs. 1540 ; *C*, Rs. 2380. **8.** 8 hours. **9.** 720 kilometres. **10.** 4 and 3 litres. **11.** 253. **12.** 3 half-crowns ; 8*s*. ; 9 six-pences. **13.** 20 persons ; Rs. 6. **14.** Each of the equal cocks in 32 hours, and the other in 24. **15.** Rs. 8 and Rs. 5 respectively. **16.** 75 and 25 litres. **17.** 6 qrs. of wheat ; 10 qrs. of barley. **18.** 79·2 and 39·6 kilometres per hour. **19.** 20 bushels of rye, and 52 of wheat. **20.** 50 rupees and 50 twenty-five paise at first, 20 rupees and 30 twenty-five paise left. **21.** $2\frac{1}{2}$ kilometres per hour. **22.** *A*, 5 minutes ; *B*, 6 minutes. **23.** 10 and 12 kilometres per hour. **24.** $\frac{b(n-1)}{a-c}$ kilometres per hour. **25.** 100 kilometres. **26.** Started reading at $26\frac{62}{143}$ minutes past 3 p.m., stopped reading at $17\frac{28}{143}$ minutes past 5 p.m. and read for 1 hour $50\frac{118}{143}$ minutes.

Exercise 100. [Page 370]

1. $x=5, y=4$. **2.** $x=7, y=-5$. **3.** $x=8, y=6$. **4.** $x=9, y=11$. **5.** $x=10, y=13$. **6.** [Take ten times the side of a small square as the unit of length.] $x=1·2$. **7.** $x=7$. **8.** $x=7$. **9.** 9. **10.** 4. **11.** $(-6, 4)$; $(8, 2)$; $(6, 8)$; area $=40$ sq. units. **12.** $(5, 4)$. **13.** (i) $(3, 0)$; $(0, 3)$; $(-3, 0)$; $(0, -3)$; area $=18$ sq. units. (ii) $(1, 5)$; $(12, 5)$; $(12, 10)$; $(1, 10)$; area $=55$ sq. units. (iii) $(3, 0)$; $(8, 0)$; $(0, 5)$; $(0, 12)$; area $=40·5$ sq. units. **14.** (i) $(0, 0)$, $(5, 0)$, $(0, 6)$; area $=15$ sq. units ; (ii) $(2, 1)$, $(2, 4)$, $(5, 1)$; area $=4·5$ sq. units ; (iii) $(4, 6)$, $(-4, 2)$, $(2, -4)$; area $=36$ sq. units. **15.** $\left.\begin{matrix}x=1\\y=1\end{matrix}\right\}$ **16.** $\left.\begin{matrix}x=7\\y=-5\end{matrix}\right\}$ **17.** $\left.\begin{matrix}x=9\\y=11\end{matrix}\right\}$ **18.** $x=3, y=4$; 90°.

Exercise 101. [Pages 379-381]

1. Rs. 3 40 P. ; 3 kgs. 500 grams. 2. Rs. 4·25 ; 19. 3. $3\frac{1}{2}$ hours ; 19 kilometres. 4. 8 kilometres ; 7 miles. 5. $2\frac{1}{2}$ hours after *A* starts ; $7\frac{1}{2}$ kilometres from the place of starting. 6. 4 hours after starting ; 12 kilometres from *A*. 7. Re. 1 14 P. ; 39. 8. 5 kgs.

11. At 4-30 P.M. 13⅓ kilometres from B. **12.** Rs. 434. **13.** 16·4 minutes past 3. 14. Rs. 3200 ; Rs. 130 50 P. 15. Rs. 3600. **16.** 20·5 kms. from P at 53⅓ minutes past 5 P.M. ; 23 kilometres ; at 5 P.M. **17.** 16 hours 59·4 minutes ; 17·3 kilometres from Calcutta.

Exercise 102. [Pages 384-385]

1. $\sqrt[7]{a^5}$. 2. $\frac{1}{\sqrt{x^3}}$. 3. $3\sqrt[5]{x^4}$. 4. $\frac{3}{\sqrt[5]{x^2}.\sqrt{a}}$.
5. $\frac{8}{\sqrt[3]{m^3}}$. 6. $\frac{\sqrt[4]{a^5}}{3\sqrt[5]{x^4}}$. 7. $\frac{1}{2\sqrt[6]{x}}$. 8. $\sqrt[5]{x^{2+a}}$.
9. $\sqrt[2m]{a^{11}}$. 10. $\sqrt[a]{x^4}$. 11. $x^{\frac{7}{3}}$. 12. $\frac{1}{a^{\frac{3}{2}}}$.
13. $x^{\frac{2}{3}}$. 14. $a^{\frac{2}{7}}$. 15. $x^{\frac{8}{3}}$. 16. $a^{\frac{3}{4}}$.
17. ⅛. 18. 4. 19. 27. 20. 32.
21. $\frac{1}{27}$. 22. 36. 23. $\frac{1}{25}$. 24. 81.
25. 36. 26. x^{-m}.

Exercise 103. [Page 387]

1. a^{-6}. 2. $a^{-\frac{1}{2}}b^{\frac{5}{8}}$. 3. ab^6. 4. $a^{-6}b^{-\frac{5}{3}}$.
5. a^9b^6. 6. $x^{-\frac{9}{2}}y^4$. 7. $x^{\frac{5}{32}}$. 8. a^{-1}.
9. y. 10. $\frac{4}{3}x^2a^2$. 11. $\frac{9}{16}x^{-2}a^{-2}$. 12. $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{2}{3}}$.
13. $a^{-1}b^{\frac{2}{3}}c^{\frac{1}{6}}$. 14. $a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{\frac{1}{2}}$. 15. a^4b^2.

Exercise 104. [Pages 391-394]

1. $x-2x^{\frac{1}{2}}+1$. 2. $a-27b$. 3. $1+a^2b^{-2}+a^4b^{-4}$.
4. $x^2+6xz^{\frac{1}{3}}-4y+9z^{\frac{2}{3}}$. 5. $x^{-2}+x^{-1}y^{-1}+y^{-2}$.
6. $a+a^{\frac{1}{3}}-1+a^{-\frac{1}{3}}+a^{-1}$. 7. $x-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+y+z$.
8. $a^{2m}-9b^{2n}+12b^nc^p-4c^{2p}$. 9. a^3-64b^2.
10. $a+a^{\frac{3}{4}}x^{-\frac{1}{4}}-a^{\frac{1}{4}}x^{-\frac{3}{4}}-x^{-1}$. 11. $x-x^{\frac{1}{2}}$. 12. $2+4x^{-1}+2x^{-2}$.
13. $y+x^{\frac{1}{2}}y^{\frac{1}{2}}+x$. 14. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-b$. 15. $x^{2n}-1+x^{-2n}$.
16. $4x-2x^{\frac{1}{2}}y^{-\frac{1}{2}}+2x^{\frac{1}{2}}z^{\frac{1}{3}}+y^{-1}+y^{-\frac{1}{2}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$.
18. $x^{2^n}-a^{2^n}$. 19. $x^{2^{n-1}}-y^{2^{n-1}}$. 20. a^{m-1}.

21. $x^{\frac{3}{2}}+3x^{\frac{1}{2}}-1$. 22. $x^{\frac{3}{2}}-2x^{\frac{5}{4}}y^{-\frac{1}{4}}+xy^{-\frac{1}{2}}+2x^{\frac{3}{4}}y^{\frac{1}{2}}-2x^{\frac{1}{2}}y^{\frac{1}{4}}+y$.
23. $x^n+x^{\frac{n}{2}}a^{\frac{n}{2}}+a^n$. 24. $x^{\frac{4}{3}}-4x^{\frac{5}{6}}+4x+2x^{\frac{7}{6}}-4x^{\frac{4}{3}}+x^{\frac{6}{5}}$.
25. $a^{\frac{2}{3}}x^{-\frac{2}{3}}+a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}+a^{-\frac{2}{3}}x^{\frac{2}{3}}$. 26. $\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}$. 27. $\frac{2x+36x^{\frac{1}{3}}y^{\frac{2}{3}}}{x-27y}$.
28. $\frac{a+x}{a^2+3ax+x^2}$. 29. 1. 30. $x^{\frac{1}{4}}y^{\frac{1}{4}}-x^{-\frac{1}{4}}y^{\frac{3}{4}}$. 31. 1.
32. $\frac{a^2+b^2}{a(a+b)}$. 33. 2. 34. $\left(\frac{a-b}{a+b}\right)^n+2$. 36. $\left(\frac{x^{-2}}{4}-\frac{y^{-3}}{5}\right)\left(\frac{x^{-4}}{16}+\frac{y^{-6}}{25}\right)$.
37. $\left(\frac{p}{q}\right)^{2m}$. 38. $\left(\frac{p}{q}\right)^{p+q}$. 39. 1. 40. 1. 41. 1.
44. $m=n^{\frac{1}{n-1}}$.

Exercise 105. [Pages 395-396]

1. 3. 2. 5. 3. 2. 4. $-\frac{1}{11}$. 5. 6. 6. $\frac{b+d}{a+c}$. 7. $1\frac{1}{2}$.
8. $2\frac{1}{2}$. 9. 2. 10. 3. 11. -4. 12. 0. 13. $x=2$, or 3.
14. $x=2$, or -2. 15. $x=4$, $y=2$. 16. $x=-2$, $y=-3$.
17. $x=2$, $y=3$. 18. $x=1$, $y=3$. 19. $x=2$, $y=-1$.
20. $x=1$, $y=\frac{1}{2}$. 21. $x=1$, $y=2$.
22. $x=-\frac{1}{2}$, $y=4$. 23. $x=\left(\frac{a}{b}\right)^{\frac{b}{a-b}}$, $y=\left(\frac{a}{b}\right)^{\frac{a}{a-b}}$. 24. $m=n^{\frac{1}{n-1}}$.
30. $x=1$, $y=2$ and $z=3$. 31. $x=-1\frac{1}{3}$, $y=-9\frac{1}{3}$ and $z=-5\frac{2}{3}$.
32. $x=y=z=\frac{1}{3}a$.

Exercise 106. [Page 397]

1. $\sqrt{45}$. 2. $\sqrt[3]{24}$. 3. $\sqrt[4]{96}$. 4. $\sqrt[4]{1280}$.
5. $\sqrt[n]{a^nb}$. 6. $\sqrt[a]{x^{3a}y}$. 7. $\sqrt[5]{a^{20}b^2}$.

Exercise 107. [Page 398]

1. $3\sqrt{2}$. 2. $4\sqrt{5}$. 3. $5\sqrt[3]{2}$. 4. $2\sqrt[3]{4}$. 5. $3\sqrt[4]{5}$.
6. $7\sqrt[3]{4}$. 7. $5\sqrt[4]{3}$. 8. $a^2\sqrt[3]{b}$. 9. $x^4\sqrt[n]{a}$. 10. $-8\sqrt[3]{5}$.
11. $-4ab\sqrt[3]{3b}$. 12. $5a^2x\sqrt[3]{4ax}$.

Exercise 108. [Page 398]

1. $7\sqrt{3}$. 2. $7\sqrt{2}$. 3. $8\sqrt{5}$. 4. $2\sqrt{2}$. 5. $\sqrt[3]{2}$. 6. $5\sqrt[4]{5}$.
7. $\sqrt[4]{3}$. 8. $3\sqrt{3}$. 9. $6\sqrt{5}$. 10. 0. 11. 0. 12. $17\sqrt[3]{2}$.
13. $(7x+y)\sqrt{5x}$. 14. $(x^2-2y^2+3z^2)\sqrt[3]{a}$. 15. $4a\sqrt[4]{2x}$

Exercise 109. [Page 399]

1. $\sqrt[6]{27}$ and $\sqrt[6]{4}$. **2.** $\sqrt[12]{256}$ and $\sqrt[12]{125}$. **3.** $\sqrt[15]{8}$ and $\sqrt[15]{243}$.

4. $\sqrt[12]{27}$ and $\sqrt[12]{25}$. **5.** $\sqrt[24]{256}$ and $\sqrt[24]{216}$. **6.** The latter.

7. The former. **8.** The former. **9.** $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt{2}$.

10. $\sqrt[8]{10}$, $\sqrt[4]{3}$, $\sqrt[12]{25}$.

Exercise 110. [Page 401]

1. $5\sqrt{2}$. **2.** $4\sqrt{3}$. **3.** 9. **4.** $3\sqrt{10}$. **5.** 30. **6.** 5. **7.** $3ax\sqrt[3]{6x}$.

8. $\sqrt[6]{864}$. **9.** $\sqrt[6]{288}$. **10.** $4\sqrt[6]{2}$. **11.** $9\sqrt[6]{3}$. **12.** $\sqrt[18]{72}$.

13. $\sqrt[8]{27}$. **14.** $\sqrt[18]{32}$. **15.** $\sqrt[12]{1024}$. **16.** $40\sqrt{3}$. **17.** $288\sqrt{2}$.

18. $480\sqrt[3]{3}$. **19.** $210abx\sqrt[6]{x}$. **20.** $2\sqrt{\frac{2}{3}}$. **21.** $\frac{1}{8}$. **22.** $\sqrt[3]{\frac{3}{4}}$.

23. $\sqrt[6]{\frac{2}{3}}$. **24.** ·577. **25.** 1·341. **26.** 3·535. **27.** 26·832.

Exercise 111. [Pages 401-402]

1. $a\sqrt{b}+b\sqrt{a}$. **2.** $a-b$. **3.** $6a-10\sqrt{a}$. **4.** $16x-9y$.

5. $6x-54$. **6.** $6+\sqrt{10}$. **7.** $7+4\sqrt{6}$. **8.** $6-6\sqrt{5}$.

9. $2+6\sqrt{2}$. **10.** $5+3\sqrt[3]{12}+3\sqrt[3]{18}$. **11.** $2x-2\sqrt{x^2-a^2}$.

12. $182+80\sqrt{3}$. **13.** $83+12\sqrt{35}$. **14.** $2a^2-2\sqrt{a^4-4b^4}$.

15. $29x^2-21y^2+20\sqrt{x^4-y^4}$.

Exercise 112. [Pages 403-404]

1. $\frac{23-3\sqrt{21}}{10}$. **2.** $5+2\sqrt{6}$. **3.** $24+17\sqrt{2}$. **4.** $9+2\sqrt{15}$.

5. $\frac{a+\sqrt{a^2-x^2}}{x}$. **6.** $x^2-\sqrt{x^4-1}$. **7.** $\frac{2+\sqrt{2}-\sqrt{6}}{4}$. **8.** 5·828.

9. 6·464. **10.** 5·414. **11.** 3·650. **12.** 6·854. **13.** ·504. **14.** $2x$.

15. $\sqrt{5}(1+\sqrt{2})$. **16.** $2+\sqrt{3}$. **17.** $\frac{1}{3}(\sqrt{30}+2\sqrt{3}-3\sqrt{2})$. **18.** 198.

19. $4x\sqrt{x^2-1}$. **20.** $2x^2$. **21.** $\frac{\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4}}{5}$. **22.** $2\sqrt[3]{2}+\sqrt[3]{12}+\sqrt[3]{9}$.

Exercise 113. [Pages 407-408]

1. $\sqrt{3}-1$. **2.** $2+\sqrt{3}$. **3.** $3-\sqrt{2}$. **4.** $\sqrt{5}+\sqrt{3}$. **5.** $3-\sqrt{5}$.

6. $5+\sqrt{3}$. **7.** $4-\sqrt{5}$. **8.** $3+2\sqrt{2}$. **9.** $6+\sqrt{5}$. **10.** $5-2\sqrt{3}$.

11. $2\sqrt{7}+\sqrt{3}$. **12.** $3\sqrt{5}-2\sqrt{7}$. **13.** $2\sqrt{11}+\sqrt{3}$. **14.** $\sqrt{\frac{7}{2}}-\sqrt{\frac{1}{2}}$.

15. $\sqrt{\frac{7}{3}}-\sqrt{\frac{5}{3}}$. **16.** $\sqrt[4]{2}(\sqrt{2}-1)$. **17.** $\sqrt[4]{2}(\sqrt{3}-1)$. **18.** $\sqrt[4]{3}(\sqrt{2}+1)$.
19. $\sqrt[4]{5}(\sqrt{3}+\sqrt{2})$. **20.** $\sqrt{2}$. **21.** 1, or, $\frac{5}{3}\sqrt{3}-2$. **22.** b.
23. $x+\sqrt{a^2-x^2}$. **24.** $\sqrt{a+b}+\sqrt{a-b}$. **25.** $\sqrt{a+\frac{1}{2}x}+\sqrt{\frac{1}{2}x}$.
26. $\sqrt{x+2}+\sqrt{x-3}$. **27.** $\sqrt{x+y}+\sqrt{z}$.

Exercise 114. [Pages 410-411]

1. 9. **2.** 3. **3.** 16. **4.** $\frac{3}{4}$. **5.** $\frac{9}{20}$. **6.** 25. **7.** 8. **8.** 25.
9. 2. **10.** $\frac{a}{4}$. **11.** $\frac{(b-a)^2}{2b}$. **12.** 5. **13.** 9. **14.** 7. **15.** 5.
16. 6. **17.** 3. **18.** $\frac{5}{8}$. **19.** 81. **20.** $x=\frac{81}{a}$. **21.** $\frac{1}{a}\left(\frac{c^2}{c-1}+b\right)^2$.
22. 5. **23.** $\frac{17a}{8}$. **24.** $\frac{4a}{5}$. **25.** 36. **26.** $\frac{2a^2-2ab+b^2}{2(b-a)}$. **27.** 4.
28. $\frac{16}{17}$. **29.** $4\frac{1}{2}$. **30.** $\frac{34}{27}$. **31.** $\frac{ab^{\frac{4}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}$. **32.** $\frac{7}{12}$. **33.** 1, or, -1.
34. $\frac{2a}{\sqrt{5}}$, **35.** 8, or 0. **36.** $\frac{41a^2}{40b}$. **37.** 7. **38.** 5. **39.** $\frac{b^2-4a^2}{4a}$.
40. ± 5. **41.** ± 4.

Exercise 115. [Pages 415-416]

1. $2xz+3y$. **2.** x^2-2x+3. **3.** x^3-x+1. **4.** $2x^2-3x+4$.
5. $2x^2+2ax+4b^2$. **6.** $3x^2-\frac{xy}{3}+3y^2$. **7.** $x^2-x+\frac{1}{4}$.
8. $7x^2-\frac{x}{5}+3$. **9.** $x^2-\frac{x}{2}+\frac{2}{x}$. **10.** $\frac{a^2}{2}+\frac{a}{x}-\frac{x}{a}$.
11. $\frac{a}{2b}=1-\frac{2b}{a}$. **12.** $\frac{3a}{x}-\frac{1}{5}+\frac{2x}{3a}$. **13.** $2x^2-2xy^2-y^4$.
14. $\frac{7x}{y}-3-\frac{y}{7x}$. **15.** $\frac{x}{y}-\frac{1}{2}-\frac{y}{x}$. **16.** $\frac{2x}{7y}-5+\frac{3y}{4x}$.
17. $x-x^{\frac{1}{2}}+1$. **18.** $x^{\frac{5}{6}}-2x^{\frac{1}{2}}-x^{\frac{1}{6}}$. **19.** $ax^{-1}+1+a^{-1}x$.
20. $x^{\frac{3}{2}}-x^{\frac{1}{2}}y^{-\frac{1}{2}}+y^{\frac{1}{2}}$. **21.** $\frac{3x^{\frac{3}{2}}}{2}-\frac{5}{3}xy^{\frac{1}{2}}+\frac{2}{5}x^{\frac{1}{2}}y$. **22.** a^m-2a^n.
23. $a^{2m+1}+3a^m-5c^{m-2}$.

Exercise 116. [Pages 419-420]

1. $5xy-4$. 2. $7ax^2-3b^2$ 3. $7a^3b^4+9a^4b^3$.

4. $\frac{x^4y^2}{2}-\frac{x^3y^5}{5}$. 5. $\frac{5ab}{2}-\frac{c^2}{3}$. 6. $a+b+c$.

7. $a-b+c$. 8. $2a-b-3c$. 9. $a^2+2b^2-3c^2$.

10. $2a^2-3b^2+5c^2$. 11. $x+\frac{a}{3}-\frac{b}{2}$. 12. $x-2-\frac{1}{x}$.

13. $x^2+1+\frac{1}{x^2}$. 14. $\frac{a}{b}+1+\frac{b}{a}$. 15. $\frac{x}{y}-\frac{1}{\sqrt{2}}+\frac{y}{x}$.

16. $\frac{3x}{a}-1+\frac{a}{3x}$. 17. $x+2+\frac{1}{x}$. 18. $a^{\sqrt{2}}-a^{-\sqrt{2}}$.

19. $a-b+c-d$. 20. a^2+b^2. 21. $a^2-b^2+c^2-d^2$.

22. $a^2+a-\frac{1}{2}$. 23. $2a(b+c)+2bc$. 25. $x=10$.

26. $a=16$. 27. 3. 28. If $p^2=4q$.

Exercise 117. [Page 423]

1. $x+9$. 2. $3x-8$. 3. $4a-3b$. 4. x^2-3x+2.

5. $2x^2+x-3$. 6. $1-3x^2+2x^4$. 7. $2x^2-3cx+c^2$.

Exercise 118. [Pages 427-428]

1. The latter. 2. The latter. 3. The former. 4. The former.

5. The latter. 6. $a:d$. 7. 1 : 4. 8. 1 : 1.

9. 75 : 8. 10. 28 : 27. 11. 5 : 7. 12. 3 : 4.

13. 63 and 72. 14. 85 and 51. 15. 28 and 35. 16. 42 and 54.

17. -15. 18. 35. 19. -17. 20. $\frac{ad-bc}{c-d}$.

23. $\frac{78}{73}$. 24. $\frac{1778}{1771}$. 25. B.

Exercise 119. [Pages 429-430]

1. 4. 2. 18. 3. $37\frac{1}{2}$. 4. 36. 5. 20.

6. 60. 7. 20. 8. 6. 9. 14. 10. 18.

11. 8. 12. $\frac{bm-an}{(a+n)-(b+m)}$. 13. 2. 14. $\frac{bm-an}{(b+m)-(a+n)}$.

Exercise 120. [Page 434]

1. $x=9,\ y=6.$ 2. $x=25,\ y=9.$ 3. $x=56,\ y=30.$

4. $\frac{a}{3}$. 5. $\frac{5}{8}$. 6. $\frac{3}{4}$. 7. $\frac{4}{81}$. 8. $\frac{1}{2}$. 9. $2\frac{7}{9}$.

10. $\sqrt{2ab-b^2}$. 11. $a\left\{1-\frac{16b^2}{(b+1)^4}\right\}$. 15. 2.

Exercise 122. [Pages 439-441]

27. 0.

Exercise 123. [Pages 445-446]

1. $a^2d^2=b^2c^2$. 2. $b^3c^2=a^3d^2$. 3. $n^4p^3=m^4q^3$. 4. $ad^2-bd+c=0$.
5. $lb^2-amb+a^2n=0$. 6. $(bn-cm)(am-bl)=(cl-an)^2$. 7. $ab=1$.
8. $35pq=6$. 9. $(b_1c_2-c_1b_2)(a_1b_2-b_1a_2)^2=(c_1a_2-a_1c_2)^3$.
10. $(b_1c_2-c_1b_2)^2(a_1b_2-b_1a_2)=(c_1a_2-a_1c_2)^3$
11. $(b_1c_2-c_1b_2)^3(a_1b_2-b_1a_2)=(c_1a_2-a_1c_2)^4$.
12. $(an^2-bn+cm)(am^2-an+b)=(c+amn)^2$.
13. $(c^2+3ab)(b^2-2ab-ac)=(3a^2-2ac+bc)^2$. 14. $a^2+b^2=m^2+n^2$.
15. $(ab_1+bc_1)^2+(a_1b+b_1c)^2=(cc_1-aa_1)^2$.
16. $a^2n+b^2l=abm$. 17. $ab+bc+ca+2abc=1$.
18. $a+b+c+abc=0$. 19. $a^2+b^2+c^2=abc+4$.
20. $d^2(a+b+c)+abc=0$. 21. $x^2+y^2+z^2+2xyz=1$.

Exercise 124. [Page 447]

4. $x=a,\ y=b.$ 5. $x=1,\ y=1.$ 6. $x=y=a.$
7. $x=1,\ y=1,\ z=0.$ 8. $x=a,\ y=b.$

Exercise 126. [Page 450]

1. 8. 2. 7. 3. 6. 4. $\frac{49}{8}$. 5. 33. 6. 2. 7. $-\frac{1}{8}$.
8. $-\frac{1}{16}$. 9. $1\frac{1}{12}$. 10. $\frac{7}{8}$. 11. 16 ; 16.

Miscellaneous Exercises VI

[Pages 461-473]

I

1. $1\frac{3}{8}$. 2. 0. 3. $5b(a+b)$. 4. $2x^2-4xy+5y^2$.

5. $\frac{b^2}{(a+b)^3}$. 6. $\frac{ab}{b-a}$. 8. $5+\sqrt{6}$.

II

1. 21. **2.** $4x^2-6x-1$. **3.** 12. **4.** $\frac{3\sqrt{2}}{5}$.

5. x^2-3x+2. **6.** $\frac{1}{2}$. **7.** 11.

III

1. -30. **2.** $\frac{4x^2}{1-x^4}$. **3.** $(a+b-3c)(a-b+3c)$.

4. $x-a$. **5.** $(x-1)(x-2)(x-3)$. **6.** $\frac{x(x+2)}{x^2-2x+4}$. **7.** $\frac{1}{3}$.

IV

1. $\frac{y^2}{x^2}$. **2.** $x^6+x^5y-x^3y^3+xy^5+y^6$. **3.** (i) $(x+1)^2(x-1)$;
(ii) $(a+1)(a-1)(b+1)(b-1)$. **5.** $(64x^6-729)(3x+2)$. **6.** $-\frac{8}{7}$.
7. $x=a^2b,\ y=ab^2$.

V

1. 1. **2.** $a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b)$.

3. $\frac{9(a^2+3)}{a(a^2+27)}$. **4.** $\frac{x}{\sqrt{y}}-4+\frac{4\sqrt{y}}{x}$. **6.** $\frac{34\sqrt{5}-18}{11}$. **7.** $x=3,\ y=1$.

VI

1. 1. **2.** $b^2-a^2+\frac{b^4}{a^2}-\frac{a^4}{b^2}$. **3.** $x^2+(a-b)x-ab$.

5. $\frac{5x^2-4x-8}{3x^2+4x+24}$. **6.** $x=\frac{1}{4},\ y=\frac{1}{3}$. **7.** $x=3,\ y=5,\ z=7$.

VII

1. $2x^3y^{-3}-3x^4y$. **2.** ae^x+e^x+a+1. **4.** $\frac{ax+by}{ax-by}$. **5.** 7. **7.** $80,\ 128$.

VIII

2. (i) $(b+c-a)(b+c-5a)$; (ii) $(x+2y+a)(x-a)$.

3. $\frac{a+b}{(a-b+c)(b+c-a)}$. **5.** -6. **6.** $2x^2-3x^{-1}+4x^{-4}$.

7. $x=7\frac{1}{2},\ y=3\frac{1}{3},\ z=1\frac{1}{2}$.

IX

1. -20. **2.** $\frac{1}{x^2-1}$. **3.** $(a-b+1)(a^2+b^2+1+ab-a+b)$.

4. 6. **6.** $x=16,\ y=4$. **7.** $27\frac{3}{11}$ minutes past 8.

X

1. $9a^2+4b^2+9c^2-6bc+9ca+6ab$. 2. x^2+2x+3.

3. $\{(a+b)x+(a-b)y\}\{(a-b)x+(a+b)y\}$. 4. $\frac{a^2-b^2}{a^2+b^2}$. 6. 8.

7. 20 days.

XI

2. (i) $(a+b-c-d)(a-b+c-d)$; (ii) $(x+y-z)(x-y+z+1)$

3. $\frac{3x}{a}-1+\frac{a}{3x}$. 4. $x=y=z=1$. 5. $x^2-5xy+7y^2$.

6. 480 at 16 a shilling ; 90 at 18. 7. 1.

XII

2. 0. 3. (i) $(x-b)(x+b-2a)$; (ii) $(x+a)(x+b+c)$.

4. $3x-1$. 5. 20. 6. 10. 7. $13\sqrt{3}$.

XIII

3. 0. 4. 0. 5. 30. 6. 1. 7. $46\sqrt[3]{2}$

XIV

3. 47. 4. $a+b$. 5. $x=\frac{1}{2}(2a+b+c)$, $y=\frac{1}{2}(a+2b+c)$, $z=\frac{1}{2}(a+b+2c)$.

6. 5 days. 7. $(x^2+5ax+5a^2)^2-a^4$.

XV

1. 4. 3. $\frac{2x+3}{x^2+x+1}$. 4. $2x-3b$. 5. $\frac{1}{n}$. 6. 4. 7. 54 gallons.

XVI

1. x^2+2x+3. 2. 1. 4. $x=2\frac{1}{4}$, $y=1\frac{1}{2}$.

5. $(xy+ab)(ay^2+b^2x)$. 6. $-a^2-b^2-c^2+2ab+2ac+2bc$.

8. In the 1st, the wine is $\frac{1}{3}$ of the whole, in the second, $\frac{3}{5}$.

XVII

1. x^2+x+1. 2. $x=16$, $y=25$. 3. $n(n-1)$. 6. 72.

7. $\frac{x^2-2x+3}{2x^2+5x-3}$.

XVIII

2. $\frac{5a}{4}$. 3. (i) $(7x-1)(2x-5)$; (ii) $2(a-c)(1-ac)$; (iii) $2m^2n(m+n)$.

4. 1920. 5. $a+b+c$. 7. $x=2$, $y=4$, $z=6$.

8. $\frac{c^2}{(b-d)^2}-\frac{a^2}{(b+d)^2}=1$.

XIX

1. $\frac{4a}{3}$. 3. $ac-bc-b^2+a^2$. 4. $mq : np$.

7. $\sqrt{6}-2$. 8. $a^3+b^3+c^3-3abc=0$

XX

3. $\frac{1}{abc}$. 4. $-(a^2+b^2+c^2+ab+ac+bc)$. 6. $x=c$, or, $=c-\frac{a+b}{2}$.

7. $x=\frac{b+c}{2a}$, $y=\frac{a+c}{2b}$, $z=\frac{a+b}{2c}$.

8. $6\frac{7}{8}$ and $4\frac{1}{8}$ kilometres an hour ; $5\frac{5}{8}$ kilometres.

XXI

5. $(a-b)(a+3b-2c)$. 6. x^2-x+3. 7. $(ac'-a'c)^3=(ba'-b'a)^2(b'c-bc')$.

XXII

4. 1020 metres. 7. $x=b+c$, $y=a+c$, $z=a+b$.

8. $a^3+b^3+c^3-3abc=0$.

XXIII

2. $(3x-1)(7x-2)(4x-1)$. 6. $\frac{a^2+b^2+c^2}{ab+bc+ca}$. 8. $\frac{a^2}{p^2}+\frac{b^2}{q^2}=1$.

XXIV

1. $x=a$, $y=b$, $z=c$. 2. 54, 81, 108. 6. $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{(a-b)^2}$.

XXV

7. (i) $abc+2fgh-af^2-bg^2-ch^2=0$; (ii) $bc+ca+ab+2abc=1$.

8. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\left(\frac{a'^2}{a}+\frac{b'^2}{b}+\frac{c'^2}{c}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$.

Exercise 127. [Page 476]

1. ± 7. 2. $\pm\frac{5}{2}$. 3. ± 2. 4. ± 9. 5. ± 2. 6. $\pm\frac{\sqrt{3}}{2}$.

7. $\pm\frac{a\sqrt{a^2-4}}{2\sqrt{a^2-1}}$. 8. $\pm\sqrt{mab}$. 9. $\pm\frac{2}{b\sqrt{4a-b^2}}$. 10. $\pm\frac{2a}{\sqrt{5}}$.

11. $\pm\sqrt{\frac{3}{2}}$. 12. $\pm\sqrt{\frac{15}{2}}$. 13. $\pm n\left(a-\frac{n^2}{4}\right)^{\frac{1}{2}}$. 14. $\pm\frac{1}{2}$.

Exercise 128. [Pages 478-479]

1. $-1, -12$. 2. $15, -14$. 3. $a, 3a$. 4. $b, 2a-b$. 5. $\frac{c}{a}, \frac{c}{b}$.

6. $\frac{3a}{4}, -\frac{8a}{3}$. 7. $\frac{1}{3}(5a+3b)$; $\frac{1}{2}(2a+7b)$. 8. $\frac{1}{4}(2a+5b)$; $\frac{1}{3}(6a-b)$.

9. $\frac{2}{3}(a+b), -\frac{4}{3}(a+b)$. **10.** 29, -10. **11.** 1, $\frac{2b}{a-b}$. **12.** $a, -b$.

13. $\frac{a}{2}, \frac{3a}{4}$. **14.** 4, 8. **15.** 0, $\frac{2ab-ac-bc}{a+b-2c}$. **16.** $a, \frac{a}{5}$.

17. 2, $-\frac{2}{3}$ **18.** 2, $-\frac{18}{5}$. **19.** 5, $-\frac{12}{7}$. **20.** 4, $-\frac{14}{3}$.

Exercise 129. [Pages 479-481]

3. $\frac{3}{4}, 7\frac{1}{2}$. **4.** $3\frac{3}{5}, 2\frac{2}{3}$. **5.** $2\frac{1}{5}, 2\frac{1}{4}$. **6.** $6\frac{1}{2}, -1\frac{7}{8}$. **7.** $1\frac{1}{3}, -2\frac{2}{3}$.

8. ·4, ·05. **9.** $2\pm\frac{1}{3}\sqrt{3}$. **11.** 9, 8. **12.** $\frac{8}{5}, \frac{4}{7}$. **13.** $\frac{2}{3}, \frac{3}{10}$.

14. 29, -10. **15.** 10, -29. **17.** 2, -3. **19.** $\frac{4}{5}$, 0.

20. 10, $-\frac{2}{3}$. **21.** 24, $8\frac{2}{5}$. **22.** $\frac{1}{2}, 4\frac{1}{12}$. **23.** 3.

24. 6, $3\frac{1}{15}$. **25.** 1, $-\frac{40}{17}$. **26.** $\frac{-11\pm\sqrt{13}}{6}, a$.

Exercise 130. [Page 482]

1. 3, $2\frac{2}{3}$. **2.** $-4, -5$. **3.** $\frac{4}{5}, -\frac{5}{2}$. **4.** $\frac{5}{8}, -\frac{7}{3}$. **5.** $2\frac{1}{2}, -\frac{3}{4}$.

6. 5, $\frac{5}{8}$. **7.** $-1, \frac{5}{4}$.

Exercise 131. [Page 483]

1. $\frac{3}{2}, -6$. **2.** $\frac{7}{5}, -\frac{4}{5}$. **3.** $\frac{5}{18}, -8$. **4.** $\frac{3}{4}$, 34.

5. $9\frac{15}{17}, -11$. **6.** $\frac{a}{2}, \frac{3}{c}$. **7.** $ab, -\frac{a}{3}$.

Exercise 132. [Page 487]

1. 1, 2, 3. **2.** 1, $\frac{1}{2}(3+\sqrt{17}), \frac{1}{2}(3-\sqrt{17})$.

3. 1, $-\frac{1}{2}, -3$. **4.** $-1, -2-\sqrt{10}, -2+\sqrt{10}$.

5. $2\pm\sqrt{3}, \frac{1}{2}(1\pm\sqrt{-3})$. **6.** $\frac{1}{2}(3\pm\sqrt{-7}), 1\pm\sqrt{-3}$.

7. $-2\pm\sqrt{6}, -2\pm\sqrt{-6}$. **8.** 1, $-8, \frac{1}{2}(-7\pm\sqrt{-71})$.

9. $-1\pm\sqrt{5}, -1\pm2\sqrt{2}$. **10.** $1\pm\sqrt{2}, 1\pm\sqrt{5}$.

11. $\frac{1}{2}(3\pm\sqrt{-11}), \frac{1}{2}(3\pm\sqrt{5})$. **12.** 1, $1\pm\sqrt{2}, \frac{1}{2}(-1\pm\sqrt{17})$.

13. 1, $-1, \pm\sqrt{-1}$. **14.** 1, 6, $-1, -6$. **15.** 2, 3.

16. 2, 3. **17.** 0, 1, 2. **18.** 1, $-1, \frac{1}{2}(1\pm\sqrt{-3}), \frac{1}{2}(-1\pm\sqrt{-3})$.

19. 2, -2. **20.** 2, $\frac{1}{2}, \frac{1}{4}(5\pm\sqrt{201})$. **21.** 2, $\frac{1}{4}, \frac{1}{8}(9\pm\sqrt{-31})$.

22. 1, 2, $\frac{1}{2}(3\pm\sqrt{-1})$. **23.** 4, $-6, -1\pm4\sqrt{2}$. **24.** 2, 5, $-\frac{1}{2}, -\frac{1}{5}$.

Exercise 133. [Pages 488-490]

1. Real, irrational and unequal. **2.** Real, irrational and unequal.

3. Real, rational and unequal. **4.** Real, rational and equal.

5. Real, irrational and unequal. **6.** Imaginary.

7. Imaginary. **8.** Real, irrational and unequal.

9. Real, irrational and unequal. **10.** 8. **11.** ±12.

Exercise 134. [Pages 494-496]

1. $x^2-4x+3=0$. **2.** $x^2+2x-35=0$. **3.** $3x^2-10x+3=0$.

4. (i) $x^2-8x+11=0$; (ii) $x^2-4ax+4a^2-b=0$.

5. (i) sum $=5$, product $=6$; (ii) sum $=-9$, product $=-13$;
(iii) sum $=\frac{20}{3}$, product $=-5$; (iv) sum $=\frac{7}{2}$, product $=-\frac{9}{2}$;
(v) sum $=-\frac{1}{2}$, product $=\frac{1}{12}$.

6. (ii) $x^2-p^2x+2q(p^2-2q)=0$; (iii) $q^3x^2-p^2qx+2(p^2-2q)=0$;
(iv) $qx^2+p(q+1)x+(q+1)^2=0$.

10. (i) $91x^2+8x+3=0$; (ii) $cx^2+bx+a=0$.

13. $a=12$, $b=31$, $c=181$. **14.** $k=1$. **15.** $k=a=2$.

Exercise 135. [Pages 503-505]

1. 16 ; £5. **2.** 18. **3.** 3 centimetres.

4. A's capital = £5 ; B's capital = £120. **5.** 5 kilometres per hour.

6. 12, 5 ; $\frac{17}{\sqrt{2}}$, $\frac{7}{\sqrt{2}}$. **7.** 5, 3. **8.** A, 120 ; B, 80. **9.** 7, 2.

10. Rs. 90. **11.** Small wheel $1\frac{1}{3}$ metres ; large wheel $4\frac{1}{3}$ metres.

12. 4 pence. **13.** 56. **14.** 20 and 30 kilometres per hour. **15.** £60, or, £40.

16. 12, 16, 18. **17.** 26 and 38 metres. **18.** 25, 13, 6. **19.** 40 and 45 kilometres per hour. **20.** 256 sq. metres. **21.** 14, 10, 2. **22.** 6400.

23. A, 10 kilometres per hour ; B, 12 kilometres per hour. **24.** $\frac{(a-b)^4}{a^3}$.

25. The sides were 30 metres and 19 metres and the height 4 metres.

26. 100 shares at £15 each. **27.** 15, 12, 10, 7. **28.** $m+1 : n+1 : p+1$.

29. 625. **30.** 324 square metres.

Exercise 136. [Page 515]

8. $\left.\begin{array}{l}x=8\\y=6\end{array}\right\}$; $\left.\begin{array}{l}x=6\\y=8\end{array}\right\}$. **9.** $\left.\begin{array}{l}x=4\\y=3\end{array}\right\}$; $\left.\begin{array}{l}x=-3\\y=-4\end{array}\right\}$. **10.** $\left.\begin{array}{l}x=5\\y=7\end{array}\right\}$; $\left.\begin{array}{l}x=6\\y=6\end{array}\right\}$.

11. $x=-2, 6$. **12.** -2 ; 8.

14. $x=8$, $x=-4$, $y=9$, $y=-3$. **15.** (5, 0) ; (0, 5).

Exercise 137. [Pages 531-532]

16. (i) 1 ; (ii) 4 ; (iii) -7 ; (iv) 2·5.

17. (i) 4 ; (ii) 4 ; (iii) 21 ; (iv) 1·5.

18. $\left.\begin{array}{l}x=4\\y=1\end{array}\right\}$ and $\left.\begin{array}{l}x=1\\y=4\end{array}\right\}$. **20.** $\left.\begin{array}{l}x=4\\y=-2\end{array}\right\}$ and $\left.\begin{array}{l}x=-2\\y=4\end{array}\right\}$.

21. 1 ; 3. **22.** 1 ; $-\frac{2}{3}$. **23.** 1 ; 2·5. **24.** 1 ; $-\frac{5}{7}$.

25. (i) $\left.\begin{array}{l}x=1\cdot2\\ y=\cdot6\end{array}\right\}$ and $\left.\begin{array}{l}x=-1\cdot2\\ y=-\cdot6\end{array}\right\}$; (ii) $\left.\begin{array}{l}x=-5\\ y=-1\end{array}\right\}$ and $\left.\begin{array}{l}x=-1\\ y=-5\end{array}\right\}$;
(iii) $\left.\begin{array}{l}x=0\\ y=0\end{array}\right\}$ and $\left.\begin{array}{l}x=1\\ y=2\end{array}\right\}$; (iv) $\left.\begin{array}{l}x=0\\ y=0\end{array}\right\}$ and $\left.\begin{array}{l}x=\frac{1}{2}\\ y=\frac{1}{4}\end{array}\right\}$.

Exercise 138. [Pages 534-535]

1. (i) 16, 40, $2n-6$; (ii) 15, 39, $2n-7$; (iii) $\frac{-29}{3}$, $\frac{-101}{3}$, $\frac{37}{3}-2n$; (iv) $\frac{-19}{7}$, $-\frac{67}{7}$, $\frac{25-4n}{7}$; (v) 47, 119, $6n-19$.

2. 29th, 46th, $(3n-10)$th. **3.** 6. **4.** 98.

5. −48, −44, −40, 20th term = 28. **6.** 1st term = 13; 18th term = −38.

7. 1st term = 2, com. diff. = 3. **8.** $\frac{d(p-r)-c(q-r)}{p-q}$.

Exercise 139. [Page 537]

1. 325. **2.** 900. **3.** 504 **4.** 88. **5.** $-\frac{15}{22}$. **6.** $1\frac{1}{4}$. **8.** $52\frac{1}{2}$.

9. 0. **10.** 25452 **11.** $\frac{1}{2}(n-1)$. **12.** $\frac{n}{a+b}\left\{na-\frac{n+1}{2}b\right\}$

13. 720. **14.** n. **15.** $n(a+b)^2-n(n-1)ab$. **16.** 899. **17.** 704

18. $\frac{n}{2}\{(x-2y)n+x\}$. **19.** 4080. **20.** $\frac{21n-5n^2}{2}$

Exercise 140. [Pages 539-540]

1. 3. **2.** 9. **3.** 7. **4.** 13, or, 7. **5.** Last term 3, or, −1; number of terms 10, or, 12. **6.** 18, or, 19. **7.** n^2. **8.** 1, $\frac{5}{4}$, $\frac{3}{2}$, $\frac{7}{4}$, &c.; 1470. **9.** 1, 3, 5, 7, &c., n^2. **10.** 2. **11.** 4, or 10.

Exercise 141. [Page 541]

1. (i) $6\frac{1}{2}$; (ii) 8; (iii) m; (iv) a^2+x^2. **2.** (i) $9\frac{1}{3}$, $10\frac{2}{3}$; (ii) $\frac{2}{3}$, $7\frac{1}{3}$.
3. 207, 297, 387. **4.** −2, −6, −10, −14. **5.** 1, $-1\frac{1}{2}$, &c., −39. **6.** 14.

Exercise 142. [Page 545]

1. $\frac{n}{2}(6n^2+3n-1)$. **2.** $\frac{n(n+1)(n+2)(3n+5)}{12}$. **3.** $\frac{n}{3}(4n^2+6n-1)$.

4. $\frac{n}{3}(n^2+6n+11)$. **5.** $3n(n+1)(n+3)$.

6. $\frac{n}{4}(n+5)(n^2+5n+10)$. **7.** 26876100.

8. $n^2(2n^2-1)$. **9.** $\frac{n(n+1)(n+2)}{6}$. **10.** $\frac{n(n+1)(2n+1)}{6}$

11. $\frac{n(n+1)(n+2)(n+3)}{4}$. 12. $\frac{n}{12}(9n^3+46n^2+51n-34)$.

13. (i) $-\frac{n}{2}$ (if n is even) ; (ii) $\frac{n+1}{2}$ (if n is odd).

14. (i) $-\frac{n(n+1)}{2}$ (if n is even) ; (ii) $\frac{n(n+1)}{2}$ (if n is odd).

Exercise 143. [Pages 549-551]

1. $\frac{(2n+1)(ma-nb)}{a-b}$. 2. 9, 13, 17, 21, 25. 3. 13 ; 6. 4. 70.

5. $\frac{n(n+1)(n+2)}{6}$. 6. $\frac{n}{6}(2n^2+3n+7)$. 7. $\frac{n}{6}(2n^2+9n+1)$.

8. (i) $\frac{n}{3(2n+3)}$; (ii) $\frac{n}{2n+1}$; (iii) $\frac{n}{a(a+nb)}$. 9. 8, 12, 16, 20.

10. 8, 12, 16. 11. 3, 5, 7. 12. 1, 3, 5, 7.

13. 3, 5, 7, 9, 11, 13. 18. 16. 21. $\frac{n(n+1)(n+2)}{6}$.

22. $\frac{1}{6}(n-1)n.(2n-1)$ metres. 23. 16. 24. 5.

Exercise 144. [Page 553]

1. 8748. 2. $\frac{1}{3}$. 3. 65536. 4. -243.

5. $\frac{8}{27}$; $\pm\frac{2^{n-3}}{3^{n-3}}$, + or, −, according as n is even or odd.

6. $-\frac{448}{243}$. 7. $\frac{\sqrt{3}}{3^{n-1}}$. 8. $\frac{1}{32}$, $\frac{1}{128}$. 9. (i) 6, 12, 24, 48,...... ;

(ii) 27, 9, 3, 1, $\frac{1}{3}$,...... or, -27, 9, -3, 1, $-\frac{1}{3}$,...... ;

(iii) $\frac{81}{2}$, -27, 18, -12,...... 10. $\left(\frac{c^{n-q}}{d^{n-p}}\right)^{\frac{1}{p-q}}$.

12. pth term $=\sqrt{mn}$ and qth term $=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

Exercise 145. [Pages 554-555]

1. 265720. 2. $\frac{3}{2}\left\{1-\frac{1}{3^n}\right\}$. 3. $60\frac{20}{27}$. 4. -682. 5. $\frac{181}{243}$.

6. $\frac{1}{3}(1-2^{2r})$. 7. $\frac{1}{14}\cdot\frac{5^n \pm 2^n}{5^{n-2}}$, − or, +, according as n is even or odd.

8. (i) $2\left\{1-\frac{1}{2^n}\right\}$. (ii) 16400.

Exercise 146. [Page 557]

1. 1. 2. $\frac{1}{8}$. 3. $3\frac{1}{8}$. 4. $\frac{3}{8}$. 5. $10\frac{1}{8}$. 6. $\frac{11}{12}$.
7. $\frac{11}{18}$. 8. $\frac{3\sqrt{3}}{2}$. 9. $\frac{1}{2}(4+3\sqrt{2})$. 10. $\frac{1}{11}$.

Exercise 147. [Pages 558-559]

1. 6, 12. 2. $\frac{2}{3}$, 1, $\frac{3}{2}$. 3. $-1, \frac{3}{2}, -\frac{9}{4}, \frac{27}{8}$. 4. $\frac{16}{3}$, 8, 12, 18, 27.
5. 225, 2025, 18225.

Exercise 148. [Pages 562-564]

1. $\frac{1}{36}$; $1\frac{8}{55}$; $\frac{358}{1665}$; $\frac{1}{7}$. 2. $\frac{1+x}{(1-x)^2}$. 3. $\frac{2x}{(1-2x)^2}$. 4. $\frac{(1+6x).3x}{(1-3x)^2}$.
5. $\frac{a(1-a^n)}{(1-a)^2}-\frac{na^{n+1}}{1-a}$. 6. $\frac{1-x}{(1+x)^2}$. 7. 1. 8. $4-\frac{n+2}{2^{n-1}}$
9. $2^{n-1}(2n-1)$; $2^n(2n-3)+3$. 10. $\frac{5^{n+1}-5-4n}{16\times5^{n-1}}$.
11. $\frac{40}{81}(10^n-1)-\frac{4n}{9}$. 12. $n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)$.
13. $2^{n+1}-2-n$. 14. $2(2^n-1-4n)$. 15. $\frac{1}{9}(4^n-1+15n)$.
16. $\frac{1}{2}(3^{n+1}-3-2n)$. 19. 2, 5, 8, or, 26, 5, −16. 20. 4, 8, 16.
21. $\frac{1}{4}$, 4, 20. 27. $n.2^{n+2}-2^{n+1}+2$. 33. $\frac{1}{(1-r)(1-ar)}$.

Exercise 149. [Pages 571-574]

1. $x-3y=0$. 2. 14. 3. $2\frac{1}{2}$. 4. 1. 5. $27x^2-4y^3=0$.
6. $y=2\left(x+\frac{1}{x}\right)$. 7. $12x^2-25xy+12y^2=0$. 8. $y=2x+\frac{4}{x^2}$.
9. $y=3+2x-x^2$. 10. $y=\frac{b}{a^2}\sqrt{a^2-x^2}$ 12. 45 inches.
13. £26. 5s. 15. 45 sq. metres. 16. $346\frac{1}{2}$ sq. cm.
17. 960 cubic cm. 18. $1\frac{3}{5}$ metres. 19. 10 cm.
20. 1·2426 ins. nearly. 21. ·01875 decimetres. 22. 1610 ft. ; 305·9 ft.
23. $3\frac{3}{5}$ days. 24. $224\frac{1}{2}$ days nearly. 25. 9 : 4.
26. Value of a diamond $=\frac{£mcn^2}{(m+1)a^2}$; value of a ruby $=\frac{£cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$.
29. $x=\frac{22}{15}s+\frac{2}{15s}$. 31. The cost is least when the rate is 12 miles an hour ; and the cost per mile is £$\frac{3}{32}$ and for the journey is £9. 7s. 6d.

WEST BENGAL S. E. B. MADHYAMIK PARIKSHA

1983

1 **Answer either (*a*) or (*b*)**

(*a*) Resolve into factors (*any two*) :

(i) $20x^2-7xy-6y^2$. (ii) x^8-16. (iii) $x^4+3x^3+4x^2+3x+1$.

(*b*) Find the H. C. F. of the following expressions :—

$8x^3+27,\ 4x^2-9,\ 2x^2+5x+3,\ 2x^3+7x^2+8x+3.$

2. **Answer either (*a*) or (*b*).**

(*a*) Solve :

(i) $\dfrac{x+y}{xy}=2,\quad \dfrac{x-y}{xy}=1.$ (ii) $\dfrac{5}{x}+3y=8,\quad \dfrac{4}{x}-10y=56.$

(*b*) There are some boys and girls in a room. The square of the number of the girls is greater than the number of the boys by 28. If there were two more girls in the room, the number of the boys would have been same as the number of the girls. Find the total number of the boys and girls.

3. **Answer either (*a*) or (*b*) :—**

(*a*) Draw the graphs of the two equations $2x+3y-12=0$ and $x+2y-7=0$. Solve the two equations from their graphs. (At least three points are to be taken for drawing each graph.)

(*b*) (i) If from a positive number its positive square root is substracted, 110 is obtained. Find the number.

(ii) The sum of the squares of two numbers is 29 and their product is -10. Find the numbers.

4. Draw the graphs of the following three inequations and mark the solution-region satisfying them simultaneously, if it exists :

$$x\leqslant 1,\ y\leqslant 2,\ x-4y\leqslant 12.$$

Or, If $x=\sqrt{2}+1$, find the value of

$$x^4+\frac{1}{x^4} \text{ and } x^4-\frac{1}{x^4}$$

1984

1. **Answers either Q (*a*) or (*b*).**

(*a*) Resolve into factors (*any two*) :—

(i) $27x^6-125$.

(ii) $10x^2+7xy-12y^2$.

(iii) $x^2(y-z)+y^2(z-x)+z^2(x-y)$.

(*b*) **Find the H. C. F. of the following expressions**

$x^3-1,\ x^3+x^2-7x+5,\ x^4+2x^3-2x-1.$

2. Answer either Q. (*a*) or Q. (*b*).

(*a*) Solve :

(i) $\frac{2}{x}+\frac{3}{y}=18$, $\frac{5}{x}+\frac{4}{y}=31$.   (ii) $8x-\frac{3}{y}=5$, $5x-\frac{2}{y}=3$.

(*b*) There are some pens and pencils in a box. 3 pens are removed from the box and 2 more pencils are put in it. As a result the number of pencils in the box becomes twice the number of pens. If 7 more pencils are put in the box, the number of pencils in it becomes three times the number of pens. Find the numbers of the pens and the pencils which were originally in the box.

3. Answer either Q. (*a*) or Q. (*b*) :

(*a*) Draw the graphs of the two equations $x+2y+1=0$ and $2y-x-9=0$. Solve the two equations from their graphs. (At least three points are to be taken for drawing each graph.)

(*b*) (i) The numerator of a fraction is less than its denominator by 1 and the square of the numerator is greater than the denominator by 11. Find the fraction.

(ii) The sum of the squares of two positive numbers is greater than their product by 28. If the ratio of the two numbers is 2 : 3, find the two numbers.

4. Draw the graphs of the following three inequations and mark the solution-region satisfying them simultaneously, if it exists :

$$x \geqslant 2,\ y \geqslant 3,\ 5x-4y \leqslant 0.$$

If $x=\sqrt{3}+\sqrt{2}$ and $y=\sqrt{3}-\sqrt{2}$, then find the values of x^2+y^2 and x^3-y^3.

1985

1. Answer either (*a*) or (*b*) :—

(*a*) Choose *any two* of the following expressions and resolve each of them into two factors :

(i) $4x^4+16y^4$.   (ii) $4x^3+4x^2-7x+2$.

(iii) (x^2+5x+4).   (iv) $(x^2+5x+6)-15$.

(*b*) Find the H. C. F. of the following expressions :—

$$x^3-x,\quad x^5-x^3,\quad x^6-1,\quad x^7-x^3.$$

2. Answer either (*a*) or (*b*) :—

(*a*) Solve : (i) $\frac{x}{2}+\frac{y}{4}=10$, $\frac{x+y}{x-y}=-7$ (ii) $5+3xy=33x$, $4-7xy=17x$.

(*b*) The area of a square is twice that of a rectangle. The length of the rectangle is greater than the length of each side of the square by 5 cm. and the breadth is less by 12 cm. ; find the perimeter of the rectangle.

3. Answer either (*a*) or (*b*) :—

(*a*) Draw the graphs of the two equations : $2x+5y-19=0$ and $3x-2y+19=0$. (At least three points are to be taken for drawing each graph.)

(*b*) The product of three integers is 800. The ratio of the first and the second of them is 2 : 5. If 6 is added to each of the first and the second numbers and the third is unchanged, the product of the three becomes 3,200. Find the original three numbers.

4. Answer either (*a*) or (*b*) :—

(*a*) Draw the graphs of the following three equations and mark the solution-region, satisfying them simultaneously, if it exists :

$$0 \leqslant x \leqslant 2, \quad 0 \leqslant y \leqslant 3, \quad 2x+y \leqslant 4.$$

(*b*) If $x=\frac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y=\frac{\sqrt{3}-1}{\sqrt{3}+1}$, then prove that $\frac{x^2+y^2}{x^2-y^2}=\frac{7\sqrt{3}}{12}$.

1986

1. Answer *any two* questions :—

(i) Express $[(a^2-b^2)x^2+2ax+1]$ as a product of two factors.

(ii) Simplify : $\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}$.

(iii) If $\frac{a}{b}=\frac{b}{c}$, show that $\left(\frac{a+b}{b+c}\right)^2=\frac{a^2+b^2}{b^2+c^2}$.

(iv) Find the H. C. F. of the following expressions :

$$(x^3-5x^2-2x+24)\ ;\ (x^3-9x^2+26x-24\ .$$

2. Answer *any one* question :—

(*a*) Solve : (i) $3x+2y=7$,
$4x-y=2$.

(ii) $\left(\frac{x+a}{x-a}\right)^2-5\left(\frac{x+a}{x-a}\right)+6=0$.

(*b*) (i) If $x=\frac{\sqrt{3}}{2}$, then find the value of $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$.

(ii) In a number with two digits, the digit in the unit place is twice that of the ten's place ; moreover, if 18 is added to this number the digits in the number interchange their positions. Determine the number.

3. (*a*) Using same unit and same axes, draw the graphs of the expressions $3x+4)$ and $(5x+1)$. From the graphs determine the value of x for which values of the two expressions become equal.

Or

(*a*) At ten in the morning a person starts to his destination at a speed of 4km. per hour. After two hours another person starts from the same place by cycle at a speed of 5 km. per hour to meet the former person. If the destination is at a distance of 140 km., determine when and how much ahead of destination they will meet, with the help of graphs.

(*b*) Draw the graphs of the following inequation :

$$x \leqslant y, \quad x+y \leqslant 2.$$

Mark the solution-region satisfying the inequations simultaneously.

Or

(*b*) The ratio of two liquids in a mixture is 2 : 3 and in another mixture of the same two liquids the ratio is 5 : 4. Determine in which ratio these two mixtures should be mixed so that the amount of the two liquids in this new mixture will be the same.

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GAUHATI

COMPULSORY PAPERS

1965

1. (*a*) Resolve into factors :

(i) $3x^2 - x - 2$. (ii) $a^3 + b^3 + c^3 - 3abc$.

(*b*) Find the L. C. M. of

$x^3 - 8$, $x^2 - 5x + 6$ and $x^3 - 4x^2 + 4x$.

2. (*a*) (i) I walk $3\frac{1}{2}$ miles per hour. How long do I take to go x miles ?

(ii) What is the cost of x pencils if p pencils cost q rupees ?

(iii) A boy walks x yds. to school and then twice that distance to the playground. How many miles has he walked altogether ?

(*b*) If $(a + b + c)^2 = 3(bc + ca + ab)$, show that $a = b = c$.

3. (*a*) Solve : (i) $\frac{x}{2} - \frac{x-1}{3} = 1$.

(ii) $\left.\begin{aligned} 3x + 5y &= 4 \\ 4x - 3y &= 15 \end{aligned}\right\}$

(*b*) A coal merchant has entered into a contract to deliver 200 tons of coal at Rs. 28 per ton. When the time comes, he finds that his stock, which has cost him Rs. 21 a ton, is not sufficient. Consequently he has to buy more coal at Rs. 32 per ton, and he loses Rs. 360 in the transaction. How much coal did he have to buy ?

4. (*a*) Given $x + \frac{1}{x} = 3$, find the value of

(i) $x^2 + \frac{1}{x^2}$, (ii) $x^4 + \frac{1}{x^4}$.

(*b*) If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, show that $\frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} = \frac{3xyz}{abc}$.

5. (*a*) If $a = 1 - \frac{1}{b}$ and $b = 1 + \frac{2}{c}$, prove that

$$a + b + c = \frac{b^2 + 1}{b(b-1)}.$$

(*b*) Draw the graph of $y = x$.

(*c*) *Either*, Draw the graph of $x = 2$.

Or, Draw a graph to convert inches into centimetres or *vice versa*, given that 1 inch = 2·5 cm.

1966

1. (*a*) Resolve into factors :

(i) $a^4 + 4b^4$. (ii) $6x^2 + x - 2$. (iii) $x^2 + xy + y - 1$.

(*b*) Simplify : $\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \div \left(\frac{1}{a} - \frac{1}{b}\right)$.

2. (*a*) Find the H. C. F. of
$x^3-9x^2+23x-12$ and $x^3-10x^2+28x-15$.

(*b*) *Either*, If $\left(x+\frac{1}{x}\right)=\sqrt{3}$, show that $x^3+\frac{1}{x^3}=0$,

Or, If $a=1$, $b=2$, $c=3$, find the value of $\frac{b^2-2bc+c^2}{a^2-2ab+b^2}$.

3. (*a*) Solve : (i) $2(3x-1)+(x-2)=24$.

(ii) $\left.\begin{array}{l}3x-4y-11=0\\5x+3y+1=0\end{array}\right\}$.

(*b*) A number of two digits is equal to four times the sum of its digits ; and if 18 is added to the number, the digits are reversed. Find the number.

4. (*a*) If $x(b-c)=y(c-a)=z(a-b)=1$, find the value of $yz+zx+xy$.

(*b*) The sum of three numbers is 98. The ratio of the first to the second is 2 : 3, and the ratio of the second to the third is 5 : 8. Find the numbers.

5. (*a*) If $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$, find v in terms of u and f.

(*b*) *Either*, Draw the graph of $\frac{x}{3}+\frac{y}{4}=1$.

(Plot at least three points.)

Find the co-ordinates of the points where the graph intersects the two co-ordinate axes.

Or, Draw the graph of $F=\frac{9}{5}C+32$ and find from the graph the value of C when $F=85$.

1967

1. (*a*) Resolve into factors : (i) $(a-b)^2-(2b-c)^2$. (ii) x^2-x+20.

(*b*) Find the value of $\frac{(22 \cdot 5)^2-(7 \cdot 5)^2}{22 \cdot 5-7 \cdot 5}$.

(*c*) Find the L.C.M. of $3x^2+6x+3$, $2x^2+6x+4$ and $6ax+12a$.

2. (*a*) (i) A car travels at x km. per hour ; find the time taken by it to travel y metres.

(ii) A school has p pupils ; b of these are boys. What fractions of the pupils are girls ?

(*b*) Simplify : $\frac{x+1}{x-3}-\frac{3x+7}{x^2-2x-3}$.

(*c*) Show that $(x^2+y^2)(a^2+b^2)=(ax+by)^2+(bx-ay)^2$.

3. (*a*) Solve : (i) $\frac{x^2}{3}-\frac{x-1}{5}=1$. (ii) $\left.\begin{array}{l}2x+3y=0\\3x-2y=13\end{array}\right\}$.

Verify your results by substitution.

(*b*) A boy walks from home to school at a speed of 3 miles per hour. If he had a cycle, he would be able to travel at 10 miles per hour and the journey would take him 21 minutes less than if he had walked. What is the distance of the school from his home ?

4. (a) If $a : b = c : d = e : f$, show that $\frac{a^2+c^2+e^2}{b^2+d^2+f^2} = \frac{ace}{bdf}$.

(b) Two numbers are in the ratio of 2 to 3; if 6 be added to each number, the sums are in the ratio of 3 to 4. Find the numbers.

5. (a) If $S = \frac{n}{2}(a+e)$, find e in terms of a, n and s. Hence find e when $S = 7$, $a = -5$ and $n = 7$.

Either,

(b) Using the same unit and axes, draw the graphs of the following equations: (i) $x + 2y = 3$; (ii) $2x + y = 0$.

From your graphs, find the values of x and y which simultaneously solve both the equations.

(At least *three* points should be plotted for each graph.)

Or, Draw a graph to convert metres into yards (from 0 to 5 yards) taking 1 metre = 1·09 yd.

From the graph find the value in metres of 2½ yds.

1968

1. (a) Resolve into factors: (i) $6x^2 + 7x - 5$; (ii) $(x-a)^2 - 4(x-b)^2$.

(b) Use factors to find the values of:

(i) $18 \times 98 + 36$; (ii) $\frac{1}{3} \times 5{\cdot}8 \times 7{\cdot}2 - \frac{1}{3} \times 5{\cdot}8 \times 4{\cdot}2$.

(c) Find the H. C. F. of: $2a^2 - 4a$, $3a^4 - 12a^2$, $2a^5 - 2a^4 - 4a^3$.

2. (a) (i) How many minutes does it take a man to cycle a distance of x kilometres at a speed of y kilometres per hour?

(ii) A grocer mixes p kg. of tea costing him x rupees per kg. with q kg. of tea costing him y rupees per kg. Find the cost price of the mixed tea per kg.

(b) Simplify: $\frac{2x^2 - 4ax}{2x} + \frac{4x^2 - 14ax}{4x} - \frac{3x - 6a}{6}$.

(c) If $\left(r + \frac{1}{r}\right)^2 = 3$, find the value of $r^3 + \frac{1}{r^3}$.

3. (a) Solve: (i) $\frac{x+2}{3} - \frac{x-1}{2} = 1$.

(ii) $17x + 12y = 27$

$12x + 17y = 2$.

Verify your results by substitution.

(b) A man walked the first half of the distance from A to B at a speed of 6 km. per hour and the second half at a speed of 5 km. per hour. The second half took 18 minutes more than the first half. Find the distance from A to B.

4. (*a*) (i) Simplify : $7{\cdot}54\times7{\cdot}54+15{\cdot}08\times2{\cdot}46+2{\cdot}46\times2{\cdot}46$.

(ii) If $\frac{x}{y}=\frac{3}{4}$, find the value of $\frac{x+2y}{6x-y}$.

(*b*) When 1 is added to each of two numbers, their ratio becomes 1 : 2 and when 5 is subtracted from each of them, their ratio becomes 5 : 11. Find the numbers.

5. (*a*) If $s=ut+\frac{1}{2}ft^2$, find f in terms of s, u and t. Hence find f when $s=100$, $u=25$, $t=5$.

(*b*) Using the same unit and same axes, draw the graphs of the following equations :

(i) $x=6$; (ii) $\frac{x}{3}+\frac{y}{4}=1$.

Find the co ordinates of the point where the graphs intersect.

1969

1. (*a*) Resolve into factors : (i) $6x^2-7x-5$; (ii) $a(a-2)-b(b-2)$.

(*b*) Use factors to find the value of :

(i) $15\times97+45$; (ii) $(22{\cdot}5)^2-(7{\cdot}5)^2$.

(*c*) (i) The average weight of m parcels is x kilograms. If the average weight of n of these parcels is y kilograms, find the average weight of the remainder.

(ii) A room is a metres long, b metres wide and c metres high. If the area of doors and windows in the room is bc sq. m., find an expression for the area of the walls.

2. Answer *either* (*a*) and (*b*) *or* (*c*) and (*d*) :

(*a*) (i) Find the L.C.M. of : $2x^2-8$, $3x-6$, x^2-4x+4.

(ii) Simplify : $\frac{1}{x-2}+\frac{4}{4-x^2}$,

(iii) Solve : $\frac{3x-1}{6}-\frac{2x+5}{3}=-2\frac{1}{3}$.

(*b*) By doing a journey at 12·5 km.p.h., a bicyclist completes it in 3 minutes less time than if he had travelled at 12 km.p.h. Find the length of the journey.

[Use algebraic equation to solve it.]

(*c*) Solve : $2x-3y=11$
$3x+4y=8$.

Verify your results by substitution.

(*d*) I bought a horse and a carriage for Rs. 1,200. I sold the horse at a profit of 10 per cent and the carriage at a loss of 5 per cent and found that on the whole I had gained 4 per cent. What was the original price of the horse?

[Use algebraic equations to solve it.]

3. Answer *either* (*a*) and (*b*) *or* (*c*) and (*d*) :

(*a*) (i) What must be added to x^2+7x to make a complete square ?

(ii) If $x:y=3:4$, find the value of $\frac{x+2y}{6x-y}$.

(*b*) Using the same scale of units and the same axes, draw the graphs of th following equations :

(i) $x+2y=0$;

(ii) $3x-y=7$.

Find the co-ordinates of the point where the graphs intersect.

(Make a table of values of x and y for at least 3 points on each graph.)

(*c*) (i) If $a=2\cdot5$, $b=-6$, $c=0$, find the value of $a^2b+b^2c+ab^2$.

(ii) If $s=\left(\frac{u+v}{2}\right)t$, find v in terms of s, u and t.

Hence, find v when $s=180$, $u=21$, $t=10$.

(*d*) Draw a graph for converting kilometres into miles, given that

1 km. = 0·62 mile.

Use the graph to convert 3·1 miles to kilometres.

1970

1. (*a*) Resolve into factors : (i) $4x^2-16x+7$; (ii) $ac+bd-bc-ad$.

(*b*) Use factors to find the value of : (i) $17\times96+68$; (ii) $(2\cdot5)^2-2\cdot5\times1\cdot5$.

(*c*) (i) If n is an odd number, write down the next two odd numbers.

(ii) Find the distance in metres a man goes when running at b km. p.h. for t minutes.

2. Answer *Either* (*a*) and (*b*) *or* (*c*) and (*d*) :

(*a*) (i) Find the H.C.F. of : $2a^2x-a^2$, $4ax^2-a$.

(ii) Simplify : $\frac{2x-3}{3}-\frac{2x-4}{4}$.

(iii) Solve : $\frac{3x-1}{5}-\frac{x+2}{2}=-2\frac{1}{2}$.

(*b*) A father is now three times as old as his son ; after 15 years he will twice as old as his son. How old is the father now ?

(*c*) Solve : $2x+5y=1$, $4x+3y+12=0$.

Check your results by substitution.

(*d*) A number is of two digits whose sum is 12. If the digits are interchanged, the number is increased by 18. Find the number.

3. (*a*) If $v^2 = u^2 + 2fs$, find s in terms of u, v and f; and also the value of s when $u = 20$, $v = 12$, $f = -2$.

Either, (*b*) Using the same scale of units and the same axes, draw the graphs of the following equations :

(i) $2x + y = 0$; (ii) $x - 3y = 7$.

Find the co-ordinates of the point where the graphs intersect.

(Make a table of values of x and y for at least three points on each graph.)

Or, (*b*) Draw a graph to convert inches into centimetres, given that 1 inch = 2·5 cm. Find from the graph the number of inches in 8 centimetres.

APPENDIX

INEQUATION

1. Simple and Compound inequations :

Two quantities may have, between them, three relations, one of equality and two of inequality. If they are unequal, then one of those quantities must be either greater or less than the other. For instance, if $a \neq b$ (*i.e.*, a is not equal to b), then $a > b$ (*i.e.*, a is greater than b), or $a < b$ (*i.e.*, a is less than b).

As the relation of equality, stated in the form of an equation provides a means of finding out the value of an unknown quantity, so also the relation of inequality stated in the form of an inequation makes it possible to solve out the value or values of an unknown quantity. For example, let x be an integer greater than five. In figures then,

$$x > 5.$$

What is the value of x ? It is easy to see that x may be any integer from 6 to infinity. The solution of the inequation thus does not give a single value, it gives a set of values for x. But if it is stated that x is an integer such that it is greater than five but less than seven, then x is certainly 6. In figures, if

$$x > 5 \quad \text{and} \quad x < 7,$$

then $\quad x = 6$

In this case the solution gives a single value. But it must be noted that there are two *simple* inequations here and these may be conveniently written as

$$5 < x < 7.$$

In words this means : The integer x lies between 5 and 7. This is a *compound* inequation. A few more examples of such compound inequations are given and explained below.

(i) $x \geqslant 5$ means $x > 5$, or $x = 5$.

(ii) $x \leqslant 3$ means $x < 3$, or $x = 3$.

(iii) $5 > x > 2$ means

$$5 > x \text{ and } x > 2.$$

(iv) $x \nless 4$ (*i.e.*, x is not less than 4)

means $x = 4$, or $x > 4$.

(v) $x \ngtr 2$ (*i.e.*, x is not greater than 2)

means $x = 2$, or $x < 2$.

2. Solution of Simple inequations :

In the case of an equation the equality of two sides is not affected, if (i) the same quantity is added to or subtracted from each of the sides, and also if (ii) each of the sides is multiplied or divided by the same quantity.

It will be proved hereinafter that in the case of an inequation too (i) always holds good but (ii) holds good only under a certain condition.

(i) If $a > b$ or $b < a$, then let

$$a = b + d, \text{ where } d \text{ is positive.}$$

$\because$ this is an equation,

$$a + c = b + d + c\ ;$$

or, $$a + c = (b + c) + d\ ;$$

$$\therefore \quad (a + c) > (b + c) \quad [\because d \text{ is positive }].$$

Similarly,

$$\because \quad a = b + d,$$

then $$a - c = b + d - c$$

or, $$a - c = (b - c) + d\ ,$$

$$\therefore \quad (a - c) > (b - c) \quad [\because d \text{ is positive }].$$

This proves that an inequation remains unaffected if the same quantity is added to or subtracted from each of its sides.

(ii) Again, if $a > b$ or $b < a$, then let

$$a = b + d, \text{ where } d \text{ is positive.}$$

Multiplying both sides by c, we have

$$ac = bc + dc.$$

Now, if c is positive, then dc must be positive and so ac must be greater than bc to the extent of dc.

$$\therefore \quad ac > bc.$$

But if c is negative, then dc must be negative, and in that case ac will be less than bc to the extent of dc, *i.e.*, $ac < bc$.

Similarly,

$$\because \quad a = b + d,$$

$$\frac{a}{c} = \frac{b}{c} + \frac{d}{c} \quad \text{(dividing throughout by } c\text{)}.$$

Here also, if c is positive, then

$$\frac{a}{c} > \frac{b}{c} \text{ to the extent of } \frac{d}{c};$$

but if c is negative, then

$$\frac{a}{c} < \frac{b}{c} \text{ to the extent of } \frac{d}{c}.$$

$\therefore$ an inequation remains unaffected if both its sides are multiplied or divided by the same *positive* quantity.

Example 1. If $(2x+11) > (x+15)$, find the value of x.

$$(2x+11) > (x+15),$$

or, $\{x+(x+11)\} > \{(x+11)+4\}$,

or, $\{x+(x+11)-(x+11)\} > \{(x+11)+4-(x+11)\}$,

or, $x > 4$.

$\therefore$ the required value of x is any number greater than 4.

Example 2. Solve : $(3x+15) < (5x+11)$.

$$(3x+15) < (5x+11),$$

or, $\{(3x+11)+4\} < \{(3x+11)+2x\}$,

or, $\{(3x+11)+4-(3x+11)\} < \{(3x+11)+2x-(3x+11)\}$,

or, $4 < 2x$,

or, $2 < x$, or, $x > 2$.

Example 3. A milk-man purchased 9 cows when he had twice as many cows as he had at the beginning. With the addition of the newly purchased cows the number was found to be more than three times the original number of cows. What is the highest possible original number of the cows ?

Let us suppose that the milk-man had x number of cows at first. Then, according to the question,

$$2x + 9 > 3x,$$

or, $2x + 9 - 2x > 3x - 2x,$

or, $9 > x$, *i.e.*, $x < 9$.

$\therefore$ the highest possible original number of cows is 8.

Example 4. $\frac{1}{5}$th of the number of boys of a class got plucked only in mathematics and one more than $\frac{1}{4}$th of the total number of boys of the same class got plucked only in the second language. None got plucked in more than one subject and the total number of failures was less than half the number in the class. What is the least possible number of boys in that class ?

Let the number of boys in the class be x.

Then according to the question,

$$\left(\frac{x}{5} + \frac{x}{4} + 1\right) < \frac{x}{2},$$

or, $(4x + 5x + 20) < 10x$, (Multiplying throughout by 20)

or, $20 < x$, *i.e.*, $x > 20$.

Since both $\frac{1}{5}$th and $\frac{1}{4}$th of the number are each integers, it must be divisible by 4×5 or 20 and also it must be greater than 20. The least number, therefore, must be 40.

$\therefore$ the least possible number of boys in the class is 40.

Example 5. Prove that the sum of the squares of two unequal numbers is greater than twice their product.

Let the numbers be a and b.

$\because$ $(a - b)^2 = a^2 + b^2 - 2ab$, and since a square is always a positive integer as $a \neq b$,

we have $a^2 + b^2 - 2ab > 0$,

or, $a^2 + b^2 > 2ab$.

$\therefore$ the sum of the squares of two unequal numbers is greater than twice their product.

EXERCISE 1

1. If $a = b + k$, where a and b are positive but k is negative, then, prove that $a < b$.

2. Write the following with the help of algebraic symbols :

(i) a is a positive number ; (ii) a is a negative number ;

(iii) a is equal to zero or a positive number ; (iv) a is equal to zero or a negative number ; (v) a is any number between 5 and 8.

3. State from each of the following relations whether x is positive or negative :

(i) $(x+1) > 1$; (ii) $x+2 > 3$; (iii) $x-1 > 0$;
(iv) $x+1 < 1$; (v) $2 < x+2$; (vi) $(2x+3) > (3x+3)$

Solve :

4. $(7x+23) > (2x+38)$.
5. $(7x+9) > 5(x+3)$.
6. $(3x+7) < (4x+5)$.
7. $2(x+3) < (5x+3)$.
8. $3(x+11) < (6x+15)$.
9. $(10x+5) > 5(x+1)$.
10. $(x+1) > 1$.
11. $(2x+5) < (3x+2)$.
12. $2 < (x+2) < 7$.
13. $8 > (x+2) > 3$.
14. $(x+1) \nless 1$.
15. $(x+3) \ngtr 3$.

Express symbolically :

16. x is any number between 5 and 7.

17. x is equal to or greater than 3 but less than 8.

18. x is any number from 2 to 5.

19. If x is an integer and $9 < x < 11$, then what is the value of x ?

20. If x is an integer such that $5 < x < 7$, then what is the value of x ?

21. Find the solution sets of x if x is a natural number such that $5 < x < 11$.

22. Shew that the product of two successive odd numbers is less than the square of the even number between them.

23. If $a \neq b \neq c$, then prove that $(a^2+b^2+c^2) > (ab+bc+ca)$.

24. If $\frac{1}{x}$ is a negative integer, then, prove that $x > \frac{1}{x}$.

25. A big family bought and settled down in a new house When the number of the members of the family increased to one and a half of what it was at the time of coming to the house, 25 tenants rented a portion of the house and started living there. As a result there were more than two and a half times as many heads as there were at first in that house. What was the highest possible number of members of the family at first ?

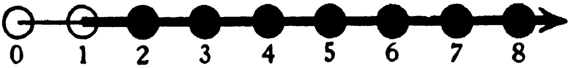
26. A boy has one rupee and a few paise in 5 and 10 paise coins totalling **15.** (i) How many 5 paise coins might he have at most ? (ii) What are the least and the greatest amounts of money that he might have had ?

3. Graph of Simple inequation :

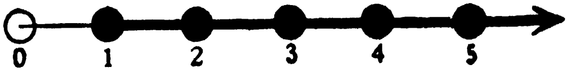
It is seen that the solution of a simple inequation gives a set of values. These may be graphically shown on a number-line. For example, if, $x+1 > 2$,

then $x > 1$.

Now, on a line from left to right, points indicating, 0, 1, 2, 3, 4, 5, etc. are marked successively at equal distances and each such point is placed in a circle. All but the circles round 0 and 1, are dotted solid and the portion of the line just-beyond the point 1 extending to the right is marked deep and fat as below. This fat portion of the line, every point on which gives a value for x greater than 1, is the graph of the inequation, $x+1 > 2$.

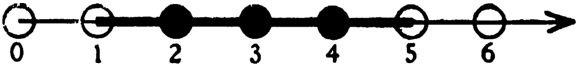


It is easy to see that the graph of the inequation, $x > 1$, will be the fat portion of the line below on which the circle round 1 is also solid. For, $x = 1$ is a solution set of this inequation.



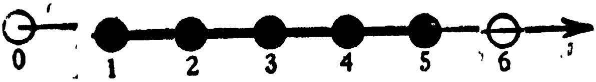
Example 1. Draw the graph of $1 < x < 5$.

Let the points 0, 1, 2, 3, 4 and 5 be plotted on the number-line below successively at one unit's distance apart. Each such point is placed within a circle. The circles round 2, 3 and 4 are made solid and the portion of the line beyond the point 1 upto a point just adjacent to 5 is marked fat. Since 1 and 5 are not on the graph, circles containing 1 and 5 are not solid. This fat portion of the number-line is the graph of the inequation.



Example 2. Draw the graph of $1 \leqslant x \leqslant 5$.

Here both the points 1 and 5 are on the graph. The portion of the number-line from the point 1 to the point 5, therefore, is the graph of the inequation, $1 \leqslant x \leqslant 5$. Since 1 and 5 are on the graph, circles containing 1 and 5 are shown solid. This is given below :



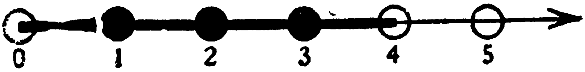
EXERCISE 2

Draw the graph of :

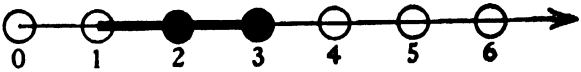
1. $x > 3$.
2. $x < 7$.
3. $(2x+5) \geqslant (x+7)$.
4. $0 < x < 3$.
5. $x > 4$ or $x = 4$.
6. $x > 7$ and $x \leqslant 11$.
7. $5 > x > 1$.
8. $2 < (x+1) < 6$.
9. $x \ngtr 4$.
10. $x \nless 0$.
11. $x \geqslant 3$.
12. $x \leqslant 5$.

Write down the inequation for each of the followings graphs :

13.



14.



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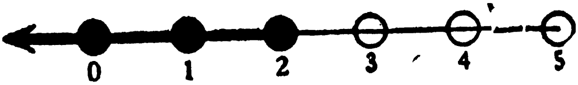


4. **Simultaneous inequation :** It has been seen earlier that a simple inequation gives a set of values for the unknown and that these values plot a segment of a straight line. It will be presently seen in the following examples that every pair of simultaneous inequations

involving two unknowns also give quite a few pairs of values for the variables. These pairs indicate points lying in a region and not on a straight line. It is, therefore, necessary to draw graphs of each of the simultaneous inequations so as to indicate the region, the co-ordinates of every point of which give a pair of solution of the inequations concerned. The following examples will explain the matter in full.

Example 1. Solve : $x+y-3>0$
and $2x-y-5>0$.

We have from

$$x+y-3>0,$$
$$\text{or,}\quad y>3-x \qquad \cdots \quad (1)$$

Let us compare this with the equation, $y=3-x$.

It is easy to see that for a particular value of x, the value of y of the inequation will be greater than that of the equation. This shows that the paired values of x and y of the inequation will give points lying on the upper region of the straight line, $y=3-x$. We, therefore, draw the graph of the equation with two mutually perpendicular straight lines $\overleftrightarrow{XOX'}$ and $\overleftrightarrow{YOY'}$ as the axes of co-ordinates and 5 times the length of a side of the smallest square of the graph-paper as the unit of length.

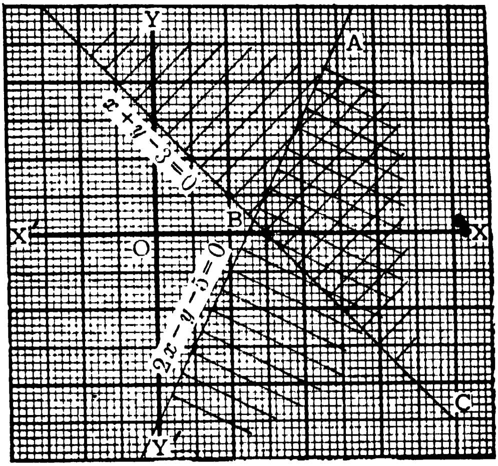
(0, 3), (1, 2) and (2, 1) are points on the graph indicated by $\overleftrightarrow{BC}$. The region shaded /// above $\overleftrightarrow{BC}$, then, is the graph of the inequation (1).

Similarly, we have from $2x-y-5>0$,

$$\text{or,}\quad 2x-5>y,$$
$$\text{or,}\quad y<2x-5. \qquad \cdots \quad (\text{ii})$$

Comparing (ii) with the equation $y=2x-5$, we find that the ordinate y of every point on (ii) is less than that of the equation, $y=2x-5$. The region indicating the graph of the inequation (ii), therefore, shall lie below the line, $y=2x-5$. With the same axes of co-ordinates and the unit of length we draw the graph of this equation and indicate it by $\overleftrightarrow{AB}$, the lower region of which is shaded \\\ to indicate the graph of the inequation (ii).

It is now easy to see that the region within the angle ABC is common to both the inequations and that the co-ordinates of every point of this region give the solution of the inequation (i) and (ii).



Example 2. Solve : $2x - 3y - 1 > 0$ ··· (1)

and $2x + 3y - 7 < 0$ ··· (2)

From (1), we have

$$2x - 1 > 3y, \text{ or, } y < \frac{2x-1}{3}$$

Therefore, the ordinate y of every point of (1) is less than that of $y = \frac{2x-1}{3}$, and hence the points on (1) will lie in a region below the line, $y = \frac{2x-1}{3}$.

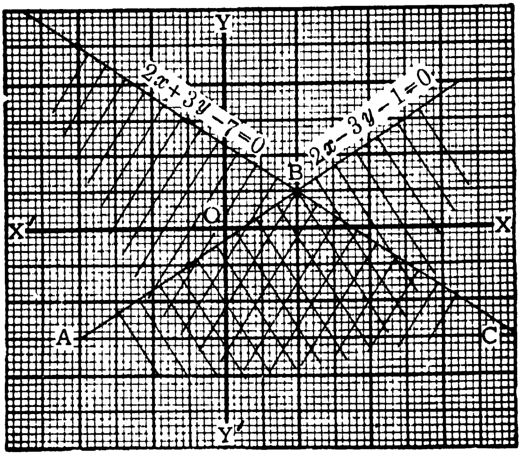
Again from (2), we have

$$2x + 3y - 7 < 0, \text{ or, } 3y < 7 - 2x,$$

$$\text{or, } y < \frac{7-2x}{3}.$$

This shows that the ordinate of every point on (2) is equal to or less than that of $y = \frac{7-2x}{3}$ and hence the points on (2) will lie either on $y = \frac{7-2x}{3}$ or in the region below the line.

Now, with two mutually perpendicular axes, $\overleftrightarrow{XOX'}$ and $\overleftrightarrow{YOY'}$ and with 5 times the length of a side of the smallest square of the graph-paper as unit, we draw the graphs of $y = \frac{2x-1}{3}$ and $y = \frac{7-2x}{3}$ and indicate them by $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ respectively.



It is easy to see that all points in the common region within the angle ABC and along $\overrightarrow{BC}$, give the solution sets of the given inequations (1) and (2).

Example 3. A weaver has to produce at least 10 pieces of 'Krishna' and 5 pieces of 'Kaveri' saris (cloth) every week to stay in business. The cost of production of each piece of 'Krishna' is Rs. 60/- and that of every piece of 'Kaveri' is Rs. 80/-. If the weaver is unable to invest more than Rs. 1500/- per week, then, how many of each kind should he produce so as to produce the largest number of pieces together ?

Let the weaver produce x pieces of 'Krishna' and y pieces of 'Kaveri' saris every week so as to meet the given conditions.

Then according to the question,

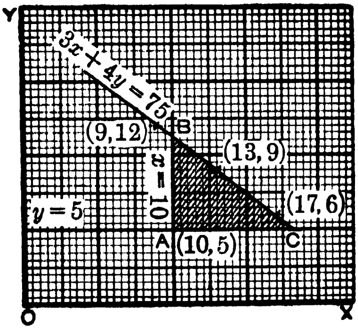
$$60x + 80y < 1500.$$

$$\text{i.e.,} \quad 3x + 4y < 75 \quad \cdots \quad \text{(i)}$$

$$\text{and} \quad x > 10, \quad \cdots \quad \text{(ii)}$$

$$y > 5. \quad \cdots \quad \text{(iii)}$$

We now draw the graphs of $x = 10$, $y = 5$ and $3x + 4y = 75$ and indicate them by $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ respectively on the graph-paper below.



It is easy to see that the co-ordinates of the points of the region bounded by the triangle ABC will give the solution sets of the above inequations.

Since the number of pieces of cloth must be integral, the values of x and y must also be integers and their greatest values must be indicated by the co-ordinates of a point on $3x + 4y = 75$ or $\overleftrightarrow{BC}$.

Again, since, $y = \dfrac{75 - 3x}{4}$ and $x > 10$ and $y > 5$,

$$\text{when } \left.\begin{matrix} x = 13 \\ y = 9 \end{matrix}\right\} \text{ and when } \left.\begin{matrix} x = 17 \\ y = 6 \end{matrix}\right\}.$$

These two points viz. (13, 9) and (17, 6) are the only points on $\overleftrightarrow{BC}$ for integral values of x and y.

Now, $13 + 9 = 22$ and $17 + 6 = 23$ and $23 > 22$.

$\therefore$ the required solution is $x = 17$

and $y = 6$

That is, the weaver must produce 17 pieces of 'Krishna' and 6 pieces of 'Kaveri' saris per week under the given conditions.

Example 4. A printing press takes 2 hours to compose 1 format of a book *A* in small pica and 1 hour to print 1000 copies of 1 format of the same book. The press takes 1 hour to compose 1 format of another pictorial book *B* in bigger types and 2 hours to print one thousand copies of the same format. If the press gain Rs 80/- by printing 1000 copies of 1 format of *A* and Rs 100/- by printing 1000 copies of 1 format of *B*, and if the working hours of the press do not exceed 12 hours a day, then how many formats of each book should be composed and printed in one thousand copies each, so as to maximise the profit of the press ?

Let x formats of *A* and y formats of *B* be composed and printed in 1000 copies each, so as to maximise the profit.

Then, the total time devoted to composing is $(2x + y)$ hours and

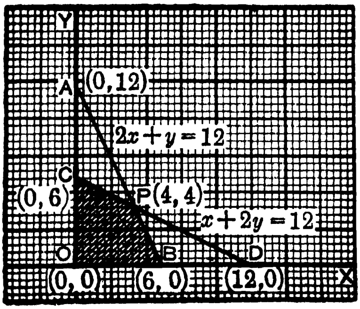
$$2x + y < 12. \quad \cdots \quad \text{(i}$$

Again the total time devoted to printing is $(x + 2y)$ hours

and $$x + 2y < 12. \quad \cdots \quad \text{(ii)}$$

Also $$x > 0 \text{ and } y > 0$$

Now, in the graph-paper below, graphs of $2x + y = 12$ indicated by $\overleftrightarrow{AB}$ and of $x + 2y = 12$ indicated by $\overleftrightarrow{CD}$ are drawn, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$



intersect at P and $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ represent graphs of $x=0$ and $y=0$ respectively.

It is evident that the co-ordinates of every point of the region bounded by $OOPB$ will give the solution sets of the above inequations. It is also seen from the graph that the values of both x and y in the region range from 0 to 6.

The profit in rupees $=80x+100y$.

It is now found from the table below that the profit is maximum, when $x=4$ and $y=4$.

Number of formats of A x	Number of formats of B y	Profit Rs. $(80x+100y)$
0	6	Rs. 600
4	4	Rs. 720
6	0	Rs. 480

$\therefore$ the press must compose 4 formats of each book and print 1000 copies of those formats so as to maximise its profit.

EXERCISE 3

Solve :

1. $x+2y-4>0$ and $2x-y-3>0$.
2. $2x+3y-7>0$ and $3x-2y-4<0$.
3. $5x+3y>11$ and $7x-2y>3$.
4. $3x-2y>5$ and $x+3y<9$.
5. $5x-3y-9>0$ and $3x-2y=5$.
6. $2x+2y>4$ and $2x+y<6$.

7. A man wants to buy at least 6 pieces of 'Dhuti' and 4 pieces of 'Sari' within 500 rupees. The cost of 'Dhutis' and 'Saris' are respectively Rs. 30 and Rs. 40 per piece. How many pieces

of each kind of cloth should the man buy so as to get the maximum number of pieces together ?

8. A man proposes to spend 100 rupees for buying a few pieces of article 'A' at 12 rupees per piece along with a few pieces of article 'B' at 8 rupees per piece. He has to purchase at least one piece of 'A' and a number of pieces of 'B' not exceeding 8. How many pieces of each kind should he purchase so as to get the maximum number of those articles together ?

9. It takes $1\frac{1}{2}$ hours in a machine A and $\frac{1}{2}$ hour in a machine B to produce one packet of articles marked x. One packet of another article marked y can be produced by working $\frac{1}{2}$ hour in machine A and $1\frac{1}{2}$ hours in machine B. Each machine may be worked upto eight hours a day. If the profit per packet of x is Rs. 5 and that per packet of y is Rs. 3, then how many packets of each kind should be produced so as to maximise the profit ?

10. 176 pilgrims have all together 35 quintals of luggage with them. They may reach their destination in two types of small buses. One type of these buses can carry 24 passengers and 5 quintals of luggage and the other can carry 16 passengers and 3 quintals of luggage. The charge per trip of the first is Rs. 250 and that per trip of the second is Rs. 200. How many trips of each of these buses should be arranged so as to carry all the passengers together with their whole lot of luggage so as to minimise the expense ? What will then be the minimum expense ?

11. A weaver has to produce at least 12 dhutis and 8 saris per week to stay in business. If the cost of production of each dhuti is Rs. 20 and that of each sari is Rs. 30 and if the weaver can invest not more than Rs. 600 per week, then how many pieces of dhutis and saris should the weaver produce per week, so as to put up maximum number of pieces for sale ?

12. Two types of cycles, low-speed and high-speed are manufactured in a factory. The cost of production of each low-speed cycle is Rs. 150 and that of each high-speed cycle is Rs. 200. The managing authority of the factory cannot spend in a week more than Rs. 5000 for raw-materials and wages. And to stay in business, the factory has to produce at least 15 pieces of low-speed and 10 pieces of high-speed cycles. If the profit per low-speed cycle is Rs. 50 and that per high-speed cycle is Rs. 75, how many of each kind should the factory produce so as to maximise the profit per week ?

ANSWERS TO EXERCISES IN THE APPENDIX

Exercise 1

2. (i) $a > 0$; (ii) $a < 0$; (iii) $a \geqslant 0$; (iv) $a \leqslant 0$; (v) $5 < a \leqslant 8$.

3. (i) positive ; (ii) positive ; (iii) positive ; (iv) negative ;
(v) positive ; (vi) negative.

4. $x > 3$. **5.** $x > 3$. **6.** $x > 2$. **7.** $x > 1$.

8. $x > 6$. **9.** $x > 0$. **10.** $x \geqslant 0$. **11.** $x \geqslant 3$.

12. $0 < x < 5$. **13.** $1 < x < 6$. **14.** $x \nless 0$. **15.** $x \ngtr 0$.

16. $5 < x < 7$. **17.** $3 \leqslant x < 8$. **18.** $2 \leqslant x \leqslant 5$. **19.** $x = 10$.

20. $x = 6$, or 7. **21.** 6, 7 8, 9, 10. **25.** 24. **26.** (i) 9 ;
(ii) Rs 1·05 ; and Rs 1·45.

Exercise 2

[Read the graphs as numbered successively 13, 14 and 15.]

13. $0 < x < 4$. **14.** $1 < x \leqslant 3$. **15.** $x \leqslant 2$.

Exercise 3

1. Co-ordinates of all points in the region above the graph of $y = \frac{4 - x}{2}$ and below the graph of $y = 2x - 3$.

2. Co-ordinates of all points in the region above the graph of $y = \frac{7 - x}{3}$ and on or above the graph of $y = \frac{3x - 4}{2}$.

3. Co-ordinates of all points in the region above the graph of $y = \frac{11 - 5x}{3}$ and below the graph of $y = \frac{7x - 3}{2}$.

4. Co-ordinates of all points in the region below the graph of $y = \frac{3x - 5}{2}$ and on or below the graph of $y = \frac{9 - x}{3}$.

5. Co-ordinates of all points in the region below the graph of $y=\frac{5x-9}{3}$ and on the graph of $y=\frac{3x-5}{2}$.

6. Co-ordinates of all points in the region above the graph of $y=2-x$ and below the graph of $y=6-2x$.

7. 10 pieces of 'dhuti' and 5 pieces of 'sari'.

8. A —3 pieces, B —8 pieces.

9. 4 packets of each kind.

10. 4 trips of the 1st type of bus and 5 trips of the 2nd type of bus; minimum cost Rs. 2,000.

11. 18 pieces of 'dhuti' and 8 pieces of 'sari'.

12. 16 low-speed and 13 high-speed.

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